Problem Set 3

18.355, Fall 2012
Due Oct.10

1. In class, we derived Cauchy's Momentum Equation by examining the momentum budget within a volume element fixed in a lab frame. Rederive this equation by considering the force balance on a fluid element moving and deforming with the flow. Do not assume that the fluid is incompressible. Apply Reynold's Transport Theorem where appropriate.

2. We have shown that, in fluid statics, the governing equation for the pressure in a liquid of density ρ in the gravity field $\mathbf{g}$ is

$$\nabla p = \rho \mathbf{g}$$

a) For a fluid of constant density, show that this equation can be integrated to give $p(\mathbf{x}) = p_0 + \rho \mathbf{g} \cdot \mathbf{x}$, where p_0 is a constant, and $\mathbf{x}$ is the position vector relative to an arbitrary origin.

b) For a submerged body of arbitrary shape, show that the force $\mathbf{F}$ and torque $\mathbf{L_O}$ (taken about some point O) exerted *on* the body by the fluid are given by

$$\mathbf{F} = -\int_S p \; \mathbf{n} \; dS \quad , \quad \mathbf{L_O} = -\int_S \mathbf{x} \wedge \mathbf{n} \; p \; dS \quad .$$

Here $\mathbf{x}$ is measured relative to the origin O, S is the body surface, and $\mathbf{n}$ is the unit normal directed from the body into the fluid.

c) Use the Divergence Theorem to determine explicit expressions for the force and torque on the body. State then prove Archimedes Principle.

3. An antisymmetric second rank tensor $\mathbf{\Omega}$ has only 3 independent components (convince yourself of this); consequently, it may be represented by using the components of a vector, say ω. Show that the relationship between $\mathbf{\Omega}$ and ω may be expressed as

$$\mathbf{\Omega} = \frac{1}{2} \epsilon \cdot \omega$$

if the vector ω is defined by

$$\omega = -\epsilon : \mathbf{\Omega} \quad .$$

Show that this definition of ω is consistent with the definition of vorticity in terms of the antisymmetric vorticity tensor $\mathbf{\Omega} = \frac{1}{2}(\nabla \mathbf{u} - (\nabla \mathbf{u})^T)$.

4. Calculate the velocity gradient tensor $\nabla \mathbf{u}$, the rate of strain tensor $\mathbf{E} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ and the vorticity tensor $\mathbf{\Omega} = \frac{1}{2}(\nabla \mathbf{u} - (\nabla \mathbf{u})^T)$, and the vorticity vector ω for:

a) simple shear flow: $u_x = Gy$, $u_y = 0$ and $u_z = 0$.

b) two-dimensional extensional flow: $u_x = Gx$, $u_y = -Gy$ and $u_z = 0$.

Sketch the velocity fields in each case. (Note that G is referred to as the shear rate).

5. Starting with the Thermal Energy Equation derived in class, deduce the Heat Equation,

$$\frac{D\theta}{Dt} = \kappa \nabla^2 \theta + \frac{\Phi}{\rho c_p} + \frac{Q_h}{c_p},$$

for the case of an ideal gas, for which the internal energy $e = c_v \theta$, coefficient of thermal expansivity $\alpha = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial \theta}\right)_P = \frac{1}{\theta}$, pressure $p = \rho R \theta$, and the ideal gas constant $R = c_p - c_v$, c_v and c_p being the specific heat at constant volume and pressure, respectively, and $\kappa = k/(\rho c_p)$ being the thermal diffusivity.

Hint: Use continuity to show that $-p\ \nabla \cdot \mathbf{u} = -\rho(c_p - c_v)\frac{D\theta}{Dt}$.

6. Assuming the above heat equation to be relevant for water, estimate how long it would take you to heat a glass of water to boiling by stirring with a spoon. Assume there is no heat loss by evaporation, radiation or evaporation. (Relevant water properties: viscosity $\nu = 0.01$ cm^2/s; heat capacity $C_p = 4$ Joules/g °K; density $\rho = 1$ g/cc).

If the equivalent energy were put into gravitational potential energy, roughly how high could you lift yourself?

7. In class, we saw the Cocktail Boat, which propels itself along the fluid surface by leaking alcohol off its stern. Given that alcohol reduces the surface tension of an air-water interface (~ 70 dynes/cm) by approximately 50%, rationalize the observed boat speed.