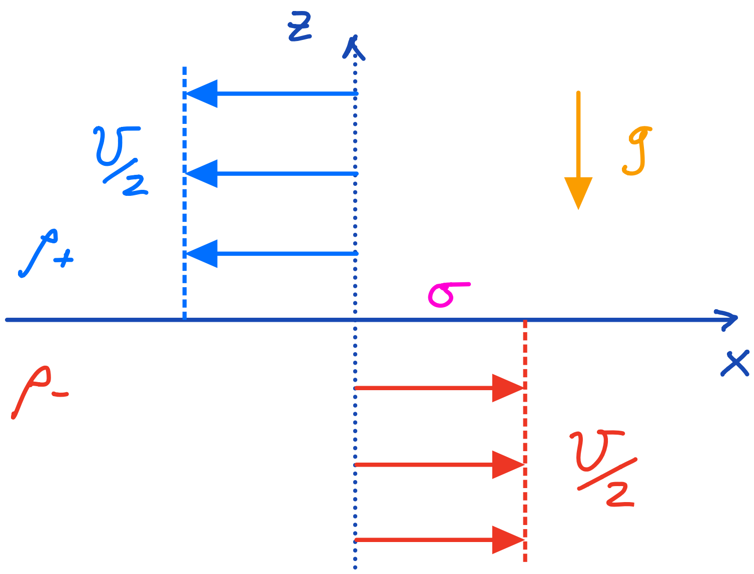
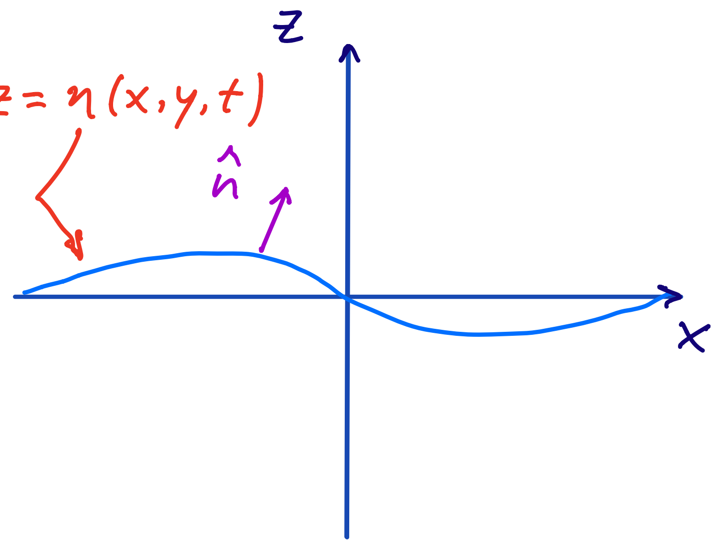
Lecture 14. Instability of superposed fluids





- in addition to the Rayleigh – Plateau instability, fluid fracture and drop formation may be caused by shear-driven or gravitationally-driven instability

Define interface: $h(x,y,z) = z - \eta(x,y) = 0$

$$\Rightarrow \quad \vec{\nabla} h = (-\eta_x, -\eta_y, 1)$$

and $$\hat{n} = \frac{\vec{\nabla} h}{|\vec{\nabla} h|} = \frac{1}{(1+\eta_x^2+\eta_y^2)^{\frac{1}{2}}}(-\eta_x, -\eta_y, 1)$$

Describe flow as inviscid, irrotational.

Basic State: $\eta = 0$, $\underline{u} = \underline{\nabla}\phi$, $\phi = \mp \frac{1}{2} U x$ in $z \gtrless 0$

Perturbed State: $\phi = \mp \frac{1}{2} U x + \underbrace{\phi_\pm}_{\text{PERTURBATION FIELD}}$ in $z \gtrless 0$

Solve $\vec{\nabla} \cdot \underline{u} = \nabla^2 \phi_\pm = 0$ subjects to BCs

1. $\phi_\pm \to 0$ as $y \to \pm\infty$

2. Kinematic BC : $\frac{\partial \eta}{\partial t} = \underline{u} \cdot \hat{n}$ ☆

where $\underline{u} = \underline{\nabla}\phi = \underline{\nabla}\left(\mp \frac{1}{2} U x + \phi_\pm\right)$

$$= \mp \frac{1}{2} U \hat{i} + \frac{\partial \phi_\pm}{\partial x}\hat{i} + \frac{\partial \phi_\pm}{\partial y}\hat{j} + \frac{\partial \phi_\pm}{\partial z}\hat{k}$$

and $\hat{n} = (-\eta_x, -\eta_y, 1)$ when linearized w.r.t small perturbations.

Into ☆ $\Rightarrow$ $\frac{\partial \eta}{\partial t} = \left(\mp \frac{1}{2} U + \frac{\partial \phi_\pm}{\partial x}\right)(-\eta_x) + \frac{\partial \phi_\pm}{\partial y}(-\eta_y) + \frac{\partial \phi_\pm}{\partial z}$

Linearize : assume all perturbation fields, η, $\phi_\pm$ and their derivatives are small $\Rightarrow$ neglect their products

$\Rightarrow$ $\frac{\partial \phi_\pm}{\partial z} = \frac{\partial \eta}{\partial t} \mp \frac{1}{2} U \frac{\partial \eta}{\partial x}$ at $z = \eta$, but linearize: on $z = 0^\pm$.

3. Normal Stress Balance

$$p_- - p_+ = \sigma \underline{\nabla} \cdot \hat{n} \quad \text{on } z = \eta$$

Linearize : $p_- - p_+ = -\sigma(\eta_{xx} + \eta_{yy})$ on $z = 0$

We now deduce the pressure $p_\pm$ from time-dep Bernoulli:

$$\rho \frac{\partial \phi}{\partial t} + \frac{\rho}{2} u^2 + p + \rho g z = f(t)$$

where $u^2 = \underbrace{\frac{1}{2} U^2}_{\text{const}} \mp U \frac{\partial \phi_\pm}{\partial x}$ + higher order terms

Linearizing:

$$\rho_\pm \frac{\partial \phi_\pm}{\partial t} + \frac{1}{2}\rho_\pm\left(\mp U \frac{\partial \phi_\pm}{\partial x}\right) + P_\pm + \rho_\pm g \eta = G(t)$$

$$\Rightarrow P_- - P_+ = (\rho_+ - \rho_-) g\eta + \left(\rho_+ \frac{\partial \phi_+}{\partial t} - \rho_- \frac{\partial \phi_-}{\partial t}\right) + \frac{U}{2}\left(\rho_- \frac{\partial \phi_-}{\partial x} - \rho_+ \frac{\partial \phi_+}{\partial x}\right)$$

$$= -\sigma(\eta_{xx} + \eta_{yy})$$

is the linearized normal stress BC.

Seek normal modes (wave-like solns) of the form:

$$\eta = \eta_0 e^{i\alpha x + i\beta y + \omega t} = \eta_0 e^{\omega t + i \underline{k} \cdot \underline{x}}$$

where $\underline{k} = (k_x, k_y) = (\alpha, \beta)$

$$\phi_\pm = \phi_{0\pm} e^{\mp kz} e^{i\alpha x + i\beta y + \omega t}$$

where $\nabla^2 \phi_\pm = 0$ requires that $k^2 = \alpha^2 + \beta$

Apply kinematic BC: $\frac{\partial \phi_\pm}{\partial z} = \frac{\partial \eta}{\partial t} \mp \frac{1}{2} U \frac{\partial \eta}{\partial x}$ at $z = 0$

$$\Rightarrow \mp k \phi_{0\pm} = \omega \eta_0 \mp \frac{1}{2} i\alpha U \eta_0 \qquad \boxtimes$$

Normal Stress BC:

$$k^2 \sigma \eta_0 = -g(\rho_- - \rho_+)\eta_0 + \omega(\rho_+ \phi_{0+} - \rho_- \phi_{0-}) + \frac{1}{2} i\alpha U(\rho_+ \phi_{0+} + \rho_- \phi_{0-})$$

Subing in for $\phi_{0\pm}$ from $\boxtimes$ yields:

$$-k^3\sigma = \omega\left[\rho_+\left(\omega - \tfrac{1}{2}i\alpha U\right) + \rho_-\left(\omega + \tfrac{1}{2}i\alpha U\right)\right]$$
$$+ gk(\rho_- - \rho_+) + \tfrac{1}{2}i\alpha U\left[\rho_+\left(\omega - \tfrac{1}{2}i\alpha U\right) + \rho_-\left(\omega + \tfrac{1}{2}i\alpha U\right)\right]$$

$$\Rightarrow \omega^2 + i\alpha U\left(\frac{\rho_- - \rho_+}{\rho_+ + \rho_-}\right)\omega - \frac{1}{4}\alpha^2 U^2 + k^2 c_0^2 = 0$$

where $c_0^2 \equiv \left(\dfrac{\rho_- - \rho_+}{\rho_- + \rho_+}\right)\dfrac{g}{k} + \dfrac{\sigma}{\rho_- + \rho_+}k$

Dispersion Relation : $\omega(k)$

$$\omega = \frac{1}{2}i\left(\frac{\rho_+ - \rho_-}{\rho_- + \rho_+}\right)\underline{k}\cdot\underline{U} \pm \left\{\frac{\rho_+\rho_-}{(\rho_- + \rho_+)^2}(\underline{k}\cdot\underline{U})^2 - k^2 c_0^2\right\}^{\frac{1}{2}}$$

where $\underline{k} = (\alpha, \beta)$, $k^2 = \alpha^2 + \beta^2$

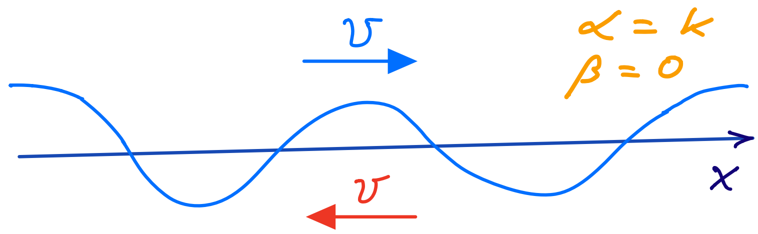
System is UNSTABLE if $Re(\omega) > 0$, i.e. if

$$\boxed{\frac{\rho_+\rho_-}{\rho_- + \rho_+}(\underline{k}\cdot\underline{U})^2 > k^2 c_0^2}$$

Squires Theorem : disturbances with wavenumber

$\underline{k} = (\alpha, \beta)$ parallel to $\underline{U}$ are the most unstable

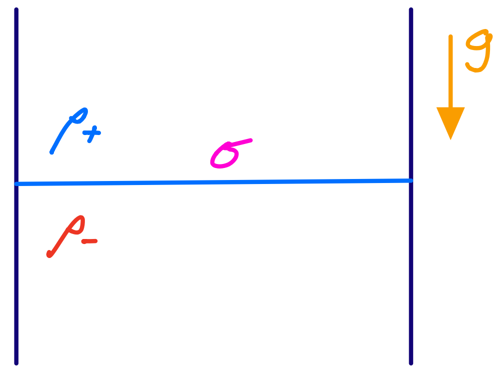
- this is a general property of shear flows: most unstable waves have crests perpendicular to wind



Two Important Special Cases

I. Rayleigh-Taylor Instability (heavy over light)

- $\rho_+ > \rho_1$, $U = 0$



Via ☆ we see that system is unstable if

$$c_0^2 = \left(\frac{\rho_- - \rho_+}{\rho_- + \rho_+}\right)\frac{g}{k} + \frac{\sigma}{\rho_- + \rho_+} k < 0$$

i.e. $\rho_+ - \rho_- > \dfrac{\sigma k^2}{g} = \dfrac{4\pi^2 \sigma}{g \lambda^2}$

$\Rightarrow$ $\boxed{\lambda > 2\pi l_c}$ for instability

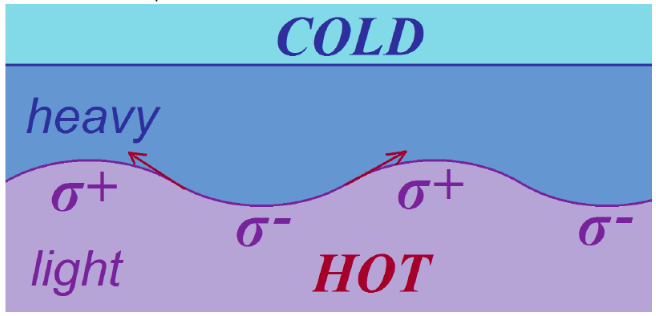
Note: 1. system stabilized to small λ disturbances by σ

2. system always unstable for suff. large λ

3. in a finite container, if its width is less than $\lambda_c = 2\pi\left[\dfrac{(\rho_+ - \rho_-)g}{\sigma}\right]^{-\frac{1}{2}}$, then the system will be stable

4. with bddies, the instability may be stabilized by a temperature gradient

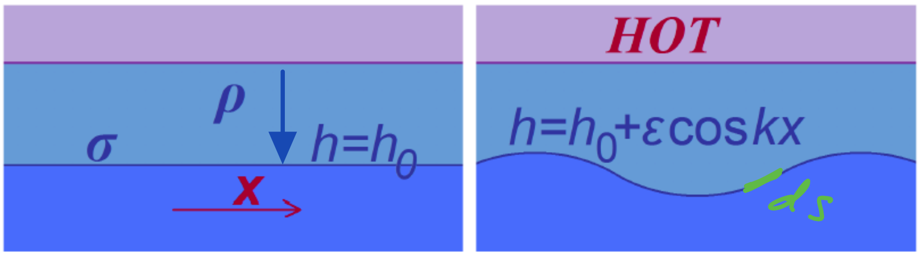
$\Rightarrow$ Marangoni flows act to resist surface deformation



Instability of a thin layer : a heuristic argument

Change in Surface Energy :

$$E_s/\text{cm} = \sigma \Delta l = \sigma\left(\int_0^\lambda ds - \lambda\right) = \frac{1}{4}\sigma \varepsilon^2 k^2 \lambda$$



Change in GPE :

$$\Delta E_g/\text{cm} = \int_0^\lambda -\frac{1}{2}\rho g (h^2 - h_0^2)\, dx = -\frac{1}{4}\rho g \varepsilon^2 \lambda$$

When is the total energy decreased ?

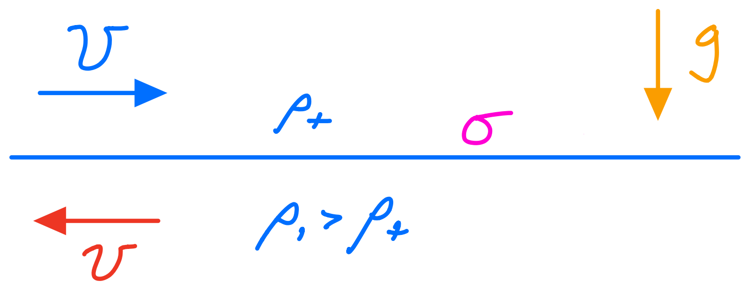
When $\Delta E_{TOTAL} = \Delta E_s + \Delta E_g < 0$

i.e. $\rho g > \sigma k^2 \Rightarrow \boxed{\lambda > 2\pi l_c}$ unstable to large λ

II. Kelvin-Helmholtz Instability

- $\rho_- \geq \rho_+$: gravitationally stable, but destabilized by shear

"WIND BLOWING OVER WATER"



Take $\underline{k}$ parallel to $\underline{v}$, so that $(\underline{k} \cdot \underline{v})^2 = k^2 v^2$

and instability criterion :

NB : $k = \frac{2\pi}{\lambda}$

$$\boxed{\rho_- \rho_+ v^2 > (\rho_- - \rho_+) g \frac{\lambda}{2\pi} + \sigma \frac{2\pi}{\lambda}}$$

Note : 1. system stabilized to large λ disturbances by g

2. " " " short λ " " σ

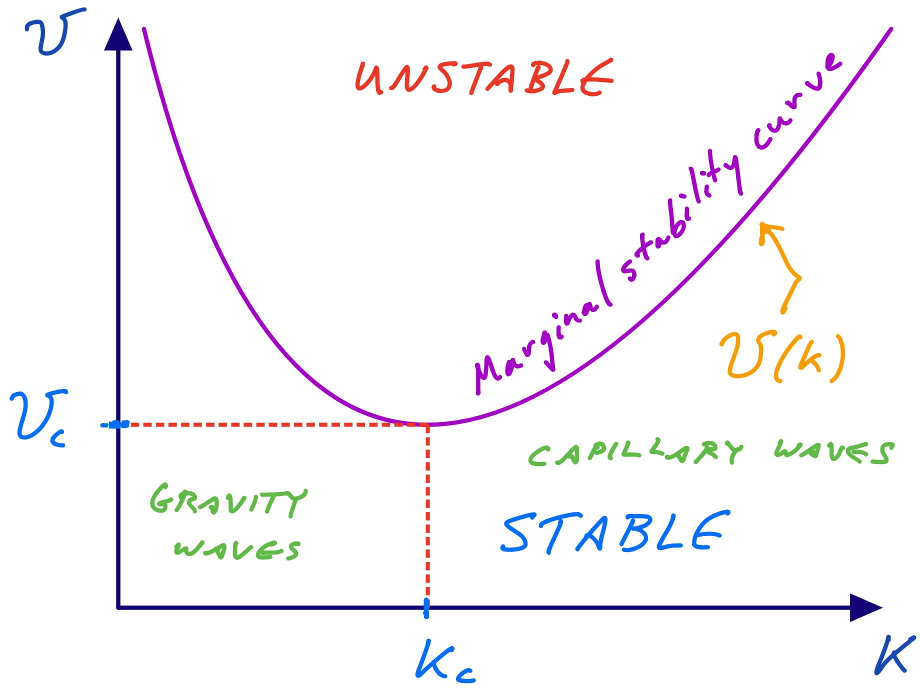
3. for any given λ (or k), one may find a critical v for which this mode is unstable.

Marginal Stability Curve :

$$v(k) = \left[\frac{\rho_- - \rho_+}{\rho_- \rho_+} \frac{g}{k} + \frac{1}{\rho_- \rho_+} \sigma k \right]^{\frac{1}{2}}$$

has a MINIMUM where $\frac{dv}{dk} = 0$ i.e. $\frac{d}{dk} v^2 = 0$

i.e. $-\frac{\Delta\rho g}{k^2} + \sigma = 0 \implies k_c = \sqrt{\frac{\Delta\rho g}{\sigma}} \sim \frac{1}{\ell_{cap}}$



Corresponding $\mathcal{V}_c = \mathcal{V}(k_c) = \frac{2}{\rho_- \rho_+}\sqrt{\Delta\rho g \sigma}$
is the MIN $\mathcal{V}$ required to generate waves

Air blowing over water :

$$\mathcal{V}_c^{\,2} = \frac{2}{1\cdot 2\times 10^{-3}}\sqrt{1\cdot 10^3\cdot 70} \Rightarrow \mathcal{V}_c \sim 650 \text{ cm/s}$$

$$k_c = \sqrt{\frac{1\cdot 10^3}{70}} \simeq 3.8 \text{ cm}^{-1} \Rightarrow \lambda_c = 1.6 \text{ cm}$$

Note : influence of surfactants on waves is to kill small λ disturbances

$\Rightarrow$ Marangoni elasticity generates vortices, dissipation on the scale of the waves