18.357 Handout 1

Recall Stokes Theorem:

$$\int_C \mathbf{F}\cdot\vec{d\ell} \;=\; \int_S \mathbf{n}\cdot(\nabla\wedge\mathbf{F})\;dS$$

Along the contour C, $\vec{d\ell}=\mathbf{m}\;d\ell$, so that we have

$$\int_C \mathbf{F}\cdot\mathbf{m}\;d\ell \;=\; \int_S \mathbf{n}\cdot(\nabla\wedge\mathbf{F})\;dS$$

Now let $\mathbf{F}=\mathbf{f}\wedge\mathbf{b}$, where $\mathbf{b}$ is an arbitrary *constant* vector. We thus have

$$\int_C (\mathbf{f}\wedge\mathbf{b})\cdot\mathbf{m}\;d\ell \;=\; \int_S \mathbf{n}\cdot(\nabla\wedge(\mathbf{f}\wedge\mathbf{b}))\;dS$$

Now use standard vector identities to see:

$$(\mathbf{f}\wedge\mathbf{b})\cdot\mathbf{m} \;=\; -\,\mathbf{b}\cdot(\mathbf{f}\wedge\mathbf{m})$$

$$\nabla\wedge(\mathbf{f}\wedge\mathbf{b}) \;=\; \mathbf{f}(\nabla\cdot\mathbf{b})-\mathbf{b}(\nabla\cdot\mathbf{f})+\mathbf{b}\cdot\nabla\mathbf{f}-\mathbf{f}\cdot\nabla\mathbf{b}$$

$$=\; -\,\mathbf{b}(\nabla\cdot\mathbf{f})+\mathbf{b}\cdot\nabla\mathbf{f}$$

since $\mathbf{b}$ is a constant vector. We thus have

$$\mathbf{b}\cdot\int_C(\mathbf{f}\wedge\mathbf{m})\;d\ell \;=\; \mathbf{b}\cdot\int_S\;[\mathbf{n}(\nabla\cdot\mathbf{f})-(\nabla\mathbf{f})\cdot\mathbf{n}]\;dS$$

Since $\mathbf{b}$ is arbitrary, we thus have

$$\int_C(\mathbf{f}\wedge\mathbf{m})\;d\ell \;=\; \int_S\;[\mathbf{n}(\nabla\cdot\mathbf{f})-(\nabla\mathbf{f})\cdot\mathbf{n}]\;dS$$

We now choose $\mathbf{f}=\sigma\mathbf{n}$, and recall that $\mathbf{n}\wedge\mathbf{m}=-\mathbf{s}$. One thus obtains

$$-\int_C \sigma\mathbf{s}\;d\ell \;=\; \int_S\;[\mathbf{n}\nabla\cdot(\sigma\mathbf{n}) \;-\; \nabla(\sigma\mathbf{n})\cdot\mathbf{n}]\;dS$$

$$=\int_S\;[\mathbf{n}\nabla\sigma\cdot\mathbf{n}+\sigma\mathbf{n}(\nabla\cdot\mathbf{n})-\nabla\sigma-\sigma(\nabla\mathbf{n})\cdot\mathbf{n}\;]\;dS$$

We note that

$\nabla\sigma\cdot\mathbf{n}=0$ since $\nabla\sigma$ must be tangent to the surface S,

$(\nabla\mathbf{n})\cdot\mathbf{n} \;=\; \frac{1}{2}\nabla(\mathbf{n}\cdot\mathbf{n}) \;=\; \frac{1}{2}\nabla(1) \;=\; 0$,

and so obtain the desired result:

$$\int_C \sigma\mathbf{s}\;d\ell \;=\; \int_S\;[\;\nabla\sigma \;-\; \sigma\mathbf{n}\;(\nabla\cdot\mathbf{n})\;]\;dS$$