Problem Set 2

18.355

The purpose of this homework assignment is to review some basic manipulations of vector calculus that will be used throughout much of this course. It is (strongly) suggested that index notation be used to prove the vector identities shown below.

1. Evaluate the following expresions :

   (i) $\delta_{ij}\delta_{ij}$ (ii) $\epsilon_{ijk}\epsilon_{kji}$ (iii) $\epsilon_{ijk}a_i a_k$ (iv) $\epsilon_{ijk}\frac{\partial^2\phi}{\partial x_i \partial x_j}$ 4

2. The meaning of $\partial A/\partial x$ is simply the rate-of-change of the scalar quantity A with respect to the direction x at a point in space. Briefly discuss the physical significance of ∇A? If a denotes a vector, what about ∇a? 2

3. Prove : $(\mathrm{a} \wedge \mathrm{b}) \cdot (\mathrm{a} \wedge \mathrm{b}) = |\mathrm{a}|^2|\mathrm{b}|^2 - (\mathrm{a} \cdot \mathrm{b})^2$ 2

4. Prove the following vector identities: a, b = vectors, ϕ = scalar function

   (i) $\nabla \cdot (\phi \mathrm{a}) = \phi \nabla \cdot \mathrm{a} + \mathrm{a} \cdot \nabla \phi$
   (ii) $\nabla \wedge (\phi \mathrm{a}) = \phi \nabla \wedge \mathrm{a} + (\nabla \phi) \wedge \mathrm{a}$
   (iii) $\nabla \cdot (\nabla \wedge \mathrm{a}) = 0$ $\Leftarrow$ true for any vector
   (iv) $\nabla \wedge (\nabla \phi) = 0$ $\Leftarrow$ true for any scalar
   (v) $\nabla \wedge (\mathrm{a} \wedge \mathrm{b}) = (\mathrm{b} \cdot \nabla)\ \mathrm{a} - \mathrm{b}\ (\nabla \cdot \mathrm{a}) - (\mathrm{a} \cdot \nabla)\ \mathrm{b} + \mathrm{a}\ (\nabla \cdot \mathrm{b})$
   (vi) $\nabla \cdot (\mathrm{a} \wedge \mathrm{b}) = (\nabla \wedge \mathrm{a}) \cdot \mathrm{b} - \mathrm{a} \cdot (\nabla \wedge \mathrm{b})$
   (vii) $\nabla \cdot (\mathrm{ab}) = (\nabla \cdot \mathrm{a})\ \mathrm{b} + \mathrm{a} \cdot (\nabla \mathrm{b})$
   (viii) $\nabla \wedge (\nabla \wedge \mathrm{a}) = \nabla(\nabla \cdot \mathrm{a}) - \nabla^2 \mathrm{a}$

   Furthermore, if C is a second order tensor, prove

   (ix) $\mathrm{a} \cdot \mathrm{C} = \mathrm{C} \cdot \mathrm{a}$ if and only if $\mathrm{C} = \mathrm{C}^T$, i.e., C is symmetric.
   (x) If $\mathrm{C} = -\mathrm{C}^T$ (C is anti-symmetric), then $\mathrm{a} \cdot \mathrm{C} \cdot \mathrm{a} = 0$. 10

5. Let x represent the usual position vector. Evaluate

   (i) $\nabla \cdot \mathrm{x}$ (ii) $\nabla \wedge \mathrm{x}$ (iii) $\nabla^2 \mathrm{x}$
   (iv) Let $r^2 = x_j x_j$; differentiate both sides with respect to x_i and show that $\partial r/\partial x_i = x_i/r$ (this is a very useful formula). 5
   (v) $\nabla^2 r$

6. Prove :

   (i) $\int_S \mathrm{n}\, dS = 0$ where n denotes the normal to the surface S.
   (ii) $\int_S (\mathrm{x} \wedge \mathrm{a}) \wedge \mathrm{n}\, dS = 2V\mathrm{a}$ where x denotes the position vector to a point on the surface S, a = constant vector and V = volume bounded by S. 5
   (iii) $\int_C \phi \nabla \phi \cdot d\mathrm{l} = 0$ where C denotes a closed curve.

2-2

7. The Reynolds Transport Theorem (RTT) states that

$$\frac{d}{dt}\int_{V(t)} f(\mathbf{x},t)\, dV = \int_{V(t)} \left[\frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{u} f)\right] dV$$

where $V(t)$ indicates a material control volume, *i.e.*, a volume element that moves with the local fluid velocity $\mathbf{u}(\mathbf{x},t)$. If the density ρ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

prove the special form of the RTT:

$$\frac{d}{dt}\int_{V(t)} \rho\, g(\mathbf{x},t)\, dV = \int_{V(t)} \rho \frac{Dg}{Dt}\, dV,$$

where D/Dt represents the material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla.$$

Homework #2, 18-355 : Solution Set

(1): What is a.) $\delta_{ij}\delta_{ij}$

b.) $\varepsilon_{ijk}\,\varepsilon_{kji}$

c.) $\varepsilon_{ijk}\, a_i a_k$

d.) $\varepsilon_{ijk}\, \dfrac{\partial^2 \psi}{\partial x_i \partial x_j}$?

Identities : i.) $(\underline{a} \wedge \underline{b})_i = \varepsilon_{ijk}\, a_j b_k$

ii.) $\varepsilon_{ijk}\,\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

a.) $\delta_{ij}\delta_{ij} = \delta_{ii} = 3$ 1

b.) $\varepsilon_{ijk}\,\varepsilon_{kji} = \underbrace{\delta_{ij}\,\delta_{ji}}_{3 \text{ by a)}} - \underbrace{\delta_{ii}}_{3}\,\underbrace{\delta_{jj}}_{3} = 3 - 9 = -6$ 1

c.) $\varepsilon_{ijk}\, a_i a_k = \varepsilon_{kji}\, a_k a_i = -\varepsilon_{kji}\, a_i a_k = 0$ 1

d.) $\varepsilon_{ijk}\, \dfrac{\partial^2 \psi}{\partial x_i \partial x_k} = \varepsilon_{kji}\, \dfrac{\partial^2 \psi}{\partial x_k \partial x_i}$ by switching i-k indices

$= -\varepsilon_{kji}\, \dfrac{\partial^2 \psi}{\partial x_i \partial x_k}$ by permuting indices on ε 1

$= -\varepsilon_{kji}\, \dfrac{\partial^2 \psi}{\partial x_k \partial x_i} = 0$

2-4

Q2: $\frac{\partial A}{\partial x}$ is the rate of change of the scalar A w.r.t. the direction x

i.) $\vec{\nabla} A = \left(\frac{\partial A}{\partial x} \hat{i} + \frac{\partial A}{\partial y} \hat{j} + \frac{\partial A}{\partial z} \hat{k} \right)$

is a vector whose components represent the rates of change of A in the x- y- and z- directions.

$\frac{}{2}$ ii.) $\vec{\nabla} \vec{A} = \vec{\nabla} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial A_1}{\partial x_1} & \frac{\partial A_2}{\partial x_1} & \frac{\partial A_3}{\partial x_1} \\ \frac{\partial A_1}{\partial x_2} & \frac{\partial A_2}{\partial x_2} & \frac{\partial A_3}{\partial x_2} \\ \frac{\partial A_1}{\partial x_3} & \frac{\partial A_2}{\partial x_3} & \frac{\partial A_3}{\partial x_3} \end{pmatrix}$ is a 2nd order tensor

The 9 components of $\vec{\nabla} \vec{A}$ represent the rate of change of the 3 components of $\vec{A} = A_i \hat{e}_i$ in the x, y and z- directions.

Q3: $(\underline{a} \wedge \underline{b}) \cdot (\underline{a} \wedge \underline{b}) = \varepsilon_{ijk} a_i b_j \hat{e}_k \cdot \varepsilon_{lmn} a_l b_m \hat{e}_n$

$= \varepsilon_{ijk} \varepsilon_{lmn} a_i b_j a_l b_m \delta_{kn}$

$= \varepsilon_{ijn} \varepsilon_{nlm} a_i b_j a_l b_m$

$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) a_i b_j a_l b_m$

$\frac{}{2}$ $= a_i^2 b_j^2 - a_i b_i a_j b_j$

$= |\underline{a}|^2 |\underline{b}|^2 - (\underline{a} \cdot \underline{b})^2$

4) : i.) $\underline{\nabla} \cdot (\phi \underline{A}) = \frac{\partial}{\partial x_i} \hat{e}_i \cdot (\phi A_j \hat{e}_j) = \frac{\partial}{\partial x_i} \phi A_j \; \delta_{ij}$

$$= \frac{\partial}{\partial x_i} (\phi A_i) = \frac{\partial \phi}{\partial x_i} A_i + \phi \frac{\partial A_i}{\partial x_i}$$

$$= \vec{A} \cdot \vec{\nabla} \phi + \phi \; \vec{\nabla} \cdot \vec{A}$$ T

ii.) $\vec{\nabla} \wedge \phi \vec{A} = \varepsilon_{ijk} \frac{\partial}{\partial x_i} \phi A_j \; \hat{e}_k$

$$= \hat{e}_k \, \varepsilon_{ijk} \left(\frac{\partial \phi}{\partial x_i} A_j + \phi \frac{\partial A_j}{\partial x_i} \right)$$

$$= \varepsilon_{ijk} \frac{\partial \phi}{\partial x_i} A_j \, \hat{e}_k + \phi \, \varepsilon_{ijk} \frac{\partial}{\partial x_i} A_j \; \hat{e}_k$$ T

$$= \vec{\nabla} \phi \wedge \vec{A} + \phi \; \vec{\nabla} \wedge \vec{A}$$

$\delta_{\ell k}$

i.) $\vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{A}) = \frac{\partial}{\partial x_\ell} \hat{e}_\ell \cdot \varepsilon_{ijk} \frac{\partial}{\partial x_i} A_j \; \hat{e}_k$

$$= \varepsilon_{ijk} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} A_j = 0$$ by properties of ε_{ijk}

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.) $\vec{\nabla} \wedge (\vec{\nabla} \phi) = \varepsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial \phi}{\partial x_j} \; \hat{e}_k = 0$ by # 0.2.)

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) $\vec{\nabla} \wedge (\vec{A} \wedge \vec{B}) = \vec{\nabla} \wedge (\varepsilon_{ijk} A_i B_j \hat{e}_k)$

$$= \varepsilon_{\ell m n} \frac{\partial}{\partial x_\ell} (\varepsilon_{ijm} A_i B_j) \hat{e}_n$$

$$= -\varepsilon_{ijm} \, \varepsilon_{m\ell n} \left(\frac{\partial}{\partial x_\ell} A_i B_j \right) \hat{e}_n$$ T

$$= + (-\delta_{i\ell} \delta_{jn} + \delta_{in} \delta_{j\ell}) \left(A_i \frac{\partial B_j}{\partial x_\ell} + B_j \frac{\partial A_i}{\partial x_\ell} \right) \hat{e}_n$$

$$= \left(-A_i \frac{\partial B_j}{\partial x_i} - B_j \frac{\partial A_i}{\partial x_i} \right) \hat{e}_j + \left(A_i \frac{\partial B_j}{\partial x_j} + B_j \frac{\partial A_i}{\partial x_j} \right) \hat{e}_i = \begin{matrix} -\vec{A} \cdot \vec{\nabla} \vec{B} - \vec{B} \; \vec{\nabla} \cdot \vec{A} \\ + \vec{A} \; \vec{\nabla} \cdot \vec{B} + \vec{B} \cdot \vec{\nabla} \vec{A} \end{matrix}$$

vii) $$\vec{\nabla}\cdot(\vec{A}\,\vec{B}) = \frac{\partial}{\partial x_k}\overbrace{\hat{e}_k \cdot (A_i\,\hat{e}_i}^{\delta_{ik}}\; B_j\,\hat{e}_j)$$

$$= \frac{\partial}{\partial x_i}(A_i B_j \hat{e}_j) = \frac{\partial A_i}{\partial x_i} B_j \hat{e}_j + A_i \frac{\partial B_j}{\partial x_i}\hat{e}_j$$

$$= \vec{B}\;\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}\vec{B}$$

viii.) $$\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{A}) = \varepsilon_{lmn}\frac{\partial}{\partial x_l}\left(\varepsilon_{ijm}\frac{\partial}{\partial x_i}A_j\right)\hat{e}_n$$

$$= -\,\varepsilon_{ijm}\,\varepsilon_{mln}\frac{\partial}{\partial x_l}\frac{\partial}{\partial x_i}A_j\;\hat{e}_n$$

$$= +\left(-\delta_{il}\,\delta_{jn} + \delta_{in}\,\delta_{jl}\right)\frac{\partial}{\partial x_l}\frac{\partial}{\partial x_i}A_j\;\hat{e}_n$$

$$= -\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}A_j\,\hat{e}_j + \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}A_j\,\hat{e}_i$$

$$= -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$$

ix.) Prove $\underline{A}\cdot\underline{\underline{C}} = \underline{\underline{C}}\cdot\underline{A}$ iff $\underline{\underline{C}} = \underline{\underline{C}}^T$

If $C_{ij} = C_{ji}$, then

$$\underline{A}\cdot\underline{\underline{C}} = A_i C_{ij}\hat{e}_j = A_i C_{ji}\hat{e}_j = C_{ji}A_i\hat{e}_j = \underline{\underline{C}}\cdot\underline{A}$$

If $\underline{A}\cdot\underline{\underline{C}} = \underline{\underline{C}}\cdot\underline{A}$, then

$$A_i C_{ij}\hat{e}_j = C_{ji}A_i\hat{e}_j \Rightarrow (C_{ij} - C_{ji})A_i\hat{e}_j = 0$$

$$\Rightarrow C_{ij} = C_{ji} = 0$$

Note : $\underline{\underline{C}} = C_{ij}\,\hat{e}_i\,\hat{e}_j$

x.) If $\underline{\underline{C}} = -\underline{\underline{C}}^T$, prove that $\underline{A} \cdot \underline{\underline{C}} \cdot \underline{A} = 0$

$$\underline{A} \cdot \underline{\underline{C}} \cdot \underline{A} = A_i C_{ij} A_j$$

$$= A_j C_{ji} A_i \quad \text{by switching dummy indices}$$

$$= -A_j C_{ji} A_i \quad \text{since } C_{ij} = -C_{ji}$$

$$= 0 \quad \text{as a result.}$$

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ew vi.) :

Prove $\underline{\nabla} \cdot (\vec{a} \wedge \vec{b}) = (\vec{\nabla} \wedge \vec{a}) \cdot \vec{b} - \vec{a} \cdot (\vec{\nabla} \wedge \vec{b})$

$$\vec{\nabla} \cdot (\vec{a} \wedge \vec{b}) = \frac{\partial}{\partial x_i} \hat{e}_i \cdot \hat{e}_\ell \, a_m b_n \varepsilon_{\ell mn}$$

$$= \frac{\partial}{\partial x_\ell} a_m b_n \varepsilon_{\ell mn}$$

$$= \left(\varepsilon_{\ell mn} \frac{\partial}{\partial x_\ell} a_m \hat{e}_n \cdot b_n \hat{e}_n\right) + \left(-\varepsilon_{\ell nm} \frac{\partial}{\partial x_\ell} b_n \hat{e}_m \cdot a_m \hat{e}_m\right)$$

$$= (\vec{\nabla} \wedge \vec{a}) \cdot \vec{b} - (\vec{\nabla} \wedge \vec{b}) \cdot \vec{a}$$

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2-8

(5): Evaluate

i.) $\nabla \cdot \underline{x} = \frac{\partial}{\partial x_i} \hat{e}_i \cdot x_j \hat{e}_j = \delta_{ij} \frac{\partial x_j}{\partial x_i} = \delta_{ij}\delta_{ij} = 3$

ii.) $\underline{\nabla} \wedge \underline{x} = \varepsilon_{ijk} \frac{\partial}{\partial x_i} x_j \hat{e}_k = \varepsilon_{ijk} \delta_{ij} \hat{e}_k$

$= \varepsilon_{iik} \hat{e}_k = 0$

iii.) $\nabla^2 \underline{x} = \underline{\nabla} \cdot \underline{\nabla}\, \underline{x} = \frac{\partial}{\partial x_i} \overbrace{\hat{e}_i \cdot \frac{\partial}{\partial x_j} \hat{e}_j}^{\delta_{ij}} x_k \hat{e}_k$

$= \frac{\partial}{\partial x_i} \frac{\partial x_k}{\partial x_j} \delta_{ij} \hat{e}_k = \frac{\partial}{\partial x_j} \delta_{kj} \hat{e}_k = 0$

iv.) If $r^2 = x_j x_j$, prove that $\frac{dr}{dx_i} = \frac{x_i}{r}$

$\frac{\partial}{\partial x_i} r^2 = \frac{\partial}{\partial x_i} x_j x_j \implies \cancel{2} r \frac{dr}{dx_i} = \cancel{2} \delta_{ij} x_j$

$\therefore \frac{dr}{dx_i} = \frac{x_i}{r}$

v.) $\nabla^2 r = \vec{\nabla} \cdot \vec{\nabla} r = \frac{\partial}{\partial x_i} \overbrace{\hat{e}_i \cdot \frac{\partial r}{\partial x_j} \hat{e}_j}^{\delta_{ij}}$

$= \delta_{ij} \frac{\partial}{\partial x_i} \frac{x_i}{r} = \frac{\partial}{\partial x_i} \frac{x_i}{r}$

$= \frac{1}{r} \frac{dx_i}{dx_i} - \frac{1}{r^2} x_i \frac{dr}{dx_i}$

$= \frac{1}{r} \delta_{ii} - \frac{x_i^2}{r^3} = \frac{3}{r} - \frac{r^2}{r^3} = \frac{2}{r}$

(6) : i.) Prove $\int_S \underline{n}\, dS = 0$

$$\int_S \underline{n}\, dS = \int_V \underline{\nabla}(1)\, dV = 0$$ 1

by Div Thm

i.) Prove $\int_S (\underline{x} \wedge \underline{a}) \wedge \underline{n}\, dS = 2 V \underline{a}$

$$\int_S (\underline{x} \wedge \underline{a}) \wedge \underline{n}\, dS = -\int_S \underline{n} \wedge (\underline{x} \wedge \underline{a})\, dS$$

$$= -\int_V \vec{\nabla} \wedge (\underline{x} \wedge \underline{a})\, dV = +2\underline{a}\, V$$

since $-\vec{\nabla} \wedge (\underline{x} \wedge \underline{a}) = \overset{\to 0}{\cancel{(\vec{x} \cdot \vec{\nabla})\vec{a}}} - \overset{\to 0}{\cancel{\vec{x}(\vec{\nabla} \cdot \vec{a})}}$
$\quad - (\vec{a} \cdot \vec{\nabla})\vec{x} + \vec{a}(\vec{\nabla} \cdot \vec{x})$

$$= -a_i \frac{\partial}{\partial x_i} x_j \hat{e}_j + a_i \hat{e}_i \frac{\partial x_j}{\partial x_j}$$ 2

$$= -a_i \delta_{ij} \hat{e}_j + a_i \hat{e}_i \delta_{jj}$$

$$= -\vec{a} + 3\vec{a} = {}^{-}2\vec{a}$$

ii.) Prove $\int_C \phi \vec{\nabla}\phi \cdot d\vec{l} = 0$

$$\int_C \phi \vec{\nabla}\phi \cdot d\vec{l} = \int_S \underline{n} \cdot (\underline{\nabla} \wedge \phi \vec{\nabla}\phi)\, dS$$

by Stoke's Thm

$$= \int_S \underline{n} \cdot (\underbrace{\underline{\nabla} \wedge \tfrac{1}{2}\underline{\nabla}\phi^2}_{\underline{\nabla} \wedge \underline{\nabla}\,(\text{scalar}) \equiv 0.})\, dS$$ 2

$$= 0$$

2-10

7. Given: $\frac{d}{dt}\int_V f(\underline{x},t)\,dV = \int_V \frac{\partial f}{\partial t} + \underline{\nabla}\cdot\underline{u}f \;\; dV$ (R.T.T.)

Prove: $\frac{d}{dt}\int_V \rho g(\underline{x},t)\,dV = \int_V \rho \frac{Dg}{Dt}\,dV$

Let $f = \rho g$ where ρ satisfies continuity eqn:

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot\rho\underline{u} = 0$$

Then:

$$\frac{d}{dt}\int_V \rho g\,dV = \int_V \frac{\partial}{\partial t}\rho g + \underline{\nabla}\cdot(\rho\underline{u}g)\,dV$$

$$= \int_V \left[\rho\frac{\partial g}{\partial t} + g\frac{\partial\rho}{\partial t} + g\,\underline{\nabla}\cdot(\rho\underline{u}) + \rho\underline{u}\cdot\underline{\nabla}g\right]dV$$

$$= \int_V \rho\left[\frac{\partial g}{\partial t} + \underline{u}\cdot\underline{\nabla}g\right]dV$$

$$= \int_V \rho\,\frac{Dg}{Dt}\,dV$$

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