

Problem Set 2: 18.S996 Hydrodynamic quantum analogs

Due April 10

Problem 1. Consider the Klein-Gordon equation:

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + \frac{m^2 c^2}{\hbar^2} \phi = 0 \quad (1)$$

a) Show that if one seeks a solution of the form $\phi(x, t) = e^{i\omega_c t} \Psi(x, t)$, then the wave envelope $\Psi(x, t)$ satisfies the linear Schrodinger equation,

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi \quad (2)$$

provided $\Psi(x, t)$ evolves slowly relative to the Compton frequency, $\omega_c = mc^2/\hbar$.

b) Calculate the phase velocity and group velocity of the Klein-Gordon equation. Compare your results to those of water waves.

c) If a particle is accompanied by a wave of wavelength λ , specifically the particle speed is equal to the wave's group velocity, show that the particle momentum satisfies the de Broglie relation $\mathbf{p} = \hbar \mathbf{k}$.

d) Discuss the relation between strobing the hydrodynamic pilot-wave system at the Faraday frequency, and strobing the Klein-Gordon wave at the Compton frequency.

Problem 2. The Madelung transformation

The non-linear Schrodinger (Gross-Pitaevskii) equation governs the evolution of dilute Bose-Einstein condensates, in which the interaction potential between particles takes a particularly simple form:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V\Psi + g_0 |\Psi|^2 \Psi \quad (3)$$

a) Execute the Madelung transformation on this nonlinear Schrodinger equation. Specifically, express the wavefunction $\Psi(\mathbf{x}, t) = \rho(\mathbf{x}, t) e^{iS(\mathbf{x}, t)/\hbar}$ in terms the probability density $\rho(\mathbf{x}, t)$ and action $S(\mathbf{x}, t)$, and so deduce equations governing the evolution of these real-valued functions.

b) For particle motion in a plane, relate the equations deduced above to those governing small-amplitude waves in shallow-water hydrodynamics in a term by term basis. Identify the probability density with the fluid depth, and the quantum velocity of probability with the depth-averaged flow velocity in the fluid layer.

c) Discuss the relation between surface tension σ in hydrodynamics and \hbar in quantum mechanics.

d) Interpret the resulting systems physically, discussing specifically the quantum potential.

3. Bohmian mechanics

a) Describe how Bohmian mechanics follows from Q2. Starting with the Madelung-transformed linear Schrodinger equation (set $g_0 = 0$ in Q2a), follow Bohm's assumption that the particle velocity is equal to the quantum velocity of probability $\nabla S/m$. Thus, deduce Bohm's second-order guidance equation, in which the quantum potential appears explicitly.

b) Solve the linear Schrodinger equation in a circular corral. Show that if a single mode is selected, Bohmian mechanics predicts zero particle velocities.

(OPEN-ENDED, POSSIBLY TRICK) BONUS QUESTIONS: choose 1 only

c) If there are two modes, compute Bohmian trajectories. Is the pdf associated with the resulting trajectories consistent with the wavefunction?

d) Calculate the probability distribution function $\rho(r, t)$ that emerges from a particle executing a random walk over the quantum potential. You choose the form of the random walk.

e) Give a coherent, rational account of Bohmian mechanics for a free particle.