18.357: Lecture 11

Fluid jets and the Rayleigh-Plateau instability

The shape of a falling fluid jet

Apply Bernoulli at points A and B:

$$\underbrace{\tfrac{1}{2}\rho U_0^2 + \rho g z + \underbrace{P_0 + \frac{\sigma}{a}}_{P_A}} = \tfrac{1}{2}\rho U^2(z) + \underbrace{P_0 + \frac{\sigma}{r}}_{P_B}$$

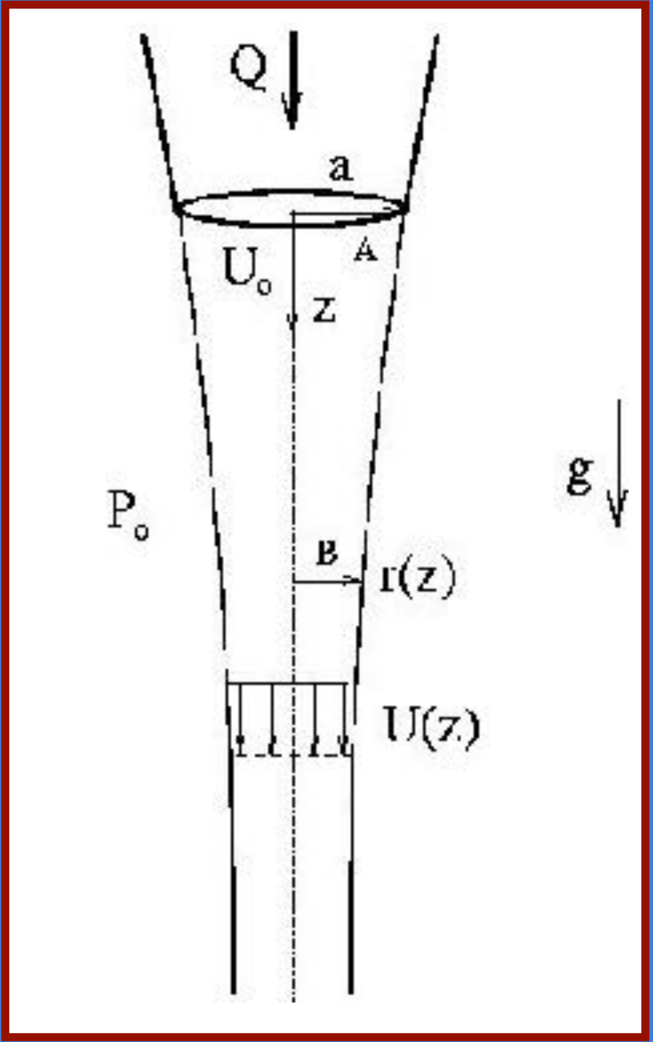
$$\longrightarrow \quad \frac{U(z)}{U_0} = \left[1 + \frac{2}{Fr}\frac{z}{a} + \frac{2}{We}\left(1 - \frac{a}{r}\right)\right]^{1/2}$$

where $Fr = \dfrac{U_0^2}{g a}$, $We = \dfrac{\rho U_0^2 a}{\sigma}$

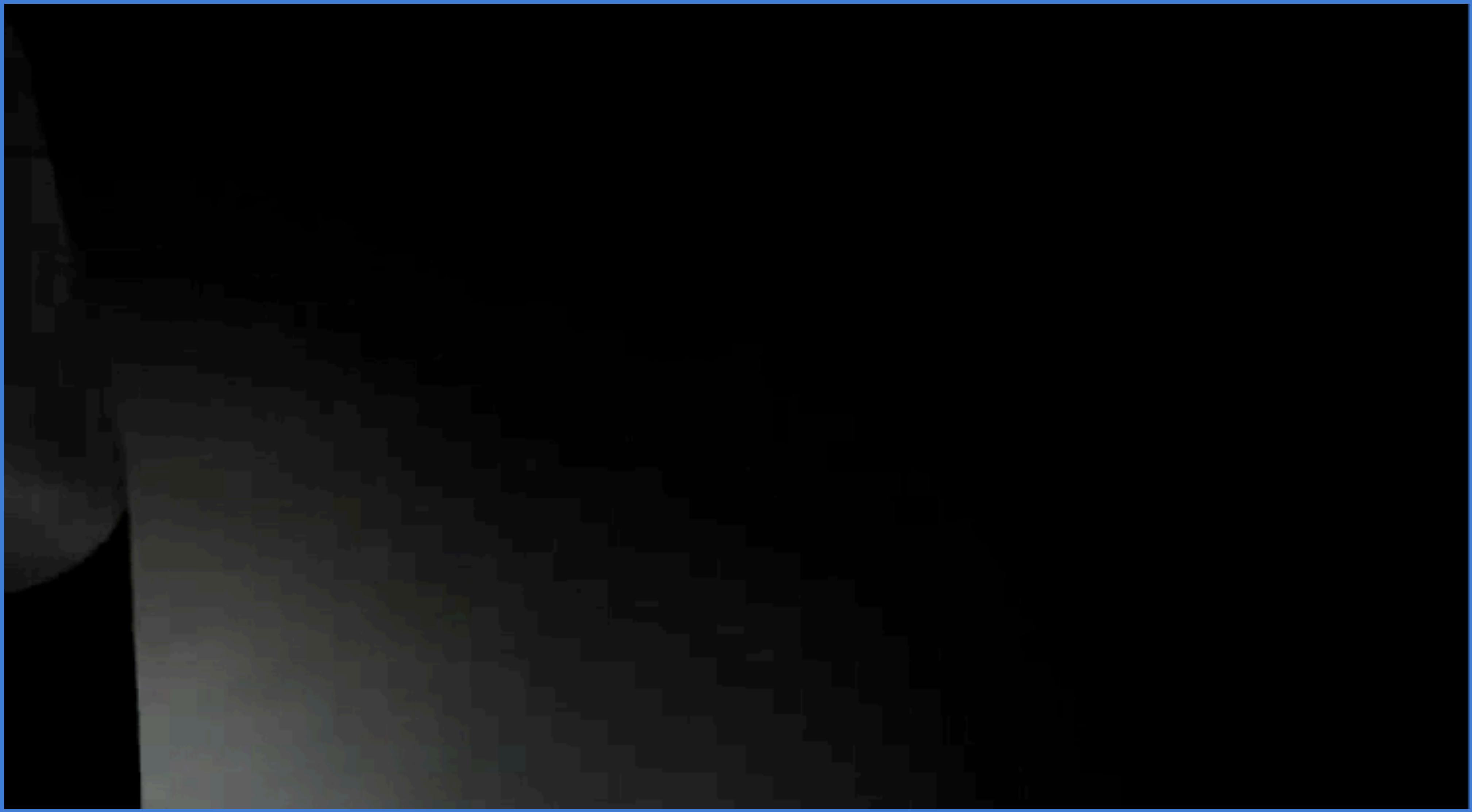
Continuity: $Q = \pi a^2 U_0 = \pi r^2 U(z)$

Jet shape: $$\frac{r(z)}{a} = \left(\frac{U_0}{U(z)}\right)^{1/2} = \left[1 + \frac{2}{Fr}\frac{z}{a} + \frac{2}{We}\left(1 - \frac{a}{r}\right)\right]^{-1/4}$$

In $We \to \infty$ limit: $$\frac{r(z)}{a} = \left(1 + \frac{2gz}{U_0^2}\right)^{-1/4} , \quad \frac{U(z)}{U_0} = \left(1 + \frac{2gz}{U_0^2}\right)^{1/2}$$

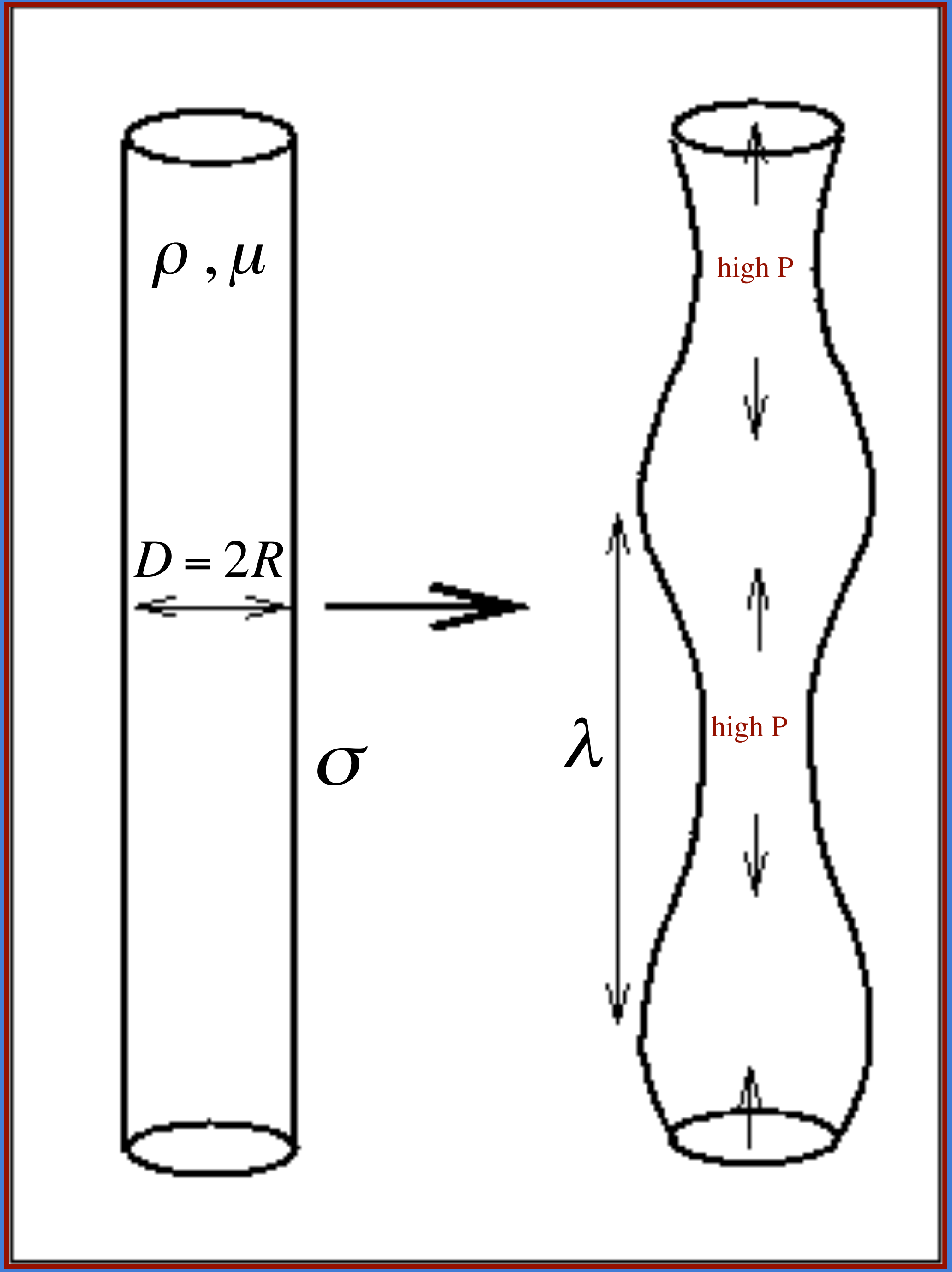


Fluid jet break-up



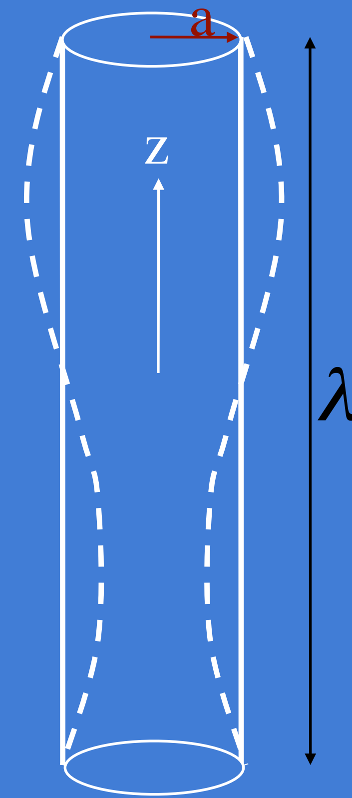
Rayleigh-Plateau Instability (Rayleigh 1900)

Capillary pinch-off of a fluid thread



Rayleigh-Plateau instability

$$k = 2\pi / \lambda$$



Seek normal modes:

$$r = a + \varepsilon\, e^{\omega t + ikz} \;,\; u_r = R(r)\, e^{\omega t + ikz}$$

$$u_z = Z(r)\, e^{\omega t + ikz} \;,\; p = P(r)\, e^{\omega t + ikz}$$

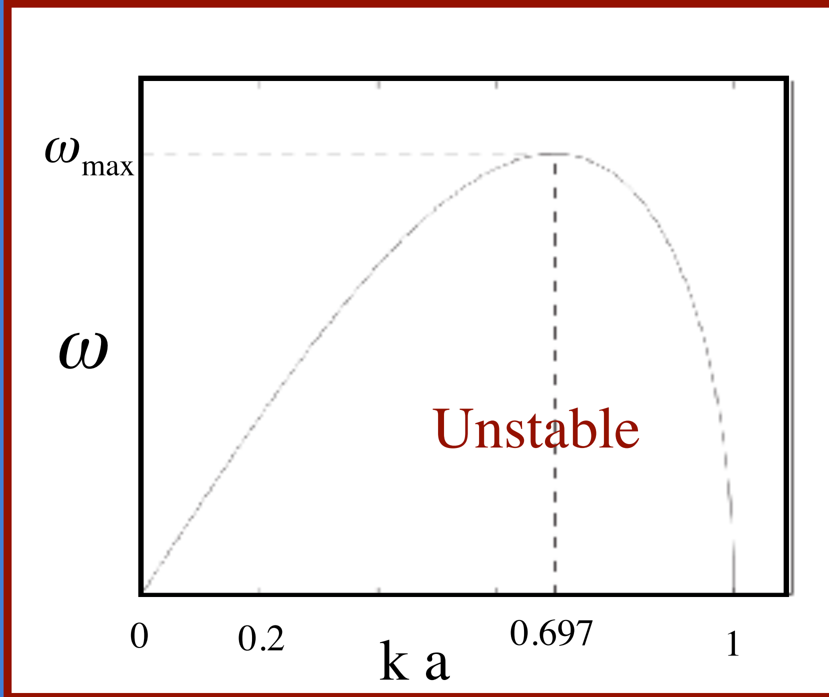
Sub into Navier-Stokes and linearize to solve for disturbance fields

Dispersion relation:

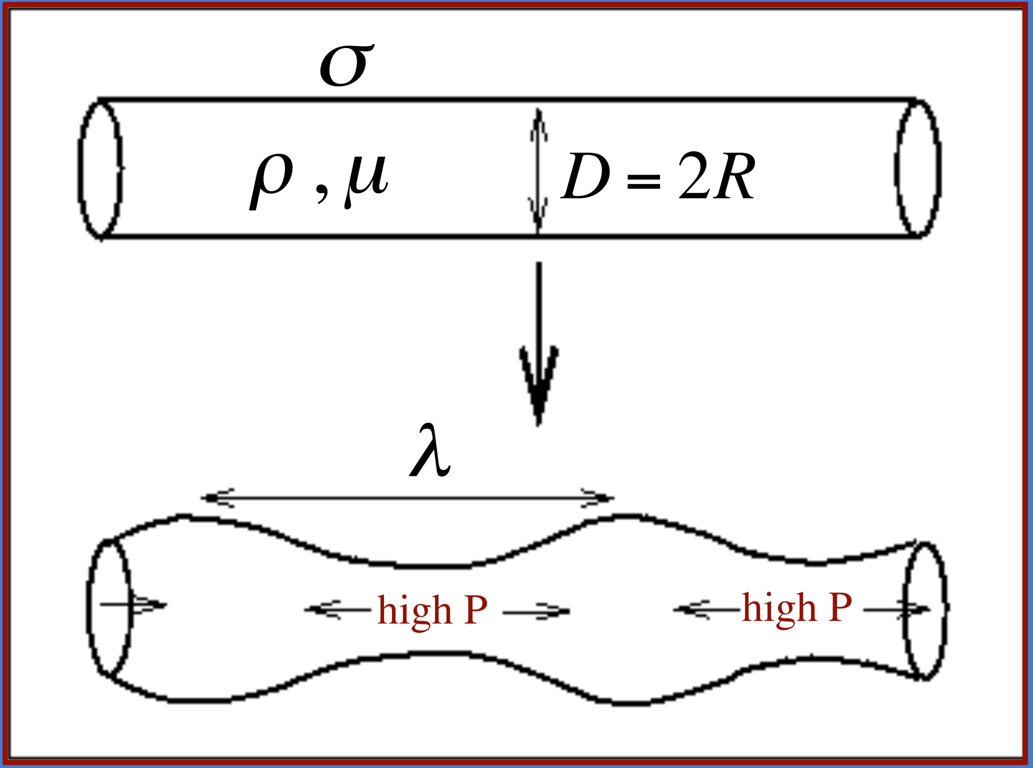
$$\omega^2 = \frac{\sigma k}{\rho a^2} \frac{I_1(ka)}{I_0(ka)} (1 - k^2 a^2)$$

- instability for modes with $\lambda > 2\pi a$
- fastest growing mode: $\lambda = 9.02\, a$

Break-up time: $$\tau_{break-up} = 2.91 \left(\frac{\rho a^3}{\sigma} \right)$$



Viscosity and the Rayleigh-Plateau Instability



- pinch-off depends on Ohnesorge number $\quad \mathrm{Oh} = \dfrac{\sigma R}{\mu \nu}$
- at high Oh: $\quad \tau_p \sim \left(\dfrac{\rho R^3}{\sigma}\right)^{1/2}$ and $\quad \lambda = 9.02\, R$
- at low Oh: $\quad \tau_p \sim \dfrac{\mu R}{\sigma}$ and $\quad \lambda$ increases with μ

Jet impingement on a reservoir

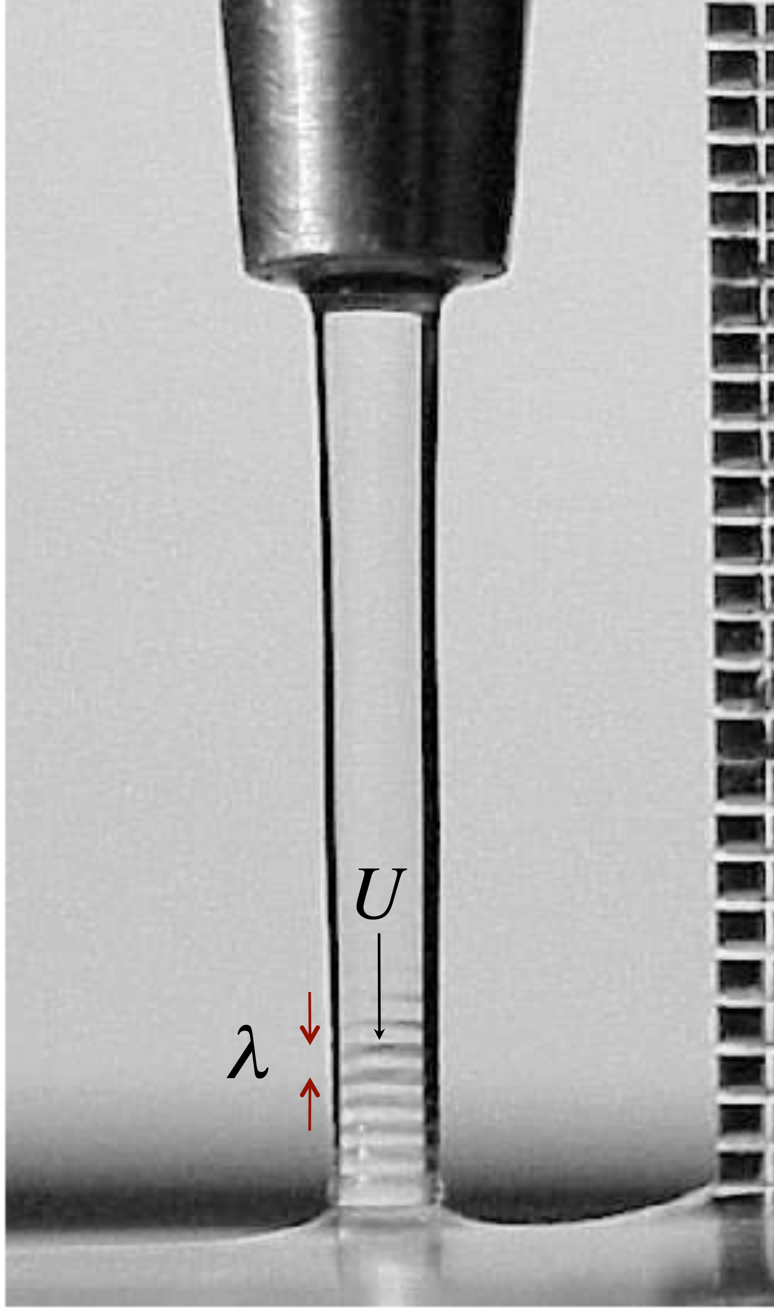
- a field of standing waves may arise at base of jet
- the wavelength $\lambda = 2\pi / k$ is set by the requirement that the phase speed correspond to the local jet speed:

$$U = -\frac{\omega}{k}$$

- using the dispersion relation $\omega(k)$ deduced for the Rayleigh-Plateau instability thus yields

$$U^2 = \frac{\omega^2}{k^2} = \frac{\sigma}{\rho k R^2} \frac{I_1(kR)}{I_0(kR)} (1 - k^2 R^2)$$

- if U is known, this may be inverted to deduce λ



What if the reservoir is contaminated?

- Marangoni stress draws surfactant up jet

- the jet enters the reservoir as if through a rigid pipe

- pipe height H set by balance of viscous and Marangoni stress

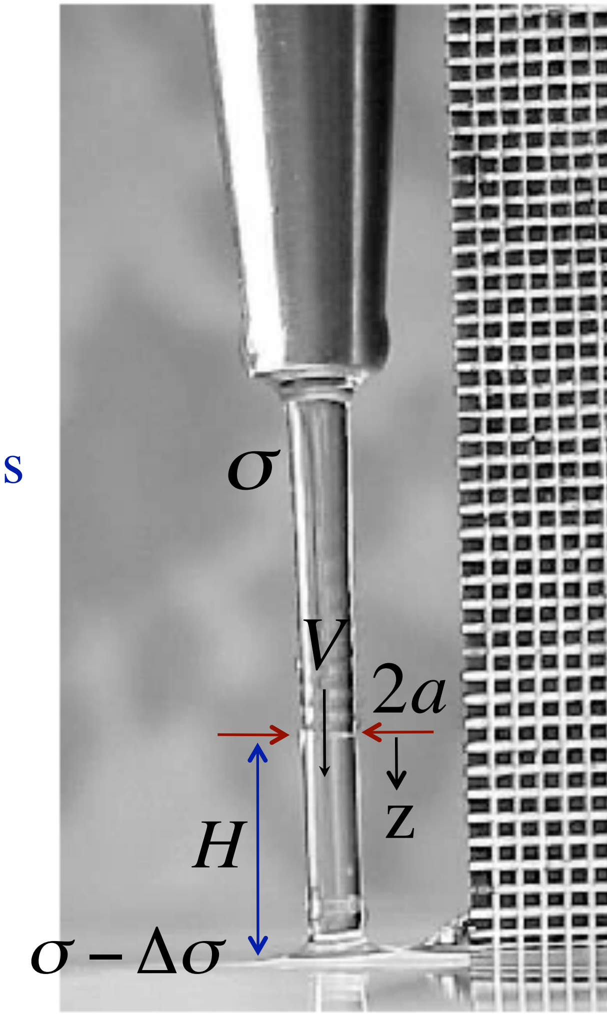
$$\rho\nu\frac{V}{\delta_H} \sim \frac{\Delta\sigma}{H}$$

  - boundary layer takes Blasius-like form:

$$\frac{\delta}{a} \sim \left(\frac{\nu z}{a^2 V}\right)^{1/2}$$

- substituting δ into stress balance yields pipe height:

$$H \sim \frac{(\Delta\sigma)^2}{\rho\mu V^3}$$



Fluid Pipes (Hancock & Bush, 2000)