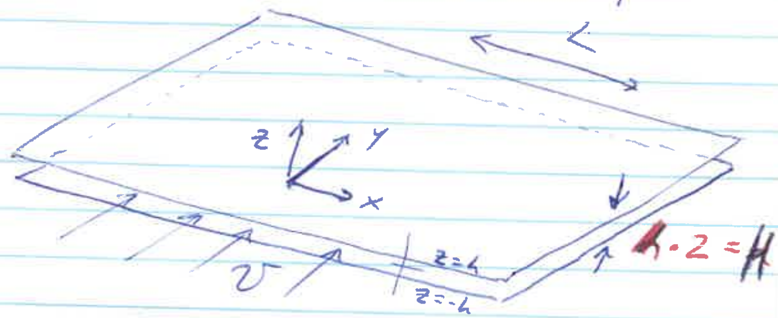
$$\frac{\partial u}{\partial t} = \frac{U^2}{L}\frac{\partial u'}{\partial t'}$$

The Hele-Shaw Cell

- we shall demonstrate that viscous flows in a thin gap geometry (ie. a "Hele-Shaw Cell"), typically comprised of fluid pressed between glass plates, ⇒ provide a simple means for experimental modeling of potential flows: high Re flows past obstacles, and flow in porous media.

Consider a steady flow in a thin gap (thickness $2h$) characterized by a lengthscale L and speed U



Writing $\underline{u} = (u, v, w)$, we have N-S eqns (steady)

$$\underline{u}\cdot\nabla\underline{u} = -\nabla p + \nu\nabla^2\underline{u}$$

$$\nabla\cdot\underline{u} = 0$$

$$t \sim \frac{L}{U}t'$$

Nondimensionalize: $(x, y) \sim L(x', y')$, $z \sim h z'$ $\cancel{z \sim Hz'}$

$(u, v) \sim (u', v')U$, $w \sim W w'$, $p \sim \frac{\mu U L}{h^2} p'$ (LUB)

Continuity: $\frac{U}{L}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) + \frac{W}{H}\frac{\partial w'}{\partial z'} = 0 \Rightarrow W \sim \frac{UL^{-1}}{H^{-1}}$

$H \to h$ throughout!

$\hat{x}$-mom: $Re\frac{H^2}{L^2}\left\{u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'}\right\} = -\frac{\partial p'}{\partial x'} + \frac{H^2}{L^2}\left\{\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right\} + \frac{\partial^2 u'}{\partial z'^2}$

$\hat{y}$: $Re\frac{H^2}{L^2}\left\{u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} + w'\frac{\partial v'}{\partial z'}\right\} = -\frac{\partial p'}{\partial y'} + \left(\frac{H}{L}\right)^2\left\{\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right\} + \frac{\partial^2 v'}{\partial z'^2}$

$\hat{z}$: $Re\left(\frac{H}{L}\right)^4\left\{u'\frac{\partial w'}{\partial x'} + v'\frac{\partial w'}{\partial y'} + w'\frac{\partial w'}{\partial z'}\right\} = -\frac{\partial p'}{\partial z'} + \left(\frac{H}{L}\right)^4\left\{\frac{\partial^2 w'}{\partial x'^2} + \frac{\partial^2 w'}{\partial y'^2}\right\} + \left(\frac{H}{L}\right)^2\left\{\frac{\partial^2 w'}{\partial z'^2}\right\}^2$

If $\left(\frac{H}{L}\right) \ll 1$, and $Re\left(\frac{H}{L}\right)^2 \ll 1$*, then we have

$$Re = \frac{UL}{\nu}$$

$$0 = -\frac{\partial p'}{\partial x'} + \frac{\partial^2 u'}{\partial z'^2}$$

$$0 = -\frac{\partial p'}{\partial y'} + \frac{\partial^2 v'}{\partial z'^2}$$

$$0 = -\frac{\partial p'}{\partial z'}$$

Solve w/ B.Cs

1. $u' = v' = 0$ at $z' = \pm 1$
2. $\frac{\partial}{\partial z'}(u', v') = 0$ at $z' = 0$

*
Note: we see the natural appearance of the "Reduced Re":

$$Re\left(\frac{h}{L}\right)^2 = \frac{UL}{\nu}\frac{h^2}{L^2} \sim \frac{U\cdot\nabla U}{\nu\nabla^2 u}$$

$$= \frac{h^2/\nu}{L/U} = \frac{\text{TIMESCALE OF DIFFUSION ACROSS GAP}}{\text{CONVECTIVE TIMESCALE OF MEAN FLOW}}$$

Integration yields: (since p' indep of z) DIMENSIONAL

$$u' = \frac{1}{2}\left(-\frac{\partial p'}{\partial x'}\right)(1-z'^2) \quad \rightarrow \quad u = \frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left[1-\left(\frac{z}{h}\right)^2\right]$$

$$v' = \frac{1}{2}\left(-\frac{\partial p'}{\partial y'}\right)(1-z'^2) \quad \rightarrow \quad v = \frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial y}\right)\left[1-\left(\frac{z}{h}\right)^2\right]$$

$$w' = 0 \quad \rightarrow \quad w = 0$$

Our flow to leading order has the 2-D description

$$\underset{\sim}{u} = \frac{-h^2}{2\mu}\underset{\sim}{\nabla}p\,\underbrace{\left[1-\left(\frac{z}{h}\right)^2\right]}_{\text{PARABOLIC}}$$

We define the mean velocity as

$$\bar{\underset{\sim}{u}}(x,y) = \frac{2}{2h}\int_0^h \underset{\sim}{u}(x,y)\,dz = -\frac{h2}{4\mu}\underset{\sim}{\nabla}p\int_0^h\left[1-\left(\frac{z}{h}\right)^2\right]dz$$

$$= -\frac{h2}{4\mu}\underset{\sim}{\nabla}p\left[z-\frac{z^3}{3h^2}\right]_0^h = -\frac{2h^2}{6\mu}\underset{\sim}{\nabla}p$$

$$\Rightarrow \boxed{\bar{\underset{\sim}{u}} = -\frac{k_{eff}}{\mu}\underset{\sim}{\nabla}p} \qquad \text{where } k_{eff} = \frac{h^2 2}{6} = \frac{h^2}{3}$$

⇒ The depth-averaged velocity field is thus a potential flow; specifically, satisfies Darcy's eqn with $K_{eff} = \frac{2h^2}{6} = \frac{h^2}{3}$

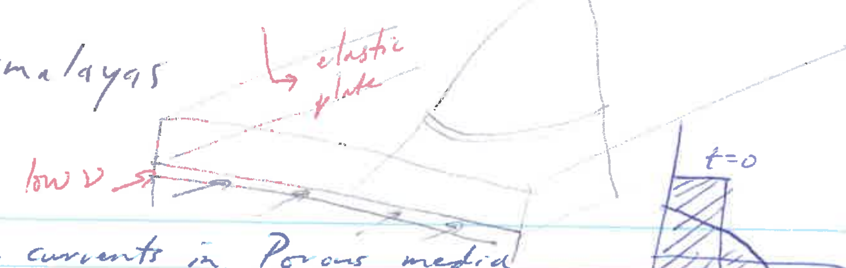
⇒ the pressure field $p(x,y)$ resulting from Stokes flow can be considered to be the velocity potential for an inviscid irrotational mean flow

⇒ the H-S cell is often used to model exptally study flow in porous media

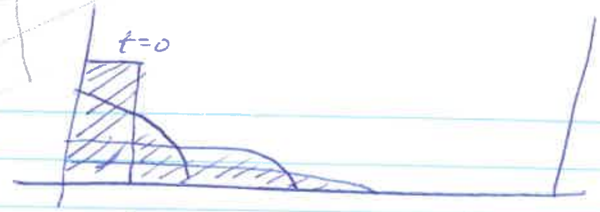
$$\underset{\sim}{\nabla}\cdot\bar{\underset{\sim}{u}} = 0 \quad \Rightarrow \quad \nabla^2 \underset{\sim}{p} = 0$$

NOTE: $\underset{\sim}{u}(z=0) = -\frac{h^2}{2\mu}\underset{\sim}{\nabla}p$ is also a potential/Darcy flow

Eg. 4 Himalayas



Eg. 1 Gravity currents in Porous media



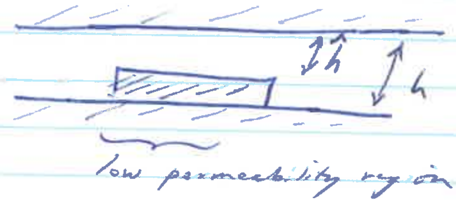
⇒ can deduce similarity sol'n describing shape (see work by Woods)

Eg. 2 Thermal plume in porous media

- Wooding 1962, JFM
- also via similarity sol'n

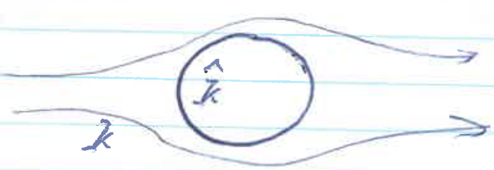
Eg. 3 Flow past porous inclusion

⇒



$$k_{eff} = \frac{h^2(x)}{3}$$

One may thus model experimentally the problem considered last class, flow past a cylindrical porosity inclusion:

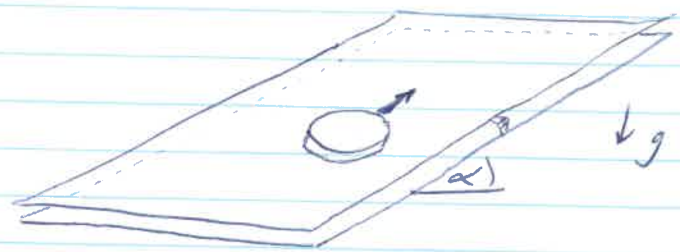


$$\hat{k} = \frac{\hat{h}^2}{3}$$

$$k = \frac{h^2}{3}$$

Note: in the $\hat{h} \to 0$ limit, $\hat{k} \to 0$ and we retain the sol'n for inviscid irrotational flow past a cylinder

A bubble rising in a H-S cell:



- to calculate drag on bubble, D, one must consider dissipation associated with the channel flow in the thin gap

$$\frac{\text{DISSIPATION}}{\text{AREA}} = \Phi/_{\text{AREA}} = 2\mu \int_{-h}^{h} \left(\frac{\partial u}{\partial z}\right)^2 dz$$

where $$\underset{\sim}{u} = \frac{h^2}{2\mu}(-\underset{\sim}{\nabla} p)\left[1 - \left(\frac{z}{h}\right)^2\right]$$

$$\Rightarrow \frac{\partial \underset{\sim}{u}}{\partial z} = +\frac{h^2}{2\mu}(\underset{\sim}{\nabla} p)\left[\cancel{1} + \frac{2z}{h^2}\right] = \frac{1}{\mu}(\underset{\sim}{\nabla} p) z$$

$$\Rightarrow \Phi/_{\text{area}} = 2\mu \int_{-h}^{h} \left(\frac{\underset{\sim}{\nabla} p}{\mu}\right)^2 z^2 \, dz$$

Note: we know there is no drag on a body in potential flow; however, here, there is shear across the gap ⇒ potential flow only describes depth-avg'd flow in plane indep of z

$$\frac{\Phi}{\text{area}} = \frac{2}{\mu}(\nabla p)^2 \left(\frac{z^3}{3}\right)\Bigg|_{-h}^{h} = \frac{4}{3\mu}(\nabla p)^2 h^3$$

but we know $\bar{u} = -\frac{k_{eff}}{\mu}\nabla p = -\frac{h^2}{3\mu}\nabla p$

$$\Rightarrow \nabla p = -\frac{3\mu}{h^2}\bar{u}$$

$$\Phi/\text{AREA} = \frac{4}{3\mu}\left(\frac{9\mu^2}{h^4}\right)h^3(\bar{u}\cdot\bar{u})$$

$$= \frac{12\mu}{h}\left(\bar{u}_r^2 + \bar{u}_\theta^2\right)$$

But $\bar{u}_r$ and $\bar{u}_\theta$ are known from potential flow generated by a uniformly translating cylinder; specifically, (see PS #5)

$$\bar{u}_r = \frac{a^2}{r^2} U \cos\theta$$
$$\bar{u}_\theta = \frac{a^2}{r^2} U \sin\theta$$

$$\bar{u}_r^2 + \bar{u}_\theta^2 = \frac{a^4}{r^4} U^2$$

$$\therefore \Phi_{TOTAL} = \int_0^\pi \int_a^\infty \Phi \, r \, dr \, d\theta$$

$$= 12\pi\mu U^2 a^2 \frac{1}{2h}$$

$$\int_a^\infty \frac{1}{r^3} dr = -\frac{1}{2}\frac{1}{r^2}\Bigg|_a^\infty = \frac{1}{2a^2}$$

We now equate the total viscous dissipation with the rate of work done by buoyancy:

$$6\pi\mu U^{\cancel{2}} \frac{\cancel{a^2}}{h} = \pi \cancel{a^2} \cdot 2h \cancel{U} \rho g \sin\alpha$$

$$\Rightarrow U = \frac{g h^2}{3\nu}\sin\alpha = \frac{g d^2}{12\nu}\sin\alpha \quad \text{where } d = 2h$$

Note: U indep of a!