

18.357: Problem Set 1

Do any 5 of the 6 following problems. Each is worth 20 points. Due March 16, 2016.

1. Drops and bubbles

a) Consider a spherical water drop of radius a , density ρ , viscosity ν and surface tension σ vibrating in a vacuum. It is perturbed from spherical, and oscillates such that its motion is characterized by a large Reynolds number. What mode of vibration do you expect to be most persistent, and why? Make an estimate for the frequency of this most unstable mode of vibration via scaling arguments. Use the above result to assess the frequency of the most unstable mode of oscillation of a rain drop.

b) Beaming. An optical effect causes incident light to be focused by drops, resulting in a beam of light being emitted by the droplet (T. Timusk, *Applied Optics*, **48**, 1212, 2009). Given the frequency of oscillation of a raindrop along with its fall speed, assess whether or not the image of rain attached below is real or fake.

c) A spherical drop of radius $a = 1\text{cm}$ is placed on a bath of the same fluid. At time $t = 0$, it merges with the bath. Assume that the drop remains spherical as its contents are evacuated. Estimate the initial exit velocity, and so a characteristic Reynolds number. On the basis of this inference, deduce the dependence of the bubble radius on time, $R(t)$, through consideration of the dominant force balance. When do you expect your solution to break down?

d) Consider now a soap bubble of radius $a = 5\text{cm}$ with the same gas inside as out. Estimate its frequency of vibration. Describe what happens when it lands on a planar soap film, merges, then evacuates its contents across the film. How does its radius evolve with time during this evacuation process? How and why are these results similar to or different from those for a droplet?

2. Capillary rise in a wedge

a) Consider an oil that wets glass, so that its equilibrium contact angle $\theta_e < \pi/2$. This oil is bound in a thin gap of thickness δ between two flat glass plates, forming a horizontal circular pancake of radius $R \gg \delta$. Assuming the capillary length $\ell_c \gg \delta$ so that gravity is negligible, calculate the cross-gap shape of the meniscus between the oil and air, and the associated Laplace pressure jump across it, in terms of δ and θ_e .

A pair of vertical glass plates are arranged so as to form a wedge with a small opening angle α . The plates are placed in a bath of the same oil such that the corner of the wedge is vertical.

b) Demonstrate that the resulting meniscus bound inside the plates takes the form of a hyperbola. Assume that the cross-gap curvature, as calculated in part (a), is dominant.

c) Calculate the work done by surface forces in forming the meniscus.

3. Capillary rise between parallel plates

A pair of vertical plates are placed in a fluid bath, the gap spacing being much less than the capillary length.

a) Deduce the rise height of the fluid between the plates as a function of the equilibrium contact angle θ_e and interfacial tension σ .

b) Describe the dynamics in the viscous limit where inertial effects are negligible.

c) Describe the dynamics in the inertial limit where viscous effects are negligible. For the sake of simplicity, consider the long-time regime where added mass effects are negligible.

4. Chemical propulsion

Gradients in wettability may lead to spatial gradients in surface energy that generate propulsive forces. We consider here a solid substrate with a linear chemical gradient such that the equilibrium contact angle θ_e increases linearly with distance along the substrate. Neglect the influence of gravity throughout.

a) If a small hemispherical drop is placed on a plane composed of this surface such that its contact line is circular, calculate the resulting force on the drop.

b) Assuming that the resulting motion arises at low Reynolds number (so is resisted primarily by viscous stresses), estimate the rate of dissipation incurred by the drop motion via a scaling argument. Use an energy balance to estimate the speed of propulsion of the droplet.

b) This surface is now rolled up into a small cylindrical tube of radius R such that there is a wettability gradient along its length. If a fluid volume $V \gg 4\pi R^3/3$ is deposited inside the tube, it takes the form of a roughly cylindrical slug of length L . What is the resulting driving force on the slug?

c) Assuming that the principal resistance to motion comes from viscous dissipation resulting from the Poiseuille-type flow within the bulk of the slug, calculate its speed via a scaling argument.

5. Geometric propulsion

Gradients in geometry may also lead to spatial gradients in surface energy that generate propulsive forces. Consider a fluid-filled wedge with small opening angle α . A bubble of volume V is placed in the wedge, forming a roughly penny-shaped form of radius $R \gg h$ (h being the local gap width) whose center is a distance x from the corner. The fluid completely wets the solid, so we assume that there is an extremely thin fluid layer separating the bubble from the plate. The curvature at the bubble's edge is thus prescribed by the local gap width.

a) Calculate the surface energy of the system as a function of x . Thereby deduce the force on the bubble. Does the bubble move towards or away from the corner?

b) As the bubble moves, it induces a flow around it. To leading order, provided the bubble is sufficiently far from the corner and α sufficiently small, this may be described as flow around a circular disc in a Hele-Shaw cell (see course notes in 18.355 or 2.25). By calculating the rate of dissipation associated with such a flow, deduce the dependence of bubble speed on distance from the corner.

c) Now orient the wedge vertically, so that its corner is horizontal, and the bubble feels the influence of gravity. Consider gravitational and surface energies in order to deduce the distance from the corner at which the bubble will be in equilibrium.

6. The origins of surface tension

The van der Waals force is an intermolecular force considered to be long-range, in that it acts over distances long relative to the molecular size, typically 10^{-10} to 10^{-8} m. The Van der Waals potential is given by $\phi = -k/r^6$, where k is some constant. (It is the Van der Waals force that initiates the coalescence of fluid drops when they get sufficiently close to a fluid surface).

a) Show that the Van der Waals potential of a fluid molecule a distance h above the horizontal surface of a fluid with ρ molecules per unit volume is $\phi = -k\rho\pi/6h^3$.

b) Show that the Van der Waals potential per unit area between two planar fluid layers separated by a thin air layer of thickness d is $\phi = -k\pi\rho^2/(12d^2)$, and that the associated attractive force per unit area $F = -A/(6\pi d^3)$, where $A = k\pi^2\rho$ is called the Hamaker constant.

c) Show that the total Van der Waals force between a small spherical fluid drop of radius b and a fluid plane separated by a distance d is given by $F = Ab/6d^2$.

d) Consider the Van der Waals potential acting on a fluid molecule inside a spherical drop of radius b , a distance $h \ll b$ from its surface. Calculate the potential arising as a result of the interfacial curvature; specifically, deduce the potential difference arising due to the removal of molecules (of density ρ) from the tangent plane. Show that this potential is $\phi = \pi\rho/(4bh^2)$. Calculate the resulting force per unit volume. Show that, in equilibrium, this must be balanced by a pressure gradient $dp/dh = A/(2\pi bh^3)$. Noting that this becomes singular at $h = 0$, integrate the pressure gradient from h to the molecular dimension ϵ . Thus show that when $h \gg \epsilon$, the pressure inside the fluid, the Laplace pressure: $P = \sigma/b$, where $\sigma = A/(4\pi\epsilon^2)$ is the surface tension.