Lecture 11. Fluid jets

The shape of a falling fluid jet

We assume the jet Reynolds number $Re = \frac{Q}{a V}$ is sufficiently large that we can neglect viscous effects. We further assume that the jet speed is independent of radius, so can be described as $U(z)$. Deduce $r(z)$ and $U(z)$.

Applying Bernoulli's $Pm$ at points $A$ and $B$:

\[ \frac{1}{2} \rho U_0^2 + \rho g z + P_A = \frac{1}{2} \rho U^2(z) + P_B \]

The local curvature of slender threads:

\[ \nabla \cdot \eta = \frac{1}{R_1} + \frac{1}{R_2} \approx \frac{1}{r} \]

\[ \Rightarrow P_A \approx P_0 + \frac{\sigma}{a}, \quad P_B \approx P_0 + \frac{\sigma}{r} \]

Bernoulli:

\[ \frac{1}{2} \rho U_0^2 + \rho g z + P_0 + \frac{\sigma}{a} = \frac{1}{2} \rho U^2(z) + P_0 + \frac{\sigma}{r} \]

\[ \Rightarrow \frac{U(z)}{U_0} = \left[1 + \frac{2}{Fr} \frac{r}{a} + \frac{2}{We} \left(1 - \frac{a}{r}\right) \right]^{\frac{n}{2}} \]

Jet accelerates due to $g$; jet slows down due to $\sigma$
\[ F_r = \frac{V_0^2}{g a} = \frac{\text{INERTIA}}{\text{GRAVITY}} = \text{Troude number} \]
\[ We = \frac{\rho V_0^2 a}{\sigma} = \frac{\text{INERTIA}}{\text{CURVATURE}} = \text{Weber number} \]

Now flux conservation ensures
\[ Q = 2\pi \int_{0}^{r} V(z) r(z) dv = \pi r^2 V(z) = \pi a^2 V_0 \]
from which we find
\[ \frac{V(z)}{a} = \left( \frac{V_0}{V(z)} \right)^{\frac{1}{2}} = \left[ 1 + \frac{2}{F_r} \frac{z}{a} + \frac{2}{We} \left( 1 - \frac{a}{r} \right) \right]^{-\frac{1}{2}} \]
This may be solved algebraically to yield \( V(z) / a \) by substituting into \( \star \) to deduce \( V(z) \).

In the \( We \rightarrow 0 \) limit, one finds
\[ \frac{V}{a} = \left( 1 + \frac{2g z}{V_0^2} \right)^{-\frac{1}{2}} , \quad \frac{V(z)}{V_0} = \left( 1 + \frac{2g z}{V_0^2} \right)^{-\frac{1}{2}} \]

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**The Rayleigh – Plateau Instability**

- the instability of a quiescent fluid cylinder bound by \( \sigma \) *static*
- neglect influence of gravity \( g \) and viscosity \( \nu \)
- consider a cylinder of radius \( R_0 \)
Internal pressure:
\[ \rho_0 = \sigma \overline{V} \cdot n = \sigma \overline{R}_0 \]
assuming \( P_{\text{atm}} = 0 \).

We consider the evolution of infinitesimal vericheose perturbation, which enables us to linearize the governing equation.

The perturbed columnar surface takes the form:
\[ \tilde{R} = \overline{R}_0 (1 + \varepsilon e^{\frac{w + i \kappa z}{2}}) \quad \text{where} \quad \kappa = \frac{2\pi}{L} \]

Find \( \kappa \) for which \( \text{Re} [w] > 0 \), in which case the perturbation will grow. Also, we'll find the mode \( \kappa \) that maximizes \( \text{Re} [w] \), the fastest growing mode.

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**Note:**
- perturbation amplitude \( \varepsilon \ll 1 \)
- \( w \) is the growth rate
- \( \kappa = \frac{2\pi}{L} \) is the wavenumber

Define:
- \( \tilde{u}_r = \text{radial perturbation velocity} \)
- \( \tilde{u}_z = \text{axial " "} \)
- \( \overline{P} = \text{Perturbation pressure} \)
Substituting $\bar{u}_r, \bar{u}_z, \bar{p}$ into Navier-Stokes equations and retaining terms only to order $\varepsilon$ yields:

$[NB: \bar{u} \cdot \nabla \bar{u} \to 0]$ \[ \]

Momentum equations:

$$\frac{d \bar{u}_r}{dt} = -\frac{1}{\rho} \frac{d \bar{p}}{dr} , \quad \frac{d \bar{u}_z}{dt} = -\frac{1}{\rho} \frac{d \bar{p}}{dz}$$

Continuity:

$$\frac{d \bar{u}_r}{dr} + \frac{\bar{u}_r}{r} + \frac{d \bar{u}_z}{dz} = 0$$

We anticipate that all perturbations have the same form as the surface disturbance, so

$$\bar{u}_r = \bar{R}(z)e^{\omega t + ikz} \quad , \quad \bar{u}_z = \bar{Z}(z)e^{\omega t + ikz} \quad , \quad \bar{p} = \bar{\rho}(z)e^{\omega t + ikz}$$

The figure illustrates the behavior of the perturbations with $\lambda = \frac{2\pi}{k}$. The domain is bounded by $R_0$ and $R_2$. The velocity field $u = (u_r, u_z)$ is shown, with $\sigma$ indicating the perturbation's phase.
Substituting $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{p}$ into linearized NS yields

Momentum: $\omega R = -\frac{1}{\rho} \frac{dP}{dn}$, $\omega Z = -i k \tilde{p}$

Continuity: $\frac{dR}{dn} + \frac{R}{n} + i k \tilde{Z} = 0$

Substituting $Z$ from $\star$, then take $\frac{d}{dn}$ to deduce

$v^2 \frac{d^2 R}{dn^2} + v \frac{dR}{dn} - \left[ 1 + (kn)^2 \right] R = 0$

This corresponds to the modified Bessel eqn of order 1, whose solutions are the modified Bessel functions of the first and second kind, $I_1(kn)$ and $K_1(kn)$. We note that $K_1(kn) \to \infty$ as $n \to 0$; thus, the well-behavedness of our solution requires

$R(n) = C f \left( I_1(kn) \right)$

where $C$ is a constant to be determined by application of appropriate boundary conditions.

Deduce $P(n)$ from $R(n)$ by integrating $\star$:

$P(n) = -\frac{\omega P C}{k} I_1(kn)$

and using the Bessel function identity $I_1'(x) = I_0(x)$.

Now we apply the boundary conditions.
1. Kinematic Condition:

\[ \frac{\partial \mathbf{u}}{\partial t} = u \cdot \mathbf{n} = \bar{u}_r \quad \text{at} \quad r = R_0 \]

\[ \Rightarrow R_0 \mathbf{\omega} = G' I_i (kR_0) \]

\[ \Rightarrow G' = \frac{\varepsilon \omega R_0}{I_i (kR_0)} \]

Thus \[ \Rightarrow P(r) = -\frac{\varepsilon \omega^2 \rho R_0}{k} \frac{I_o (kR_0)}{I_i (kR_0)} \]

2. Normal Stress BC:

\[ P_0 + \bar{p} = \sigma \mathbf{D} \cdot \mathbf{n} = \sigma \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \]

where \[ \frac{1}{R_1} = \frac{1}{R_0 (1 + \varepsilon e^{\omega t + ikz})} \approx \frac{1}{R_0 (1 - \varepsilon e^{\omega t + ikz})} \]

\[ \frac{1}{R_2} = -\frac{1}{k^2} \left[ \frac{\partial R}{\partial z} \right] \approx -\frac{d^2 R_1}{dz^2} \]

\[ \times \varepsilon k^2 R_0 e^{\omega t + ikz} \]

Subbing into \[ \Rightarrow P_0 + \bar{p} = \frac{\sigma}{R_0} - \frac{\varepsilon \sigma}{R_0} (1 - k^2 R_0^2) e^{\omega t + ikz} \]

Compare to \[ \Rightarrow \] to find:
\[ \tilde{P} = -\frac{\xi \sigma}{R_0} (1 - k^2 R_0^2) e^{\omega t + i k x} \]
\[ = -\frac{\xi \omega^2 \rho R_0}{k} \frac{I_0(kR_0)}{I_1(kR_0)} e^{\omega t + i k x} \]

**Dispersion Relation**: relates \( \omega \) to \( k \)

\[ \omega^2(k) = \frac{\sigma}{\rho R_0^2} k \frac{I_1(kR_0)}{I_0(kR_0)} (1 - k^2 R_0^2) \]

**Note**:  
1. The column is only unstable to wavelengths \( \lambda = \frac{2\pi}{k} \) that exceed the circumference of the jet.

2. Fastest growing mode occurs when \( \frac{d\omega}{dk} = 0 \), i.e. \( kR_0 = 0.697 \), when the wavelength \( \lambda = \frac{2\pi}{k} \approx 9.02 \) \( R_0 \).

3. By inverting the max growth rate, \( \omega_{\text{max}} \), one finds break-up time, \( t_{\text{break}} \approx 2.91 \sqrt{\rho R_0^3 / \sigma} \).
e.g., a water jet of diameter 1 cm,
take ~ $\frac{1}{8}$ sec, which is roughly consistent with casual observation.

4) When a vertical jet impinges on a fluid bath, a standing field of waves may be excited on its base. Requiring the wave phase speed to be equal to the jet speed $U$:

$$U^2 = \frac{\omega^2}{k^2} = \frac{5}{\pi k R_0} \frac{I_1(k R_0)}{I_0(k R_0)} (1-k^2 R_0^2)$$

Provided the jet speed $U(t)$ is known, one may rationalize the observed $T$.

Now, add soap to the bath, generating Marangoni stresses that rigidify the surface of the "fluid pipe".

Once fluid enters the pipe, a boundary layer develops on its inner wall, owing to NO-SLIP BC there.

Balance viscous + Marangoni stresses on the pipe surface:
\[ \rho \frac{V}{\mu} \frac{V}{H} \sim \frac{\Delta \sigma}{H} \quad * \]

We expect the b.d.e. to grow with distance to the pipe entrance \( z \) according to a Blasius b.d.e.

\[ \frac{S}{a} \sim \left( \frac{V_2}{a^2 V} \right)^{\frac{1}{2}} \]

Setting \( S(H) \) into * yields

\[ H \sim \frac{(\Delta \sigma)^2}{\rho \mu \bar{V}^3} \]

increases with \( \Delta \sigma \)

decreases with \( \bar{V} \)

\[ \Rightarrow \text{ Hancock + Bush, "Fluid Pipes", JFM (2003)} \]

5) We have considered here the inviscid case. Viscosity acts to increase the wavelength of the most unstable mode relative to that in the inviscid case, \( \sim 9 \text{Ro} \).