Problem Set 7

18.355, Fall 2012
Due Wednesday Nov. 21

1. Verify that the Stokes and continuity equations are satisfied by the representation

$$\mathbf{u}(\mathbf{x}) \;=\; \nabla\phi \;+\; \mathbf{x}\wedge\nabla\Psi \;+\; \nabla(\mathbf{x}\cdot\mathbf{A}) \;-\; 2\mathbf{A} \quad , \quad p(\mathbf{x}) \;=\; 2\mu\,\nabla\cdot\mathbf{A} \quad ,$$

where ϕ, Ψ and $\mathbf{A}$ are harmonic functions.

2. A rigid sphere of radius a is held stationary in a pure straining flow, so that

$\mathbf{u}(\mathbf{x}) = 0$ on the particle surface, and $\quad \mathbf{u} \to \mathbf{E}\cdot\mathbf{x} \quad$ as $\quad |\mathbf{x}| \to \infty$.

Determine the velocity and pressure fields in the fluid. Note: $\mathbf{E}$ is the constant traceless and symmetric rate-of-strain tensor of the undisturbed flow in the absence of the particle.

3. A rigid sphere of radius a_1 rotates steadily inside a larger concentric sphere of radius a_2. The rotational velocity of the inner sphere is $\mathbf{\Omega}$.

a) Assuming that the Reynolds number is small, determine the velocity and pressure fields in the fluid occupying the annular region between the spheres.

b) Deduce an expression for the torque required to rotate the inner sphere. Evaluate the torque for the case of extremely large a_2/a_1.

4. Consider the translation of an arbitrarily shaped particle through an otherwise quiescent fluid at low Reynolds number. Let $\mathbf{U}$ represent the translational velocity of the particle, $\mathbf{F}$ the hydrodynamics force exerted on the particle by the fluid and V the fluid volume surrounding the particle.

a) Show that the rate of dissipation of mechanical energy into heat in the fluid is given by

$$\Phi \;=\; 2\mu \int_V \mathbf{E}:\mathbf{E}\,dV \;=\; -\mathbf{F}\cdot\mathbf{U} \quad .$$

b) If the particle also rotates with angular velocity $\mathbf{\Omega_p}$ due to an applied torque (let $\mathbf{L}$ represent the torque exerted by the fluid on the particle), how is the above expression modified?

BONUS (1 point or $1 million from the Clay Institute)

The Clay Prize question: Prove or give a counter-example of the following statement...

In three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier-Stokes equations.