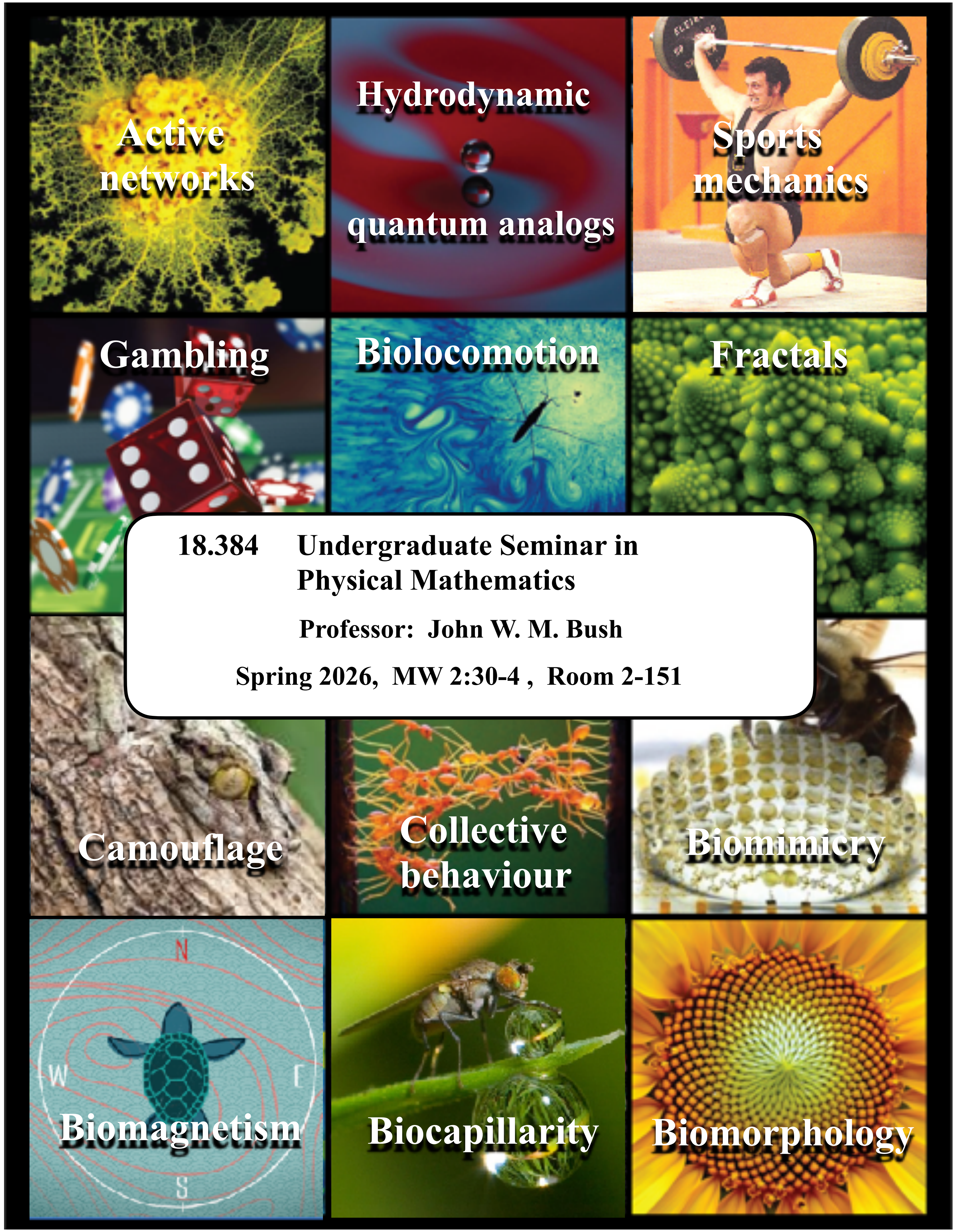


18.384 Undergraduate Seminar in Physical Mathematics

Professor: John W. M. Bush

Spring 2026, MW 2:30-4, Room 2-151

What do you find interesting?

Give this question some thought in the next few weeks as you settle upon a research theme.

After my initial 3 weeks of lecturing, you will be asked to pitch a topic to the class (in 5 minutes) and submit a brief (< 1 page) proposal to me.

“There is no such thing as uninteresting subject — only uninterested people.”

— G.K. Chesterton

18.384: Lecture 2

1) analogical thinking,

2) dimensional analysis,

3) dynamic similarity,

4) scaling arguments

5) fluid mechanics

Means of comparison and degrees of similitude

I. Metaphor

II. Physical analogy

III. Dynamic similarity

IV. Mathematical equivalence

Degrees of similitude

I. Metaphor

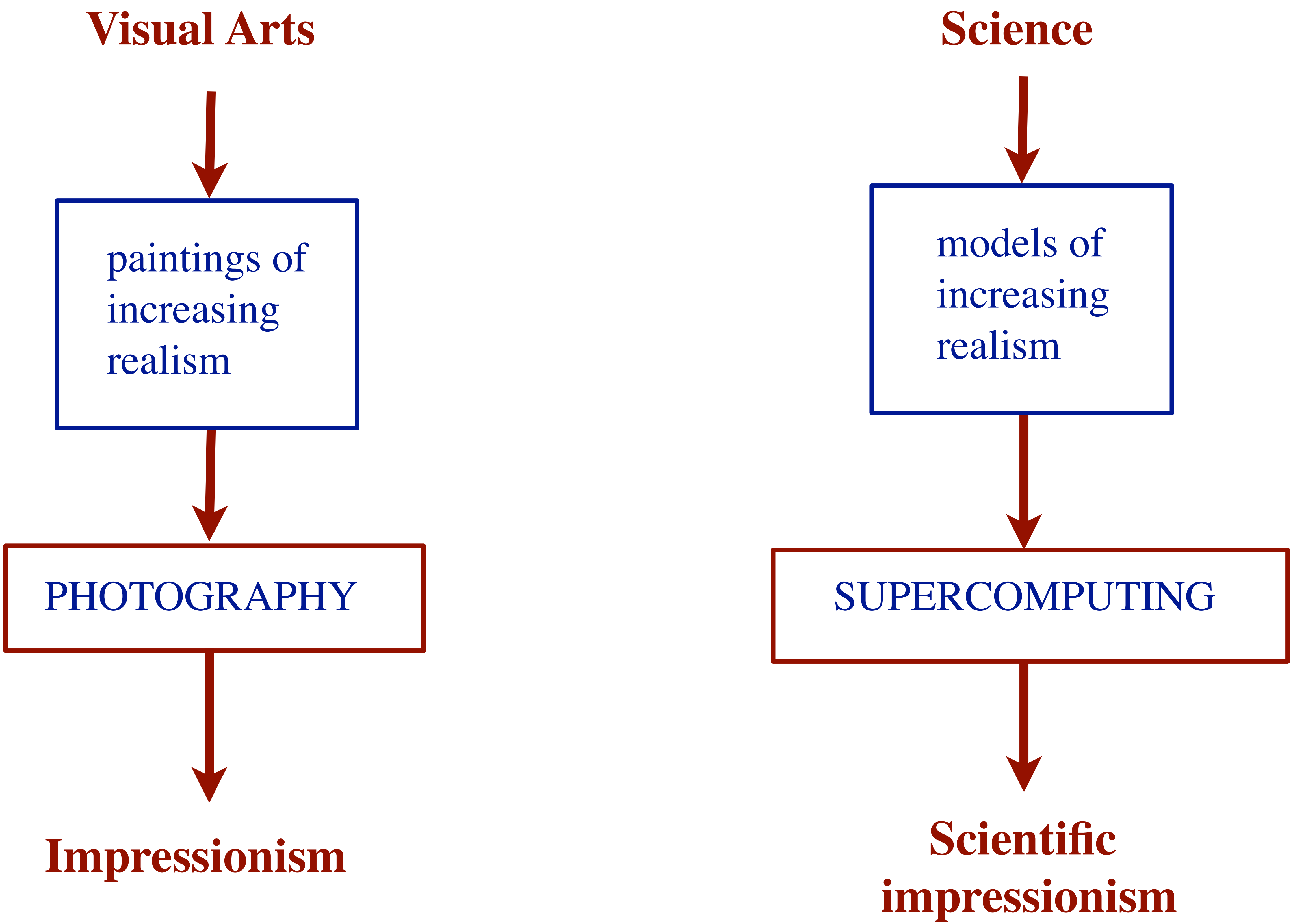
- an imprecise level of comparison that provides a means of using one system to gain insight into another

  e.g. the road unfurled before me

  e.g. the road was a ribbon of moonlight

  e.g. the stars are blue and shiver in the distance

A METAPHOR ON STYLE IN SCIENCE



A hydrodynamic metaphor

"Light corpuscles generate waves in an Aethereal Medium, just like a stone thrown onto water generates waves. In addition, these corpuscles may be alternately accelerated and retarded by the waves."

- Newton, Opticks (1704)

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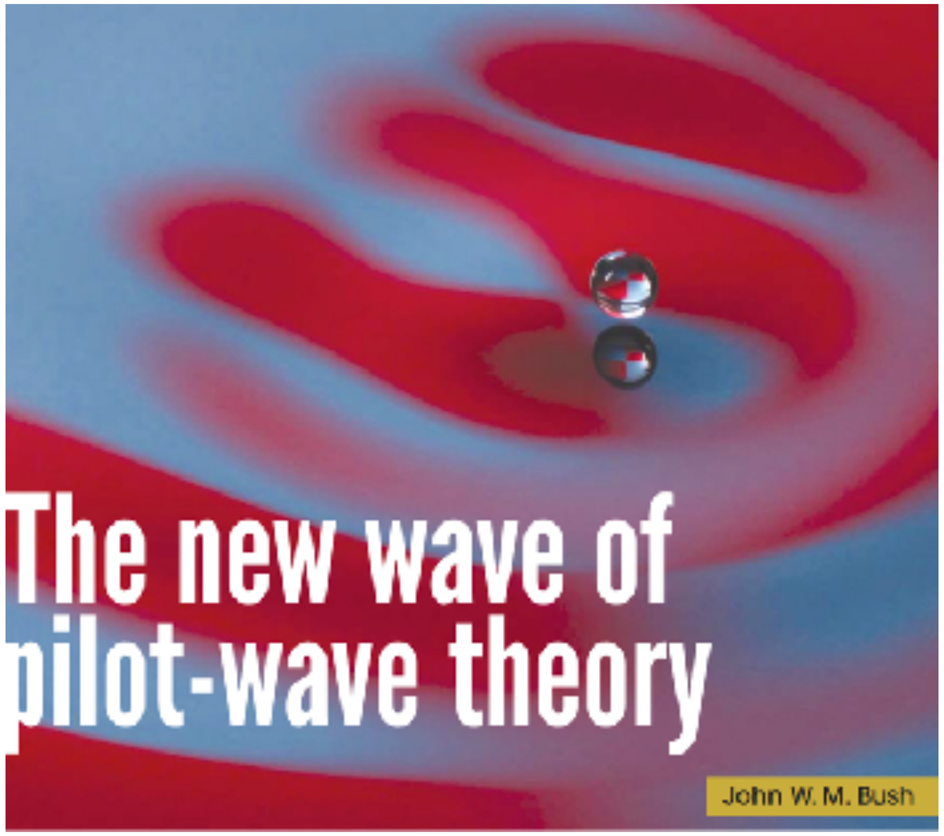
A hydrodynamic metaphor



``If particle physics is the dazzling crown prince of science, fluid mechanics is the cantankerous queen mother: While her loyal subjects flatter her as being rich, mature, and insightful, many consider her to be démodé, uninteresting, and difficult. In her youth, she was more attractive. Her inconsistencies were taken as paradoxes that bestowed on her an air of depth and mystery. The resolution of her paradoxes left her less beguiling but more powerful, and marked her coming of age. She has since seen it all and has weighed in on topics ranging from cosmology to astronautics. Scientists are currently exploring whether she has any wisdom to offer on the controversial subject of quantum foundations."

- JWMB, Physics Today (2015)

Degrees of similitude

I. Metaphor

II. Physical analogy

- may be drawn between two systems comparable in significant respects owing to similarities in their essential physics and underlying mathematical structure
- fluid mechanics provides a framework for modeling a broader class of nonfluidic systems, including electromagnetic, optical, quantum and gravitational systems

Maxwell on physical analogy

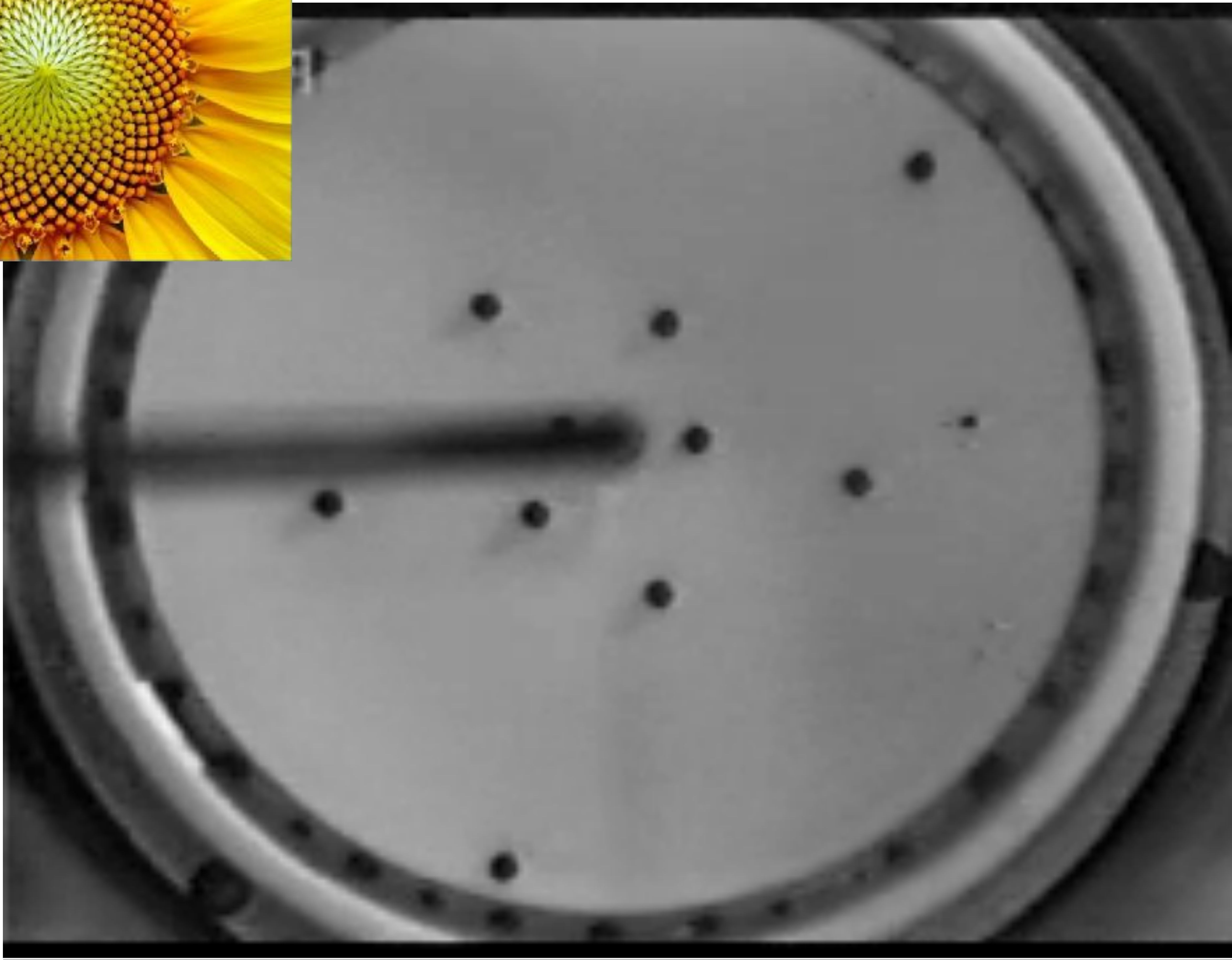
``By a physical analogy I mean that partial similarity between the laws of one science and those of another which makes each of them illustrate the other . . . . We find the same resemblance in mathematical form between two different phenomena."

—James Clerk Maxwell, On Faraday's Lines of Force (1855)

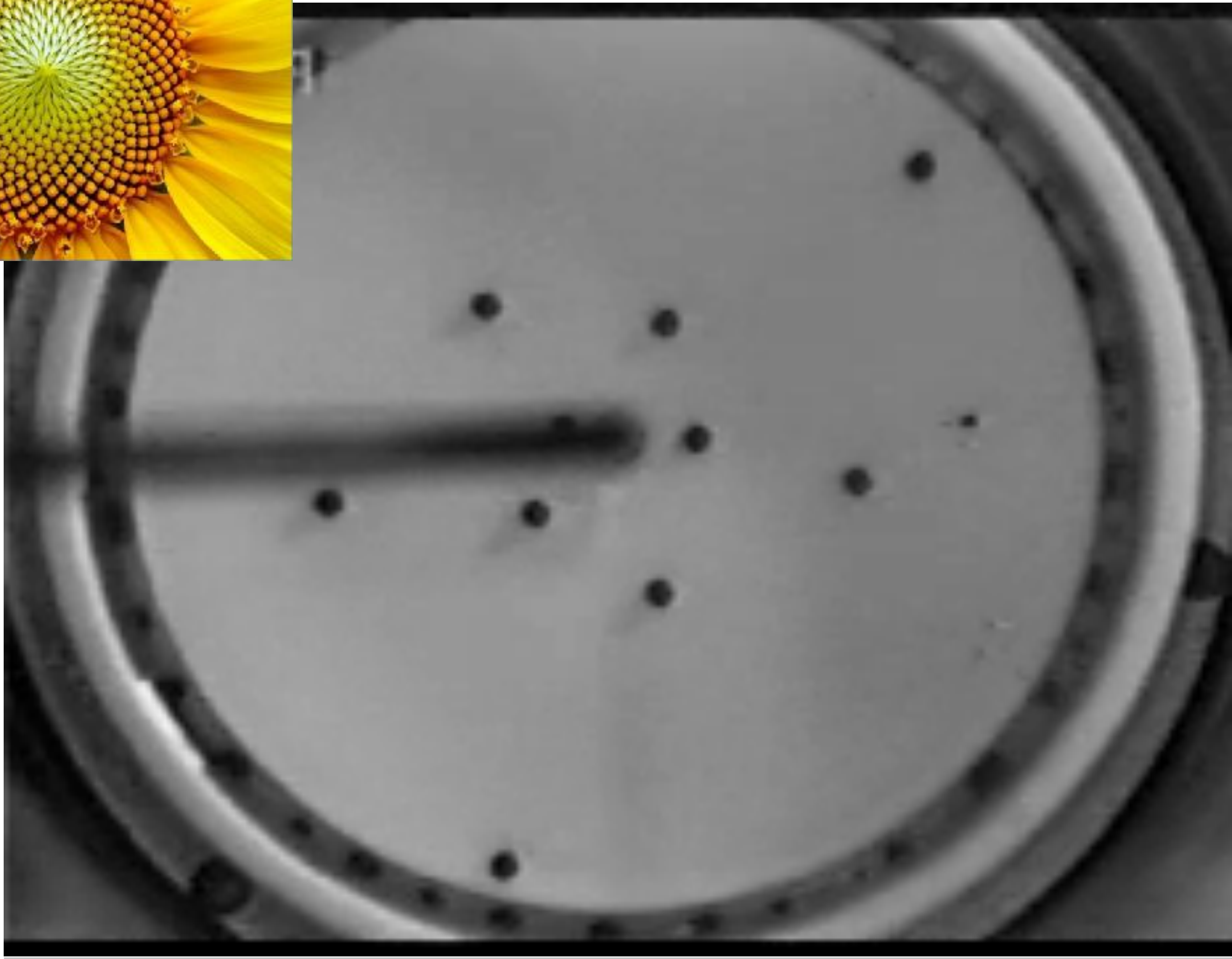
``Now, as in a pun two truths lie hid under one expression, so in an analogy one truth is discovered under two expressions . . . Every question concerning analogies is therefore a question concerning the reciprocal of puns, and the solutions can be transposed by reciprocation."

— James Clerk Maxwell, Are there real analogies in Nature? (1856)

A favorite physical analogy

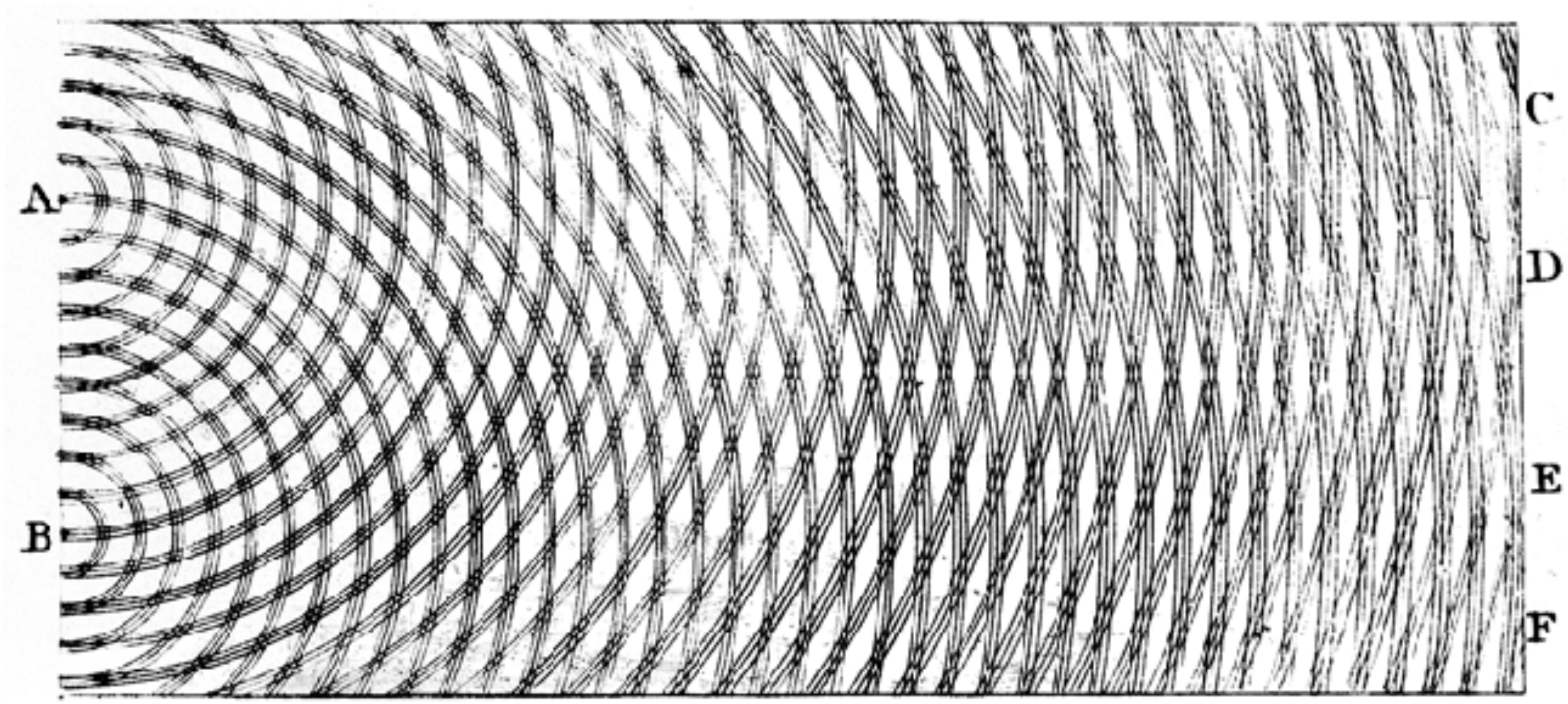


A favorite physical analogy

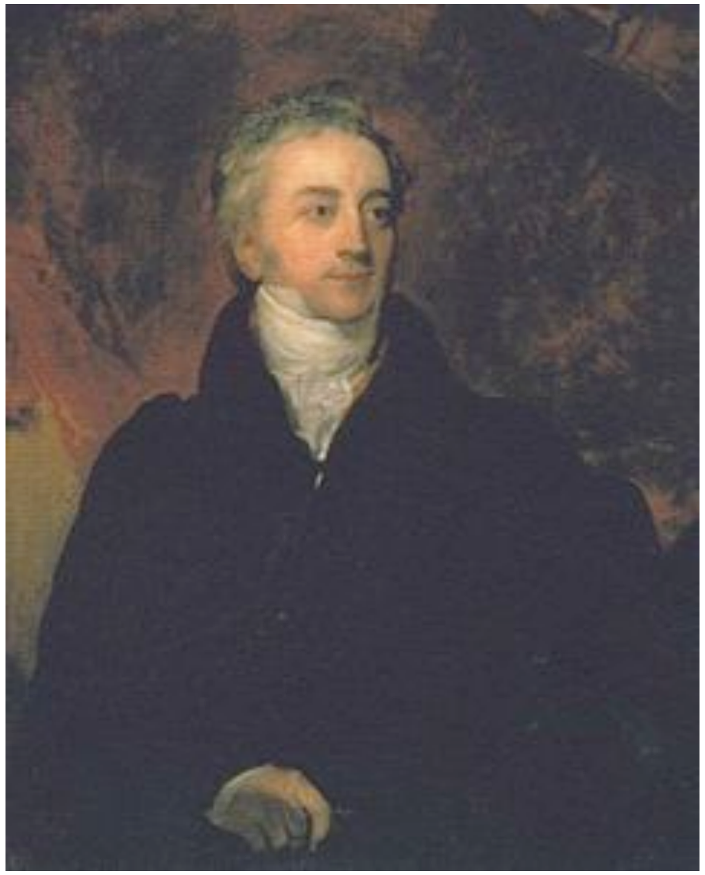


Thomas Young (1773-1829)'s diffraction analog

- contributed to theory of language, music harmony, solid mechanics, medicine, physiology, light, vision, surface tension
- deciphered the Rosetta stone *(selon les Anglais!)*
- felt that his greatest contribution was convincing the scientific community of the wave nature of light using ripple tank experiments

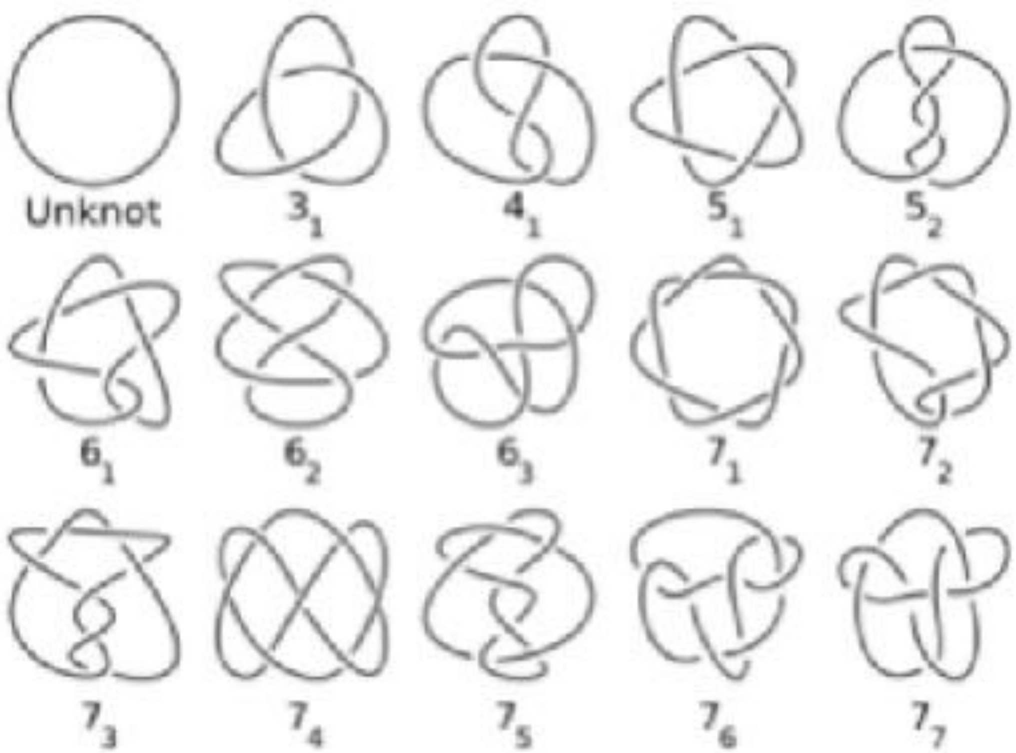


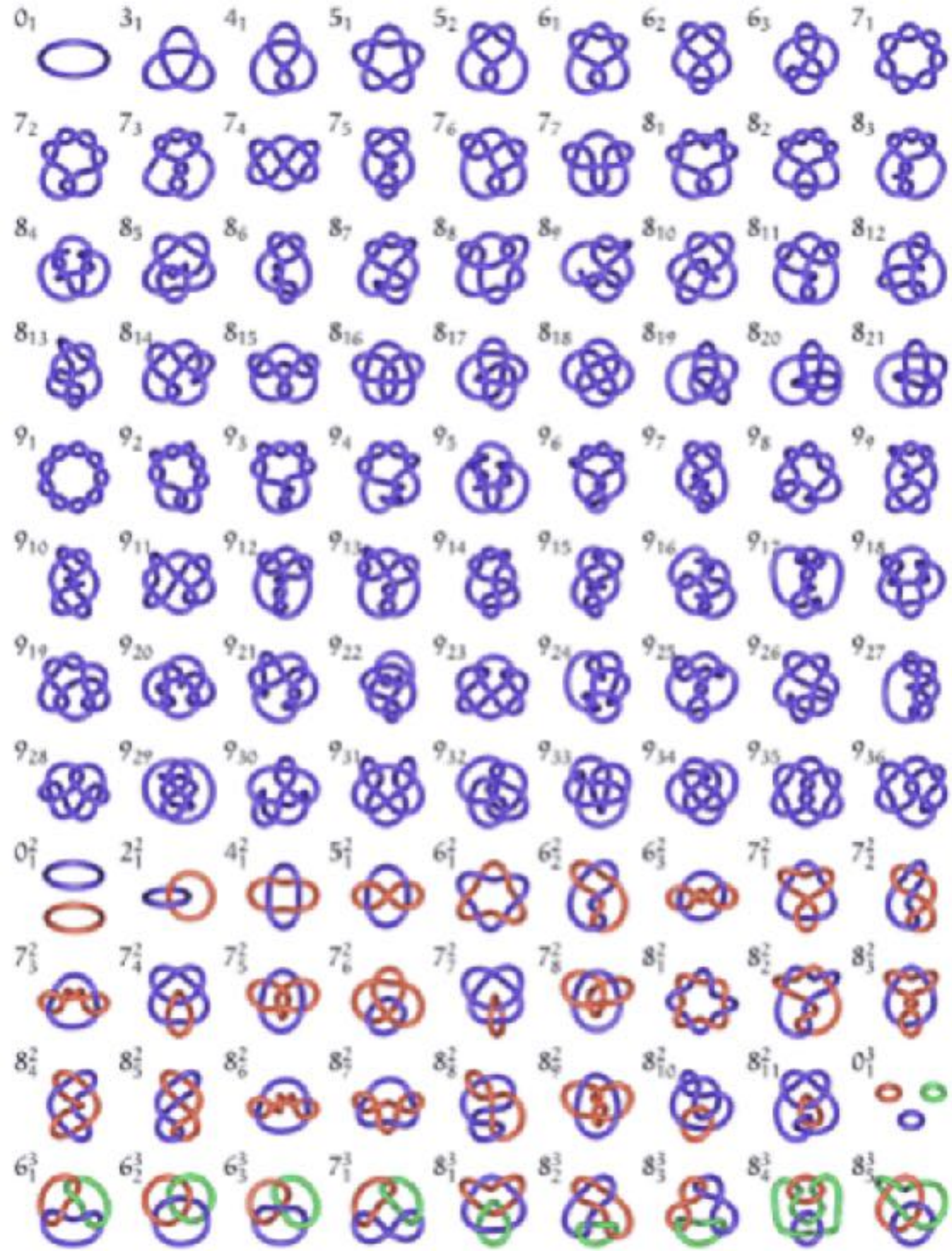
Young (1805)



Lord Kelvin (1880)

- proposed that subatomic particles were knots in the ether





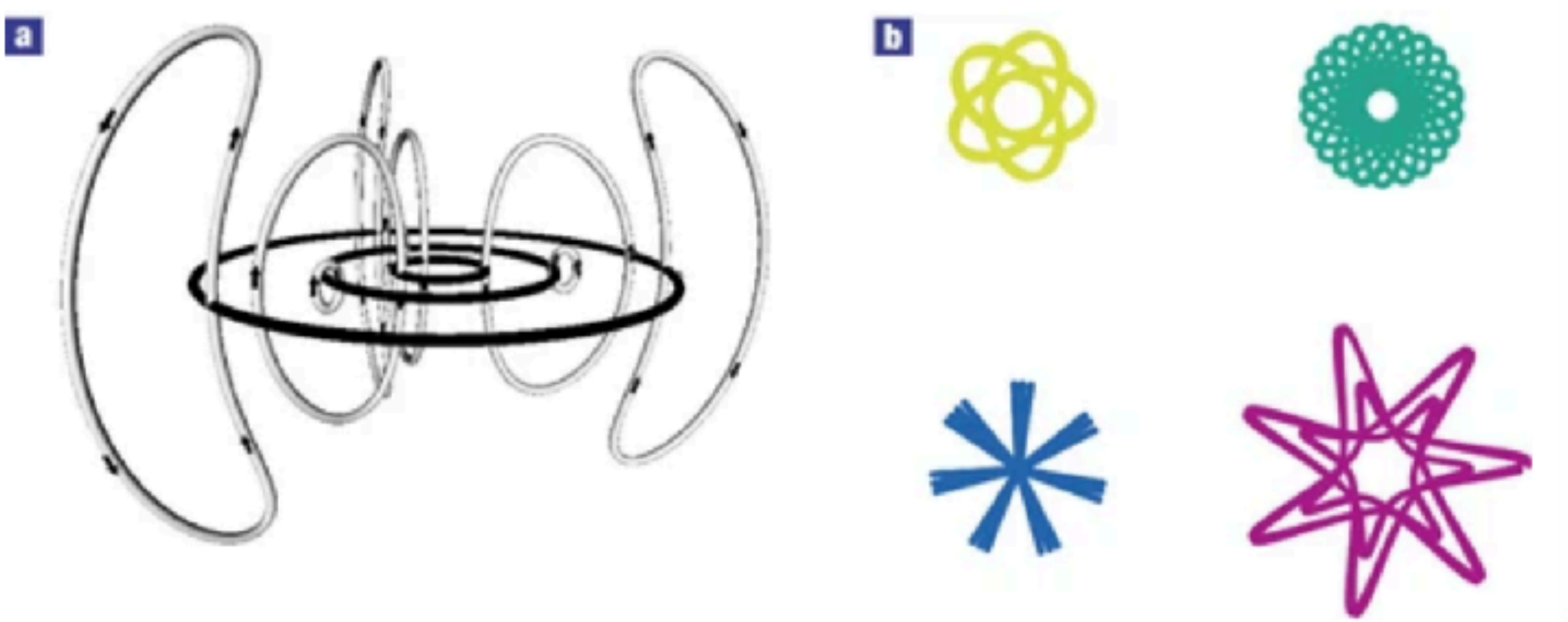
nature › nature physics › letters › article

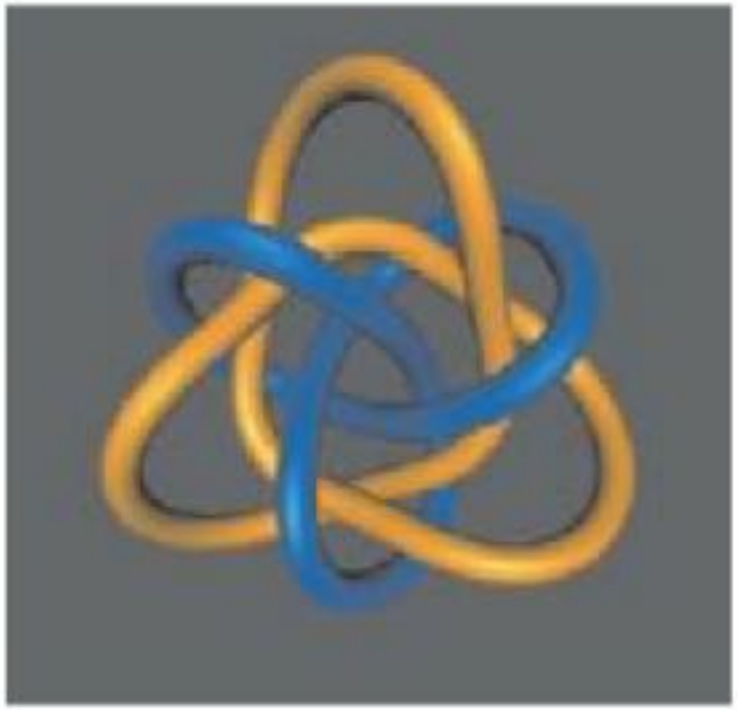
Letter | Published: 31 August 2008

Linked and knotted beams of light

William T. M. Irvine & Dirk Bouwmeester

Nature Physics **4**, 716–720 (2008) | Cite this article



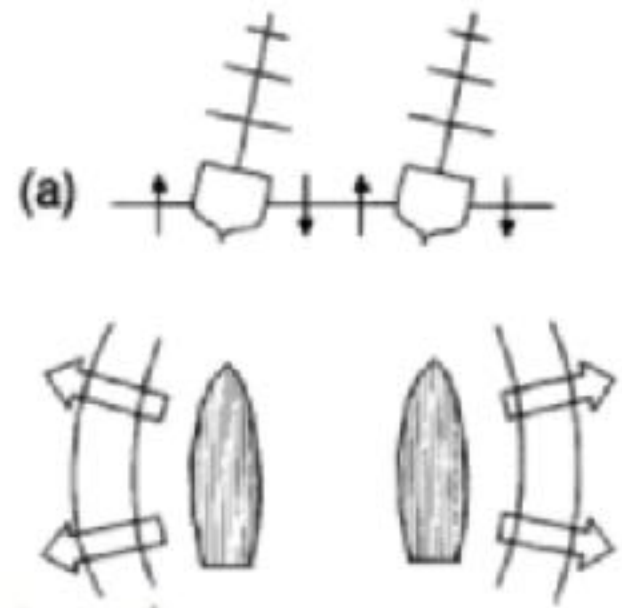


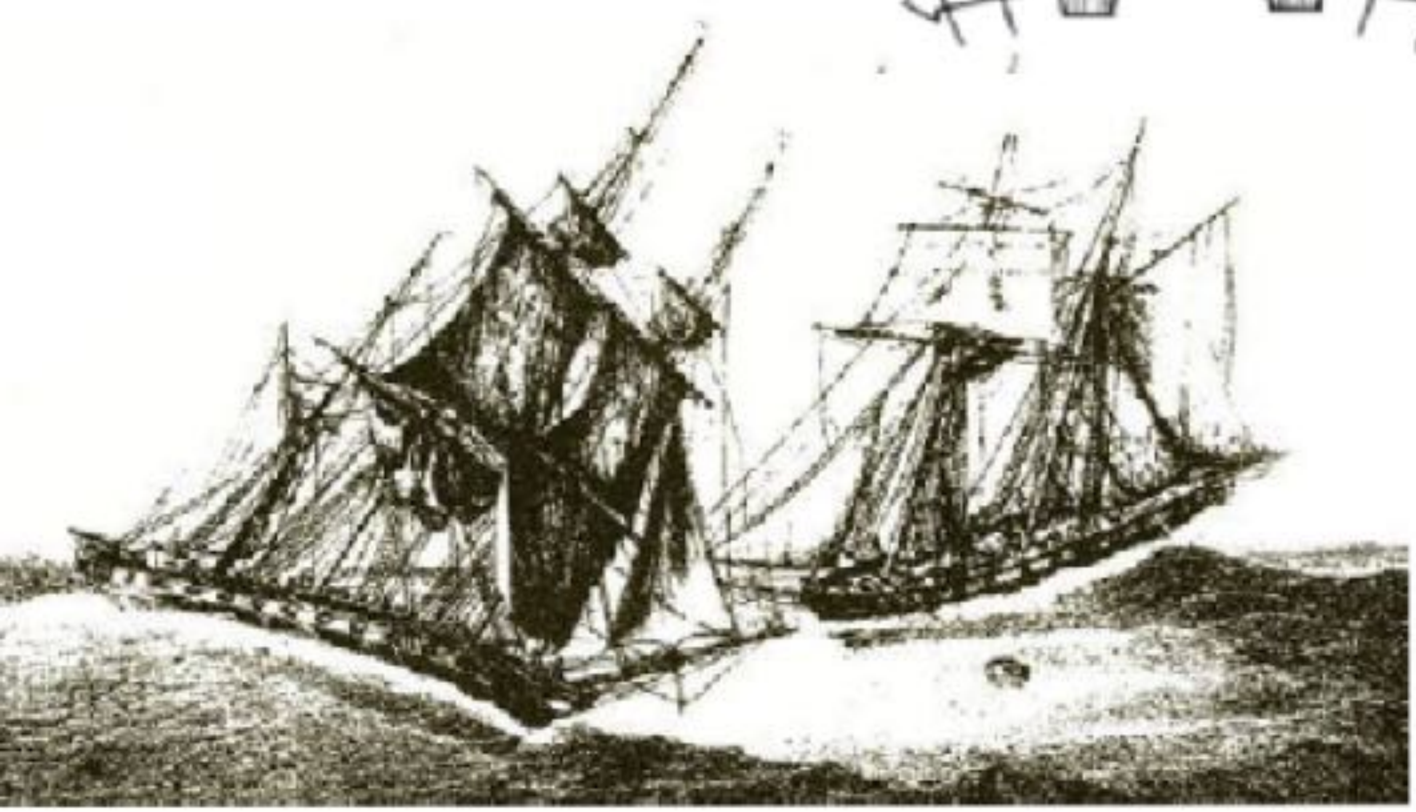
- have created such knots both optically and hydrodynamically

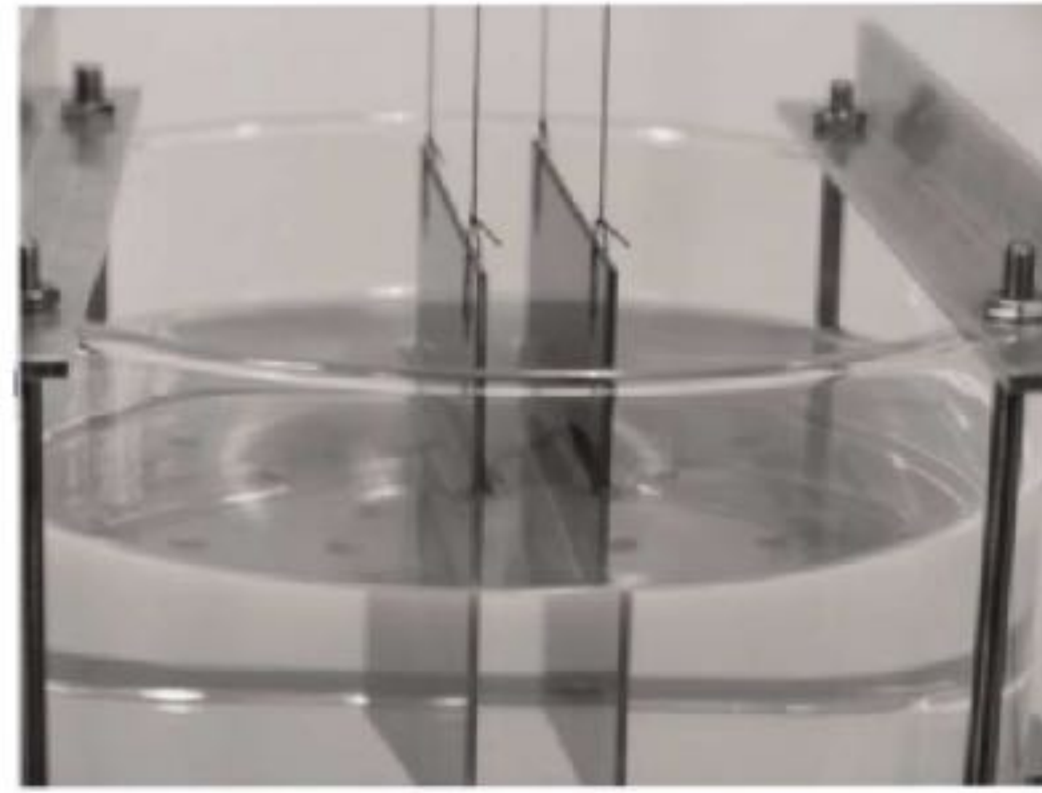
Hydrodynamic analogies of the Casimir effect

- in QM, the Casimir effect gives rise to forces between objects owing to geometric constraint on the EM quantum background field

Maritime analogy (Boersma, AJP, 1996)







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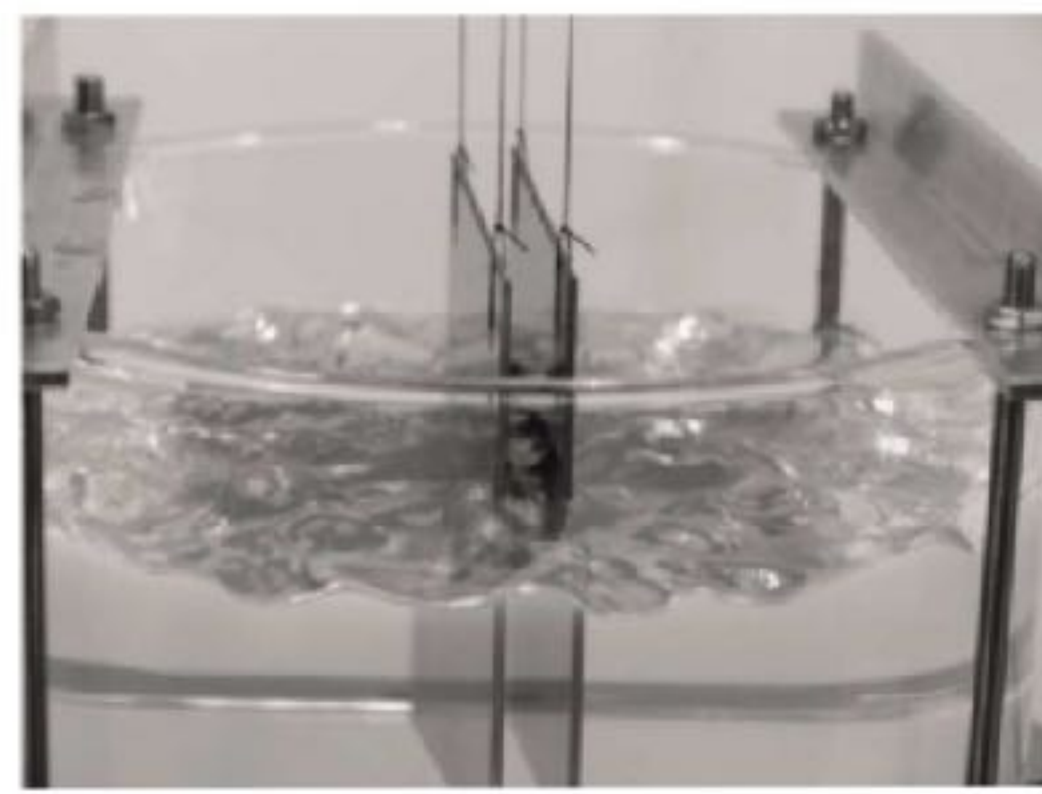
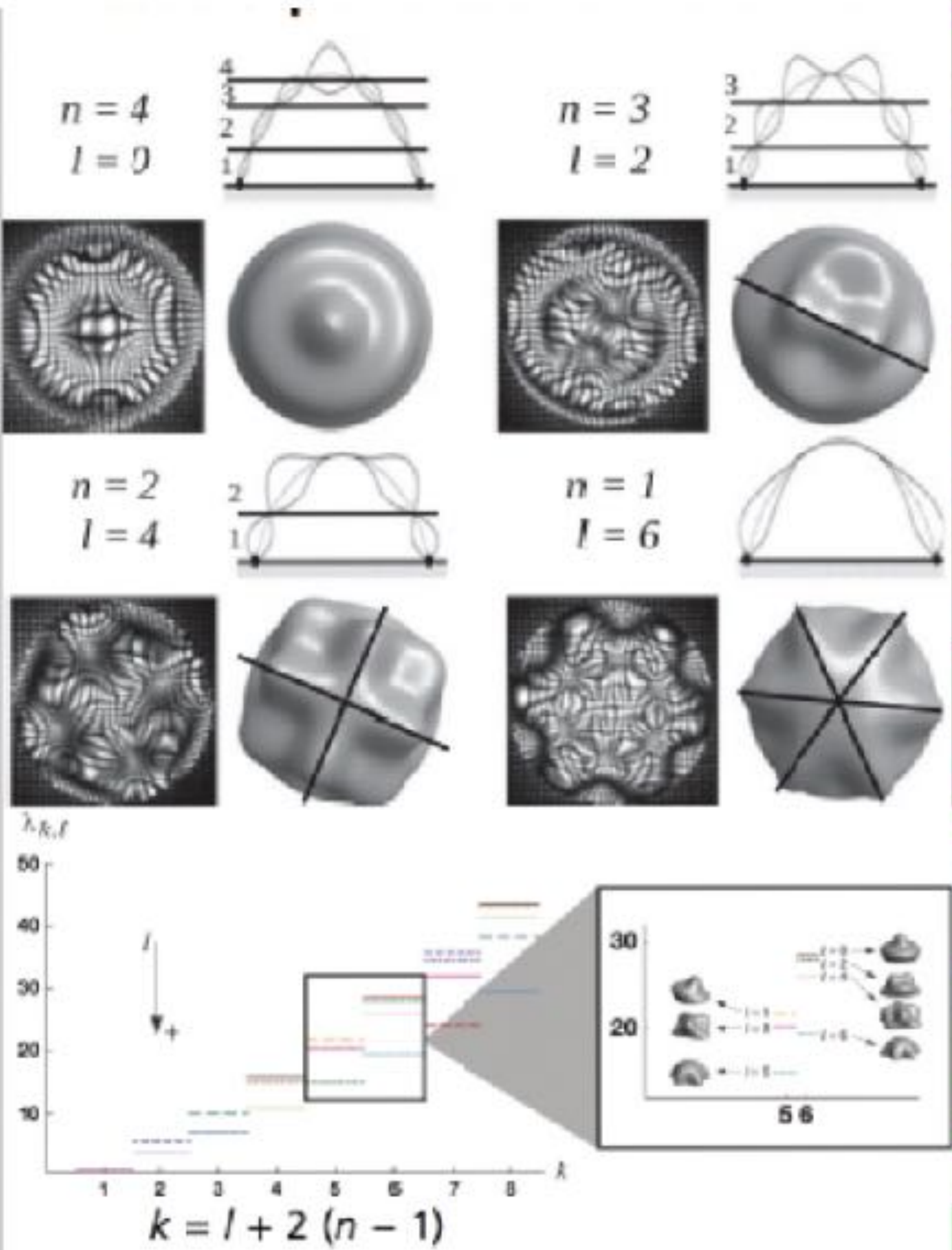
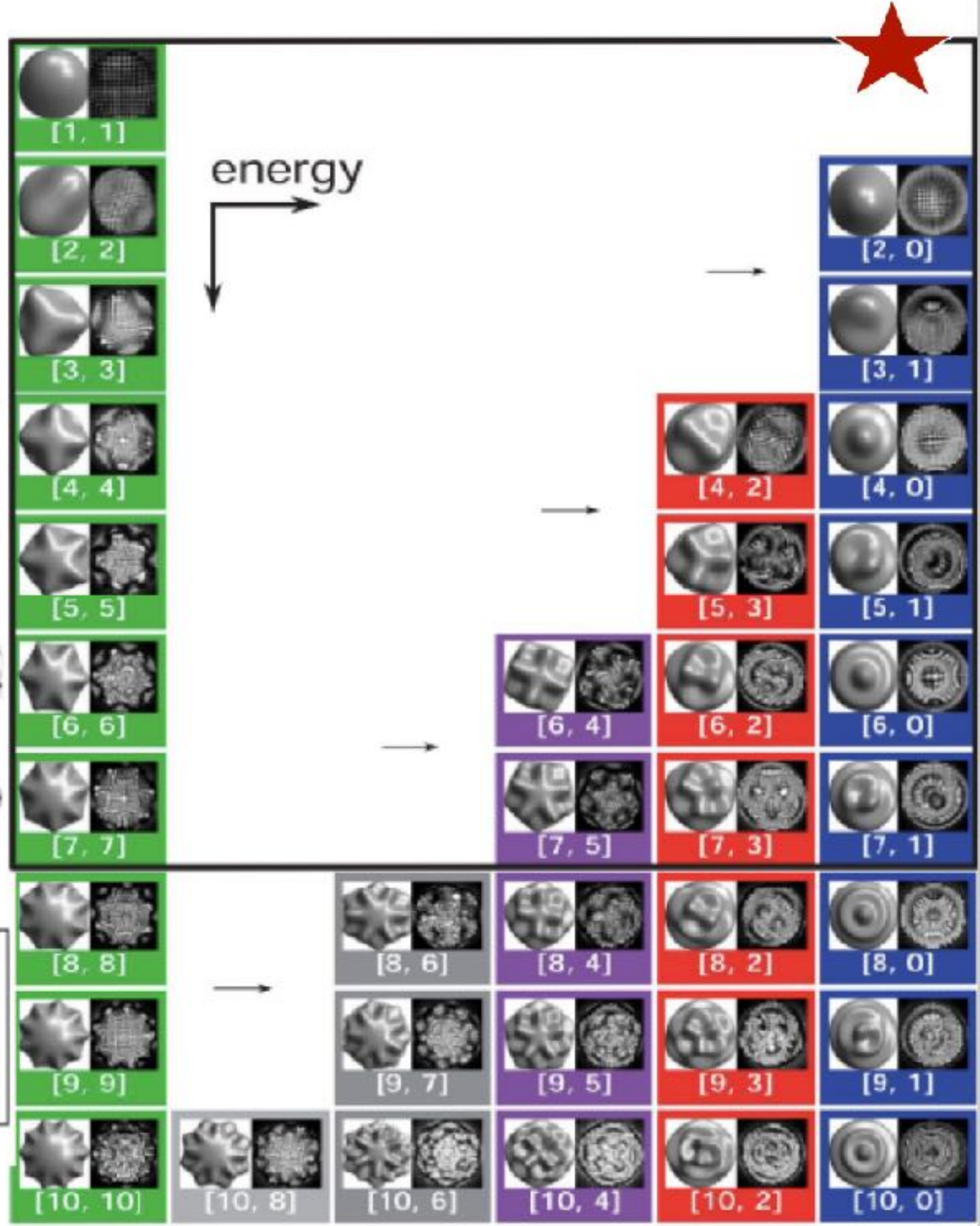


Water wave analogy (Denardo et al. AJP, 2009)

Periodic table of drop vibrations

(Steen et al., PNAS, 2019)

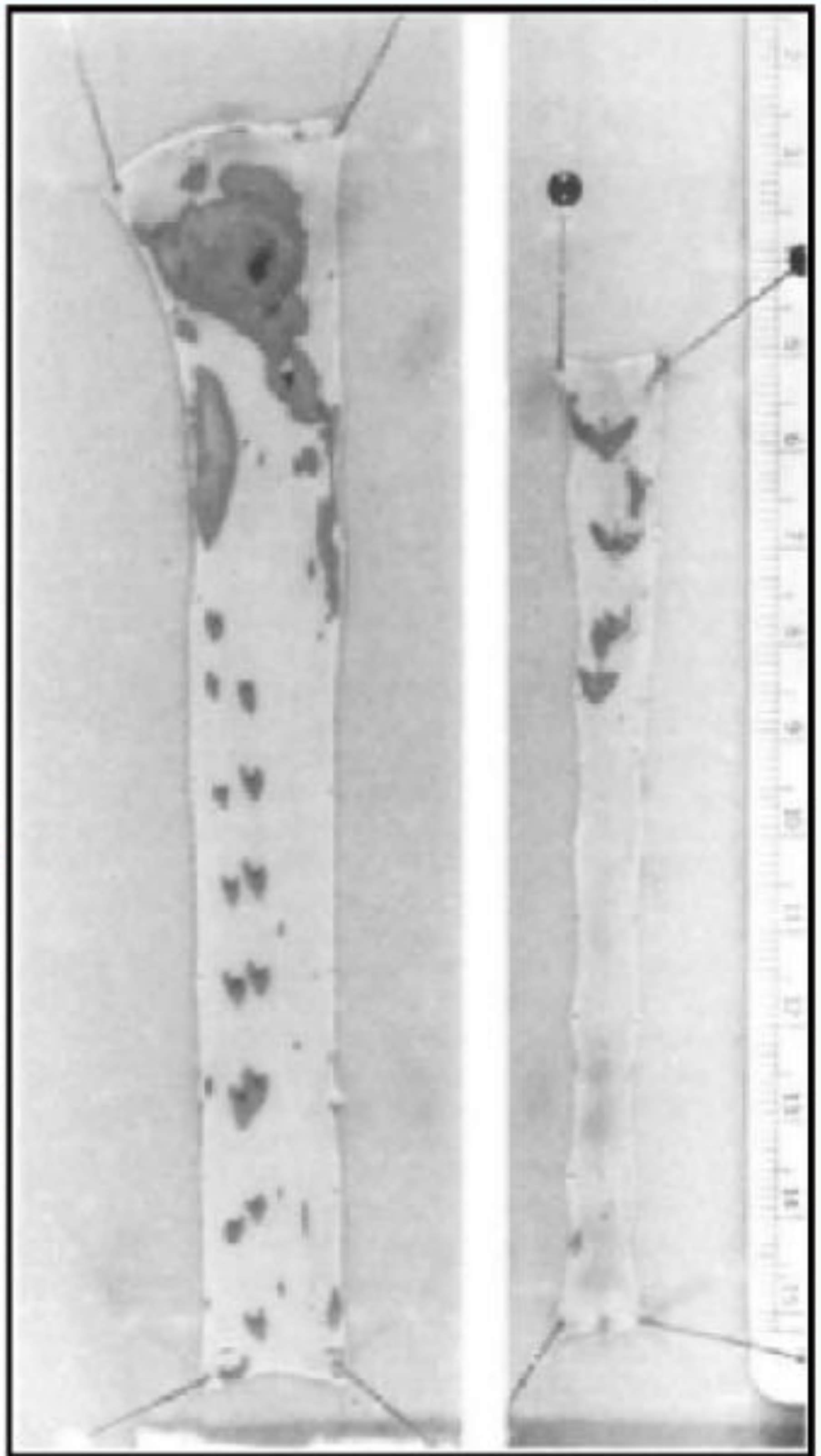




Some physical analogies are better than others

Sand dunes

Rabbit artery



Some physical analogies are better than others

How can you judge?

The similarity of the mathematical descriptions.

Degrees of similitude

I. Metaphor

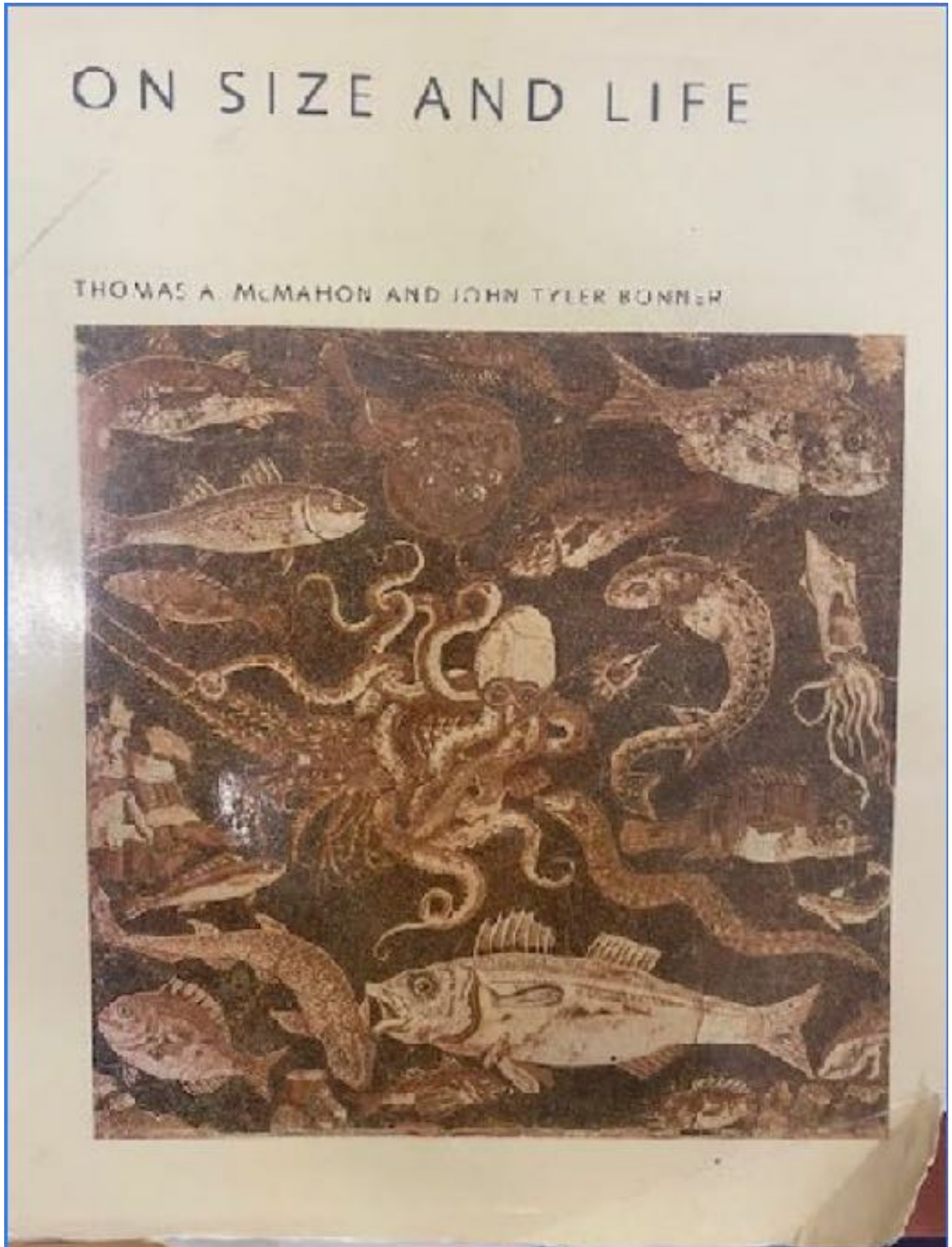
II. Physical analogy

III. Dynamic similarity

- arises between two physical systems when strict mathematical equivalence is achieved between them
- the cornerstone of laboratory modeling in fluid mechanics

… a digression into Dimensional Analysis and Scaling

Dimensional analysis and scaling arguments



Fundamental concept:

The laws of Nature cannot depend on arbitrarily chosen system of units.
A system is most succinctly described in terms of dimensionless variables.

DIMENSIONAL ANALYSIS

- the deduction of the dimensionless groups governing a physical system

Value: 1) minimizes number of parameters governing a physical system (thus facilitating experimental modeling)

2) occasionally yields scaling of physical variables directly

Deduction of Dimensionless groups: **Buckingham's Theorem**

For a system with M physical variables (e.g. density, speed, length, viscosity) describable in terms of N fundamental units (e.g. mass, length, time, temperature), there are M - N dimensionless groups that govern the system.

Corollary to Buckingham's Theorem:

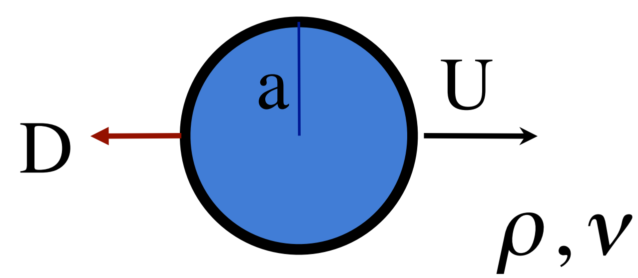
If there is only one dimensionless group, it must be constant.

DIMENSIONAL ANALYSIS

Buckingham's Theorem

For a system with M physical variables (e.g. density, speed, length, viscosity) describable in terms of N fundamental units (e.g. mass, length, time, temperature), there are M - N dimensionless groups that govern the system.

E.g. Translation of a sphere through a fluid



Physical variables: $U, \mathrm{a}, \nu, \rho, D \;\Rightarrow\;$ M = 5

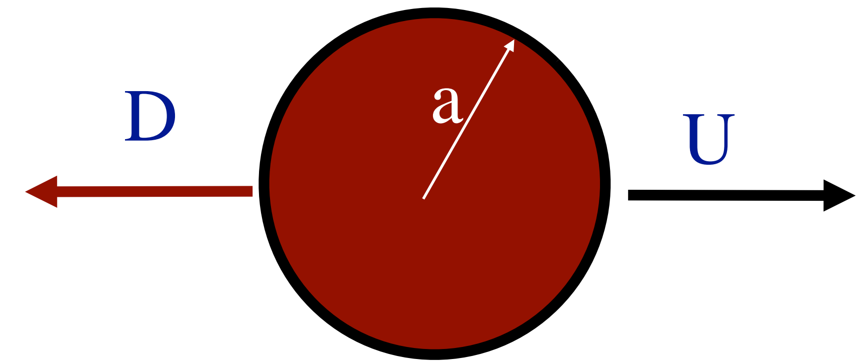
Dimensions: $[U] = \frac{L}{T}$, $[a] = L$, $[\nu] = \frac{L^2}{T}$, $[\rho] = \frac{M}{L^3}$, $[D] = \frac{ML}{T^2}$

Fundamental units: M , L , T $\Rightarrow$ N = 3

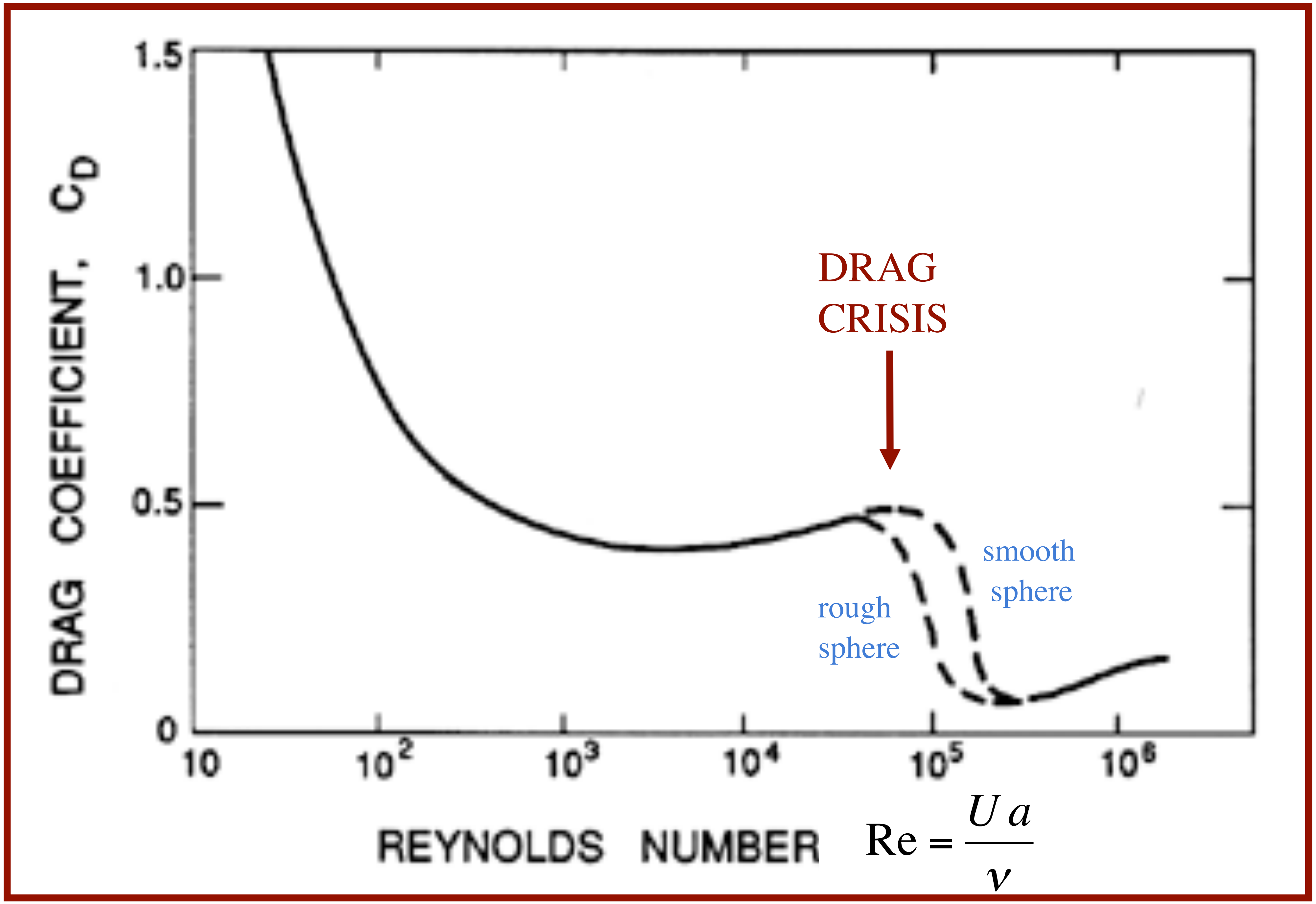
M - N = 2 dimensionless groups: $C_d = \frac{D}{\rho U^2}$, $\mathrm{Re} = \frac{U\mathrm{a}}{\nu}$

System uniquely determined by a single relation: $C_d = F(\mathrm{Re})$

Drag: $D = C_D(\mathrm{Re})\, \rho\, U^2\, a^2$

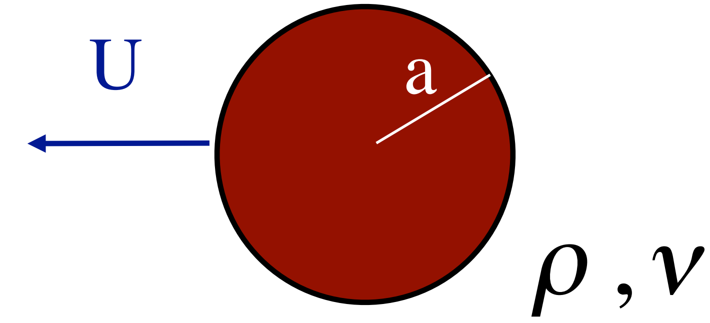


Variation of C_D with $\mathrm{Re} = \dfrac{U a}{\nu}$:



Scaling: the determination of the interdependency of variables in a physical system

Method: consider dominant force balance



Dimensional analysis gave: $D = C_D(\mathrm{Re})\, \rho\, U^2\, a^2$

- form of $C_D(Re)$ yields dependence at low and high Re

Scaling approach:

Re << 1: viscous drag dominant

$$D \sim \frac{\mu U}{a} \cdot a^2 \sim \rho \nu U a$$

VISCOUS STRESS · SURFACE AREA

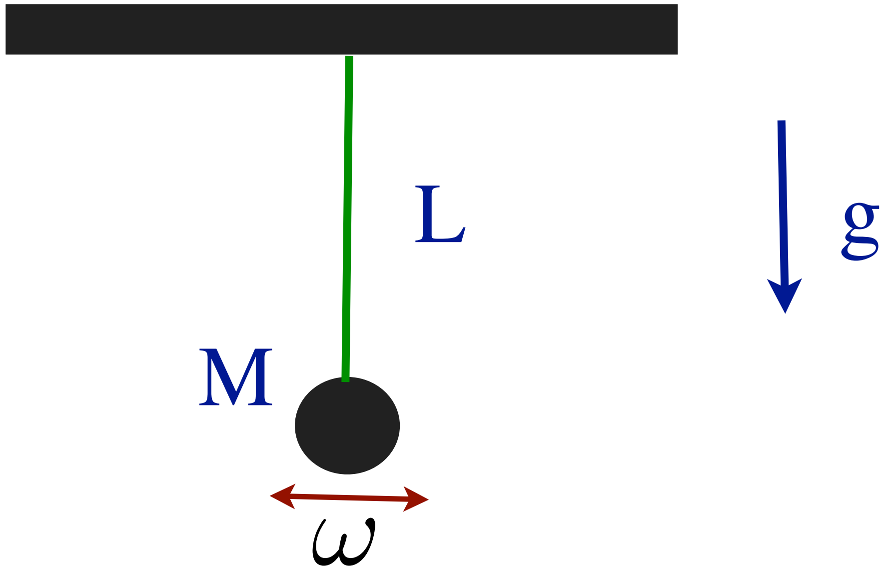
Re >> 1: pressure drag dominant

$$D \sim \rho\, U^2 \cdot a^2$$

FORE-AFT PRESSURE DROP · EXPOSED AREA

Corollary to Buckingham's Theorem:

If there is only one dimensionless group, it must be constant.



The Pendulum

Physical variables: M, L, g, ω

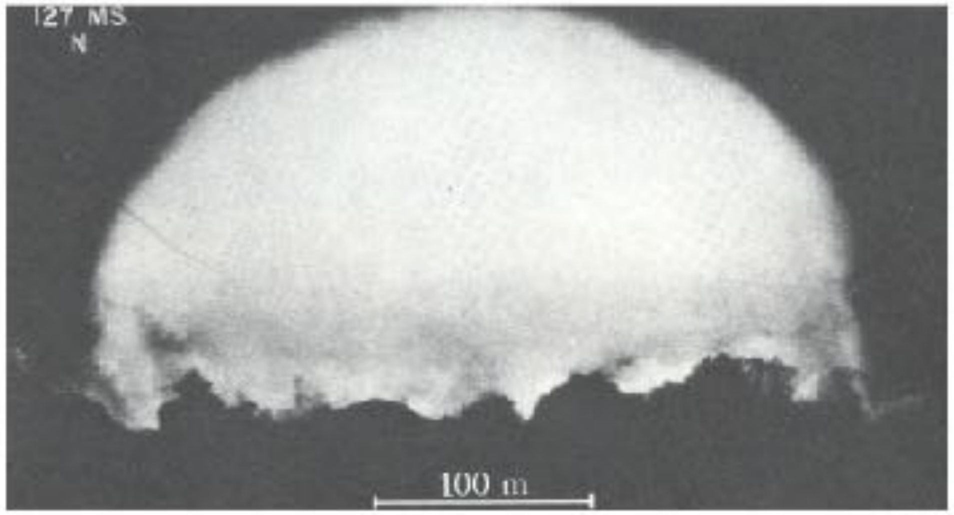
Fundamental units: M, L, T

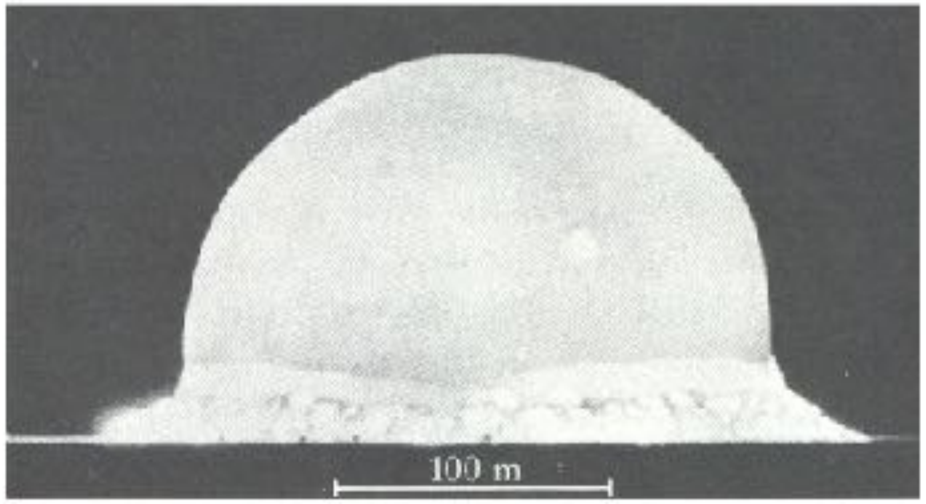
→ system prescribed by one dimensionless group

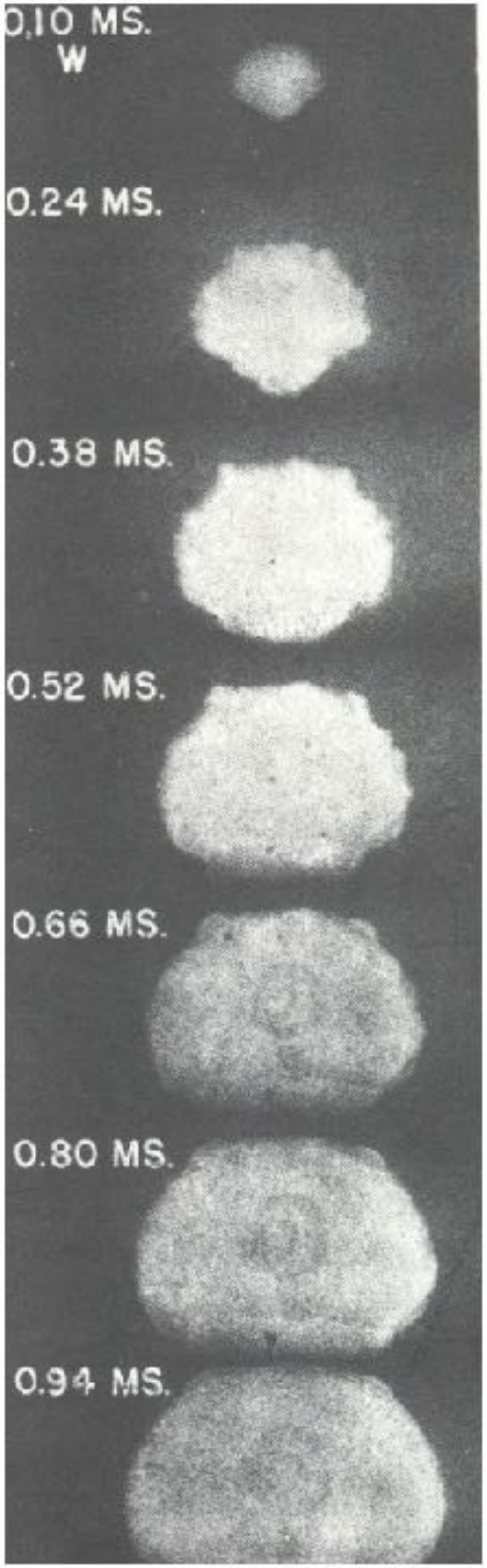
$$\Pi = \frac{\omega^2 L}{g} = constant \longrightarrow \omega \sim \left(\frac{g}{L}\right)^{1/2}$$

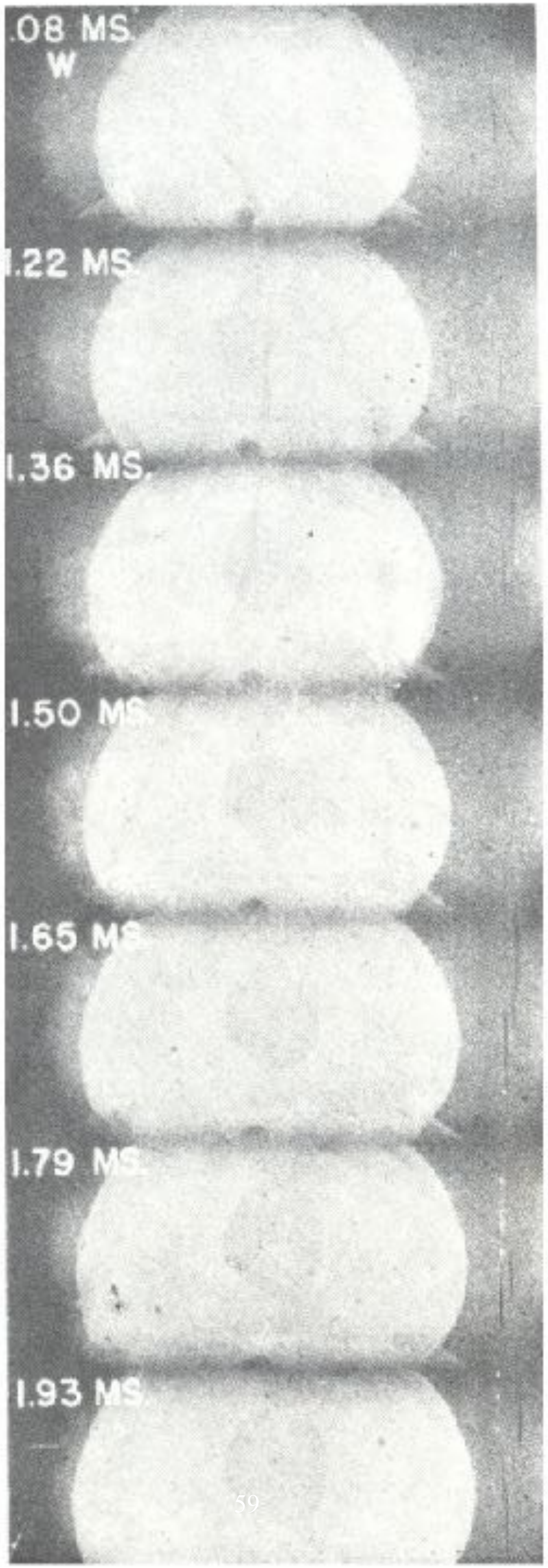
Note: finite amplitude oscillations require consideration of θ

$$\longrightarrow \omega = f(\theta) \left(\frac{g}{L}\right)^{1/2}$$







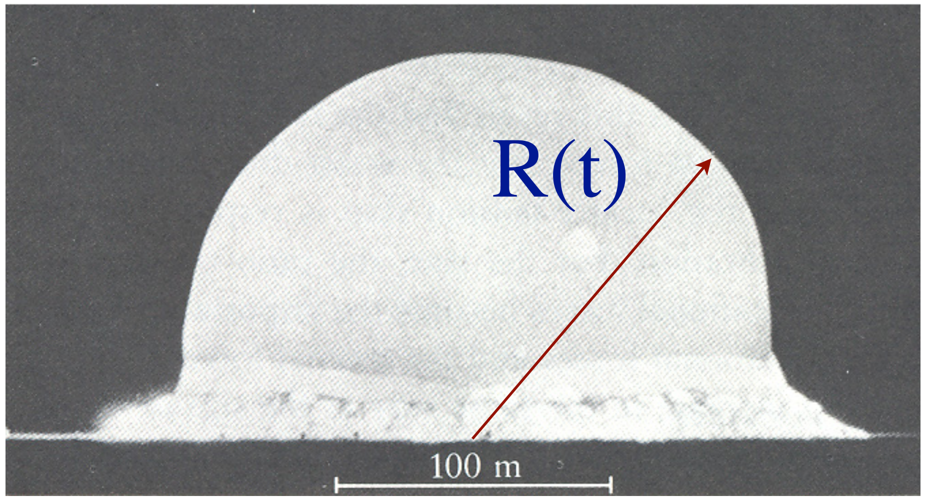


Can we predict R(t) ?

Given R(t), can we infer the energy released?

The scaling of atomic blast clouds

- G.I. Taylor, Sedov



Physical variables: R, t, E, ρ

Fundamental units: M, L, T

→ system prescribed by one dimensionless group:

$$\Pi = \frac{Et^2}{\rho R^5} = constant$$

→ yields desired scaling for radius of the blast cloud:

$$R \sim \left(\frac{E}{\rho}\right)^{1/5} t^{2/5}$$

- this inference led to arrest of G.I. Taylor for suspected spying in 1945
- this scaling led to the prosecution of a German fireworks manufacturer

Scaling in Biology

Scaling: physically informed dimensional analysis

- yields interdependency of physical variables required by balance of dominant forces or energies

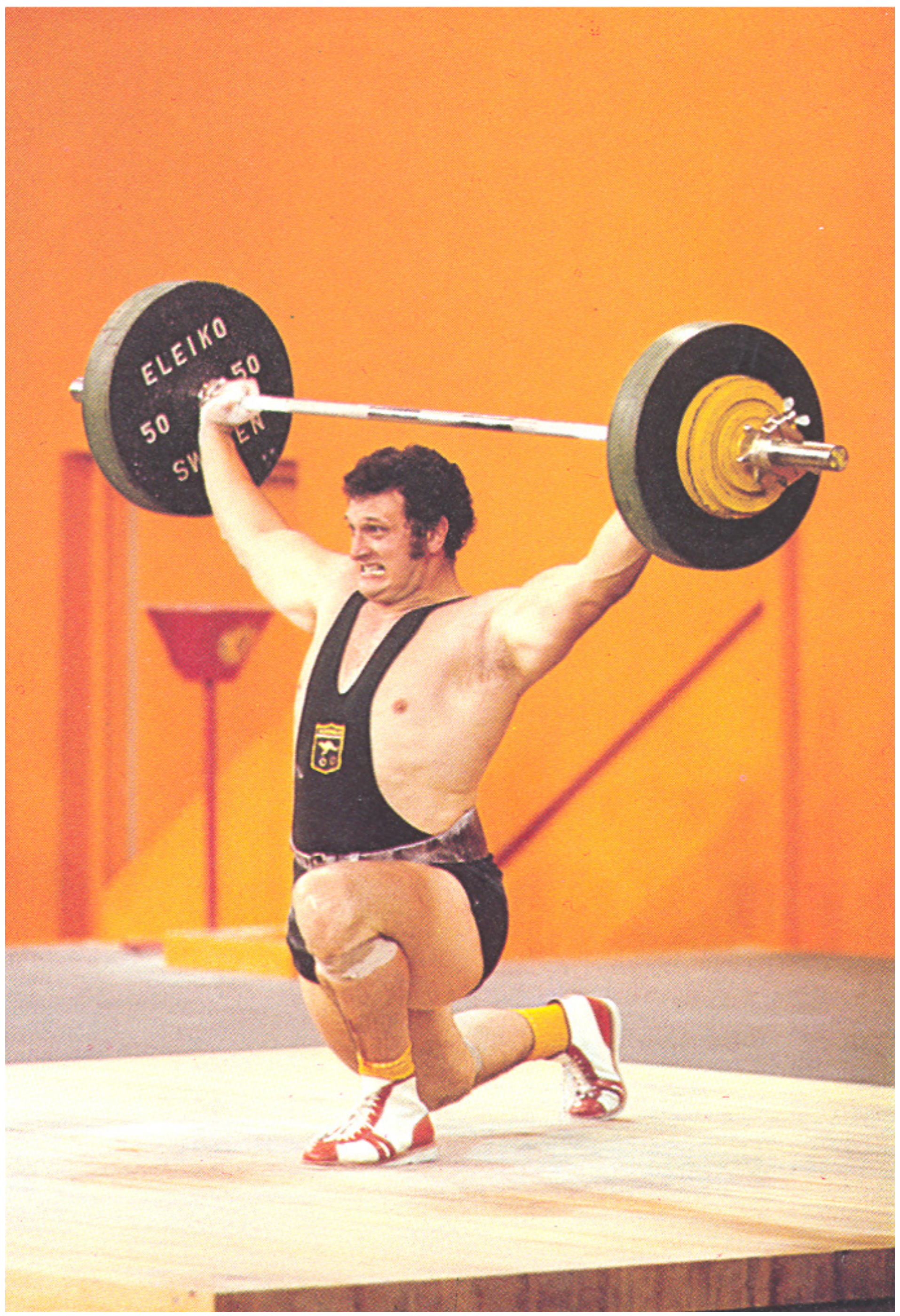
Isometry

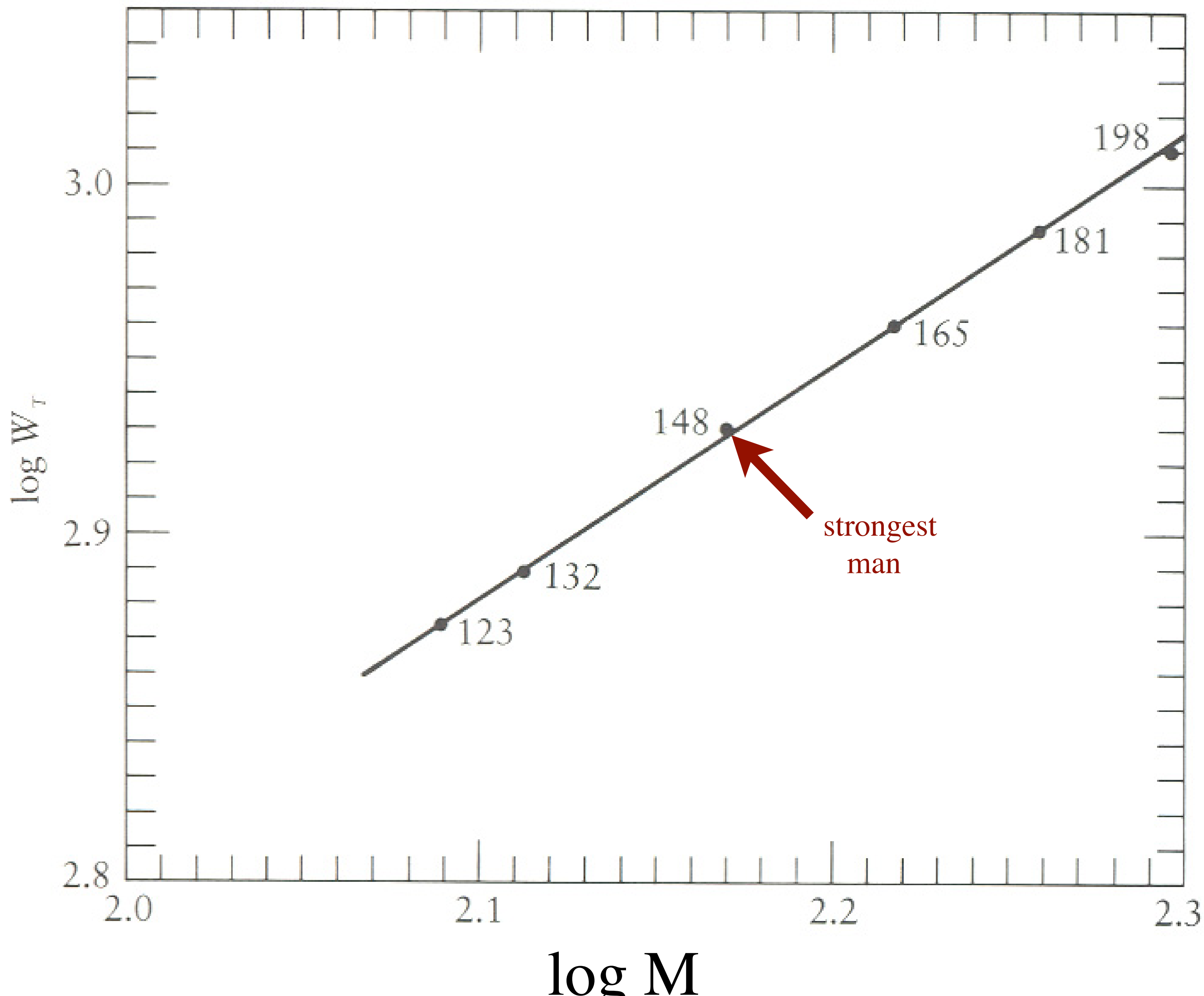
- assumption concerning the self-similar form of creatures of different size

e.g. isometry requires that for a creature of length L, surface area $A \sim L^2$, volume $V \sim L^3$

How much force can a creature of length L and mass M generate/sustain?

$$W_T \sim M^{2/3} \sim L^2$$





Ratio of force generated to size: $$\frac{W_T}{M} \sim \frac{M^{2/3}}{M} \sim M^{-1/3}$$

How does leap height depend on body size in the animal kingdom?

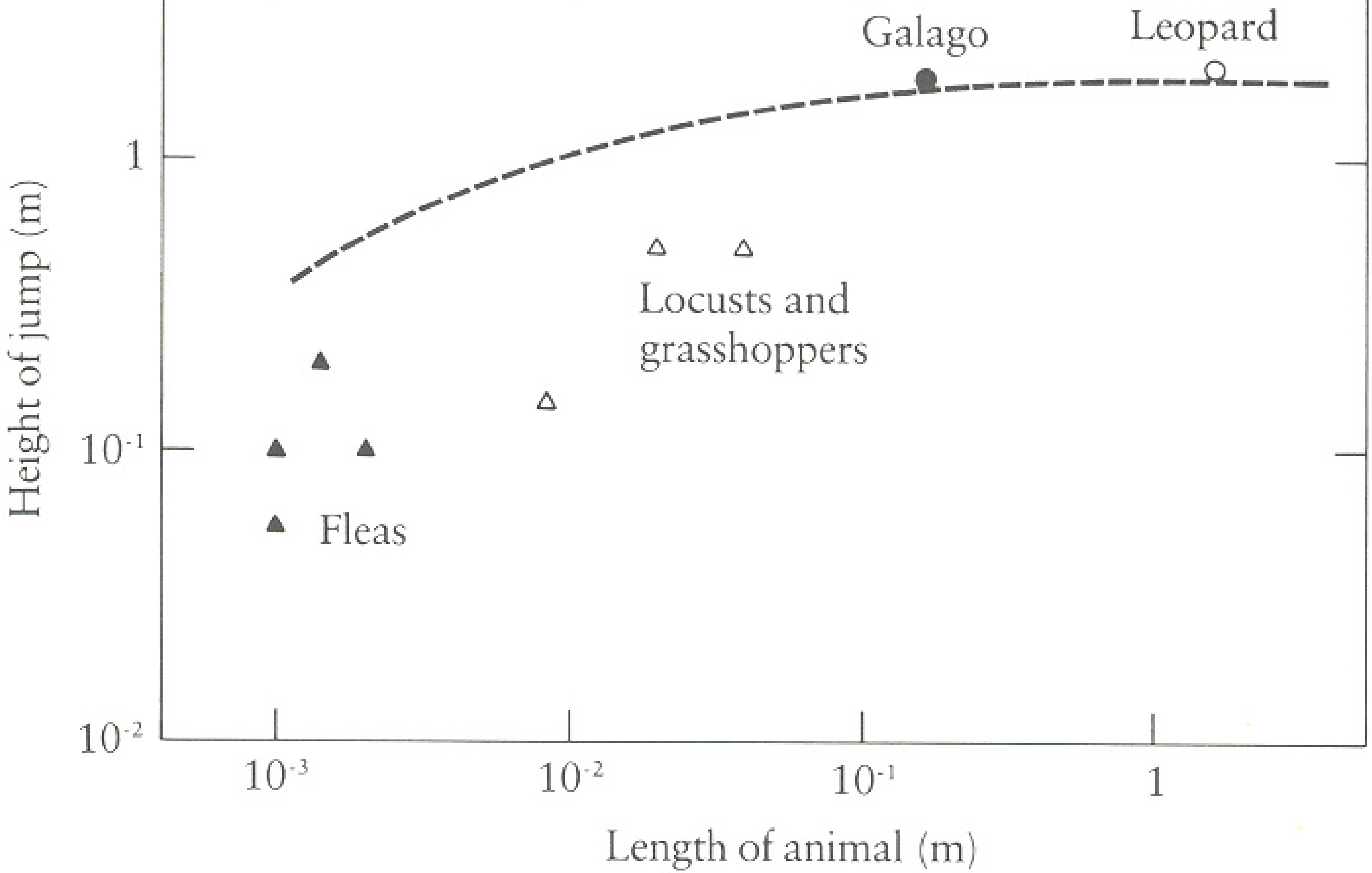
Work done at take off is converted into GPE:

$$F\ L\ \sim\ M\ g\ H$$

Thus, since $F \sim L^2$ and $M \sim L^3$,

$$\boxed{H \sim L^0}$$

Leap height independent of body size.

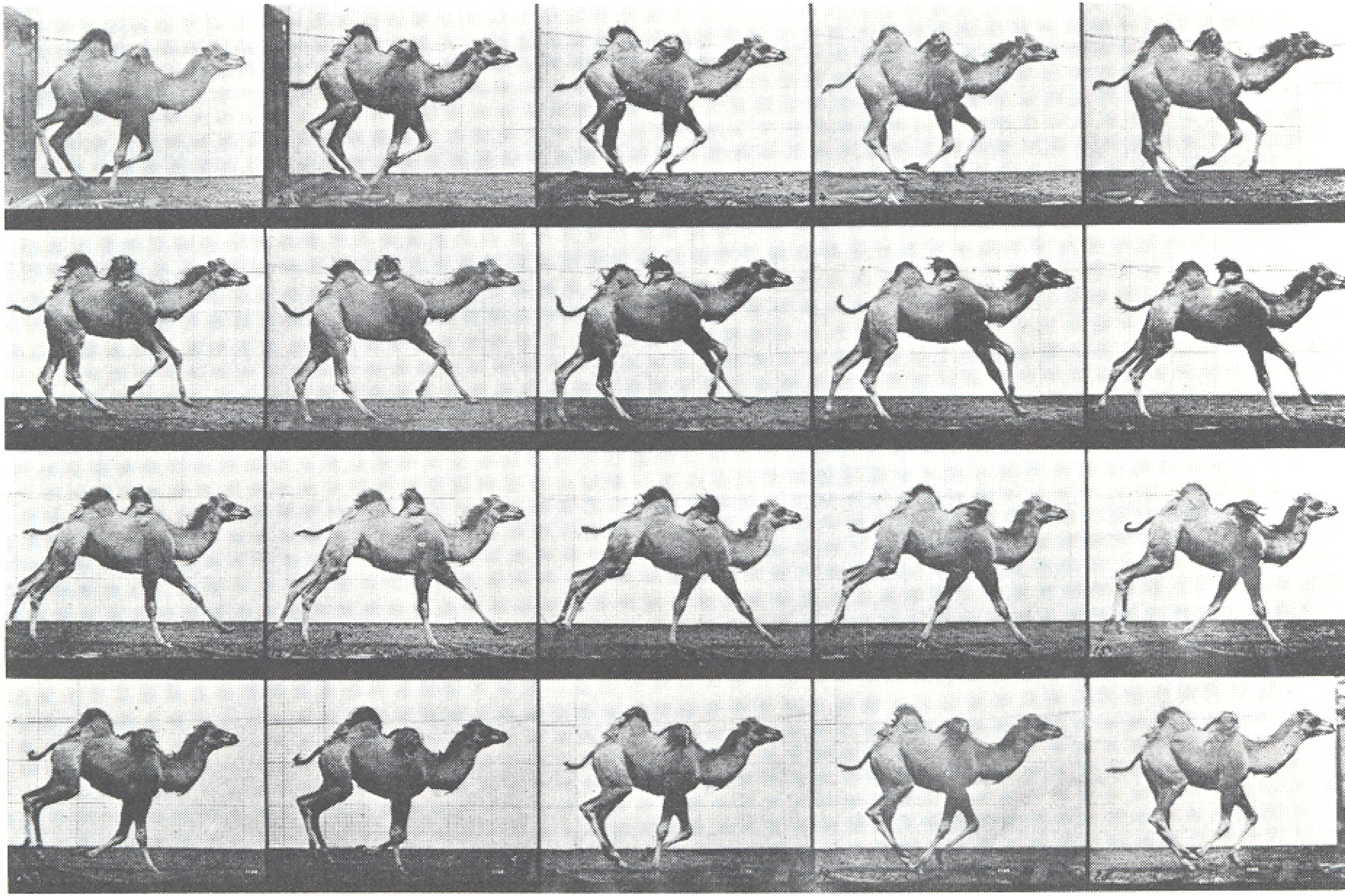


Rationale for variance

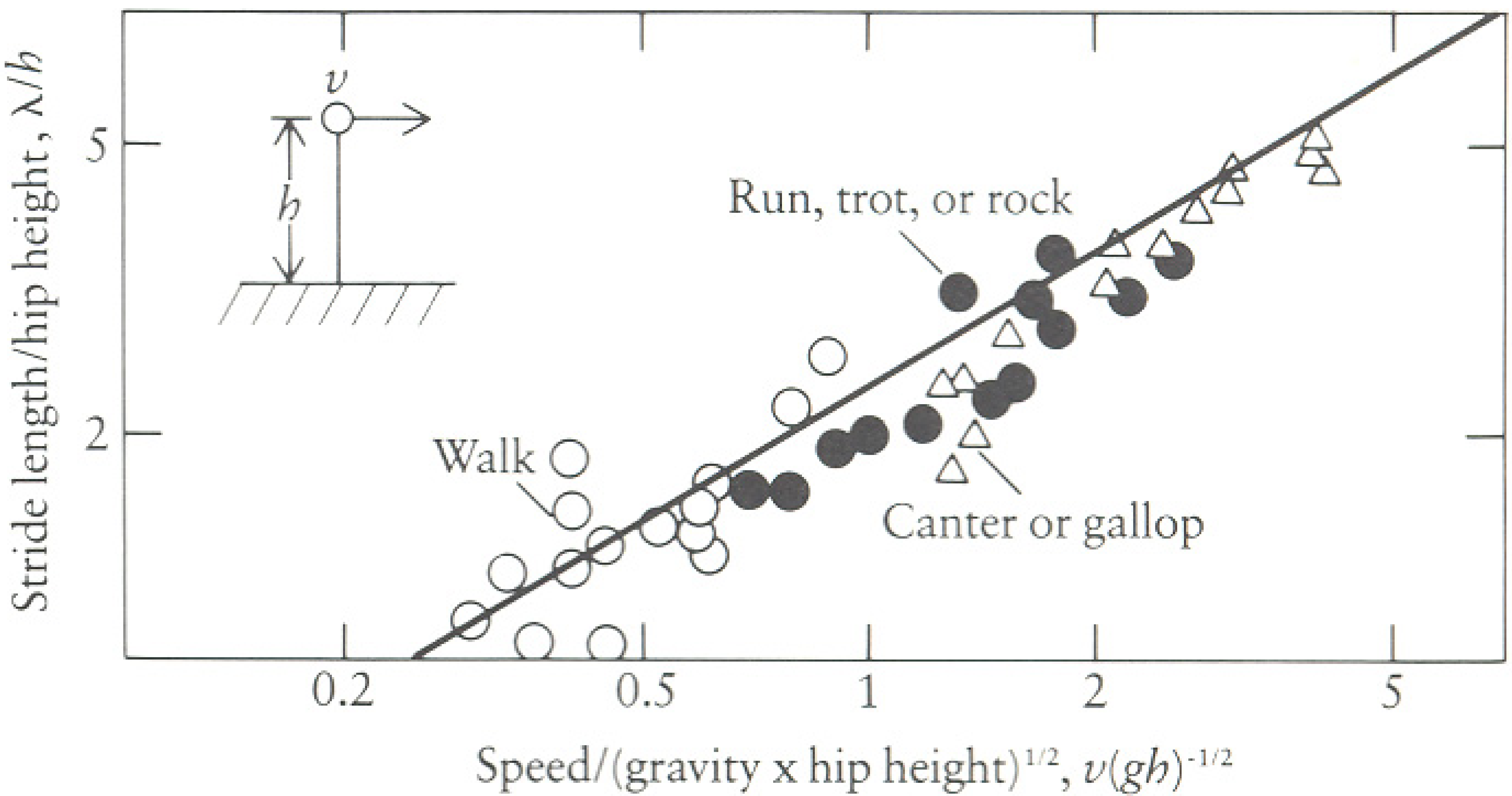
1) air resistance more important for smaller creatures

2) variance from isometry; *e.g*. some creatures have special adaptations

Gait changes in terrestrial locomotion

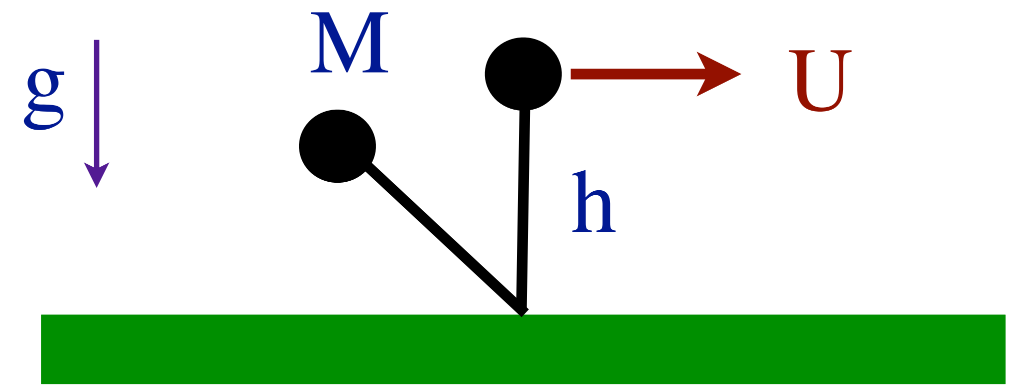


Observation: gaits change when $Fr = \dfrac{U^2}{gh} \sim 1$ for limbed terrestrials.



Rationale: the inverted pendulum

Model runner as mass M with leg length h.



At apex of stride, inertia and gravity in opposition.

$$\frac{INERTIA}{GRAVITY} = \frac{MU^2/h}{Mg} = \frac{U^2}{gh} = Fr$$

Expect lift-off, and associated gait changes, for $Fr > 1$

Reynolds numbers of swimmers and flyers

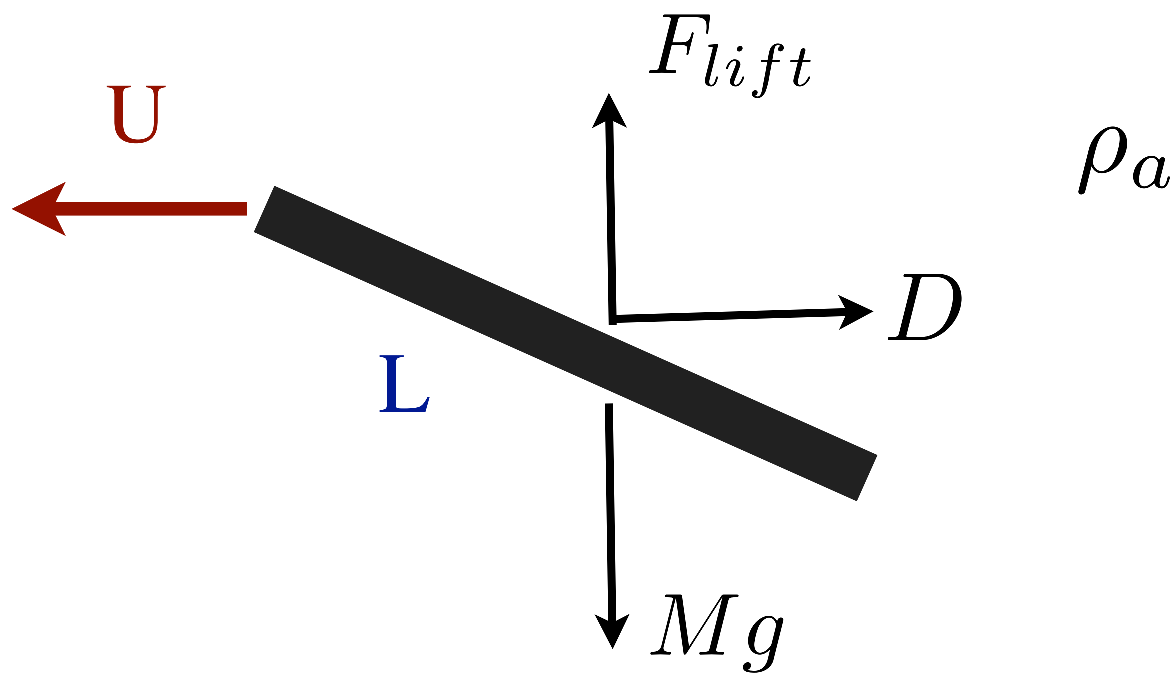
$$\frac{\text{Pressure drag}}{\text{Viscous drag}} = \frac{Ua}{\nu} \equiv \text{Re}$$

Whale	300,000,000
Tuna	30,000,000
Duck in flight	300,000
Dragonfly	30,000
Copepod	300
Smallest insect in flight	30
Swimming larvae	0.3
Sea urchin sperm	0.03
Bacteria	0.00001

The scaling of flight

How does one expect flight speed to depend on size?

Size dependence on gliding speed

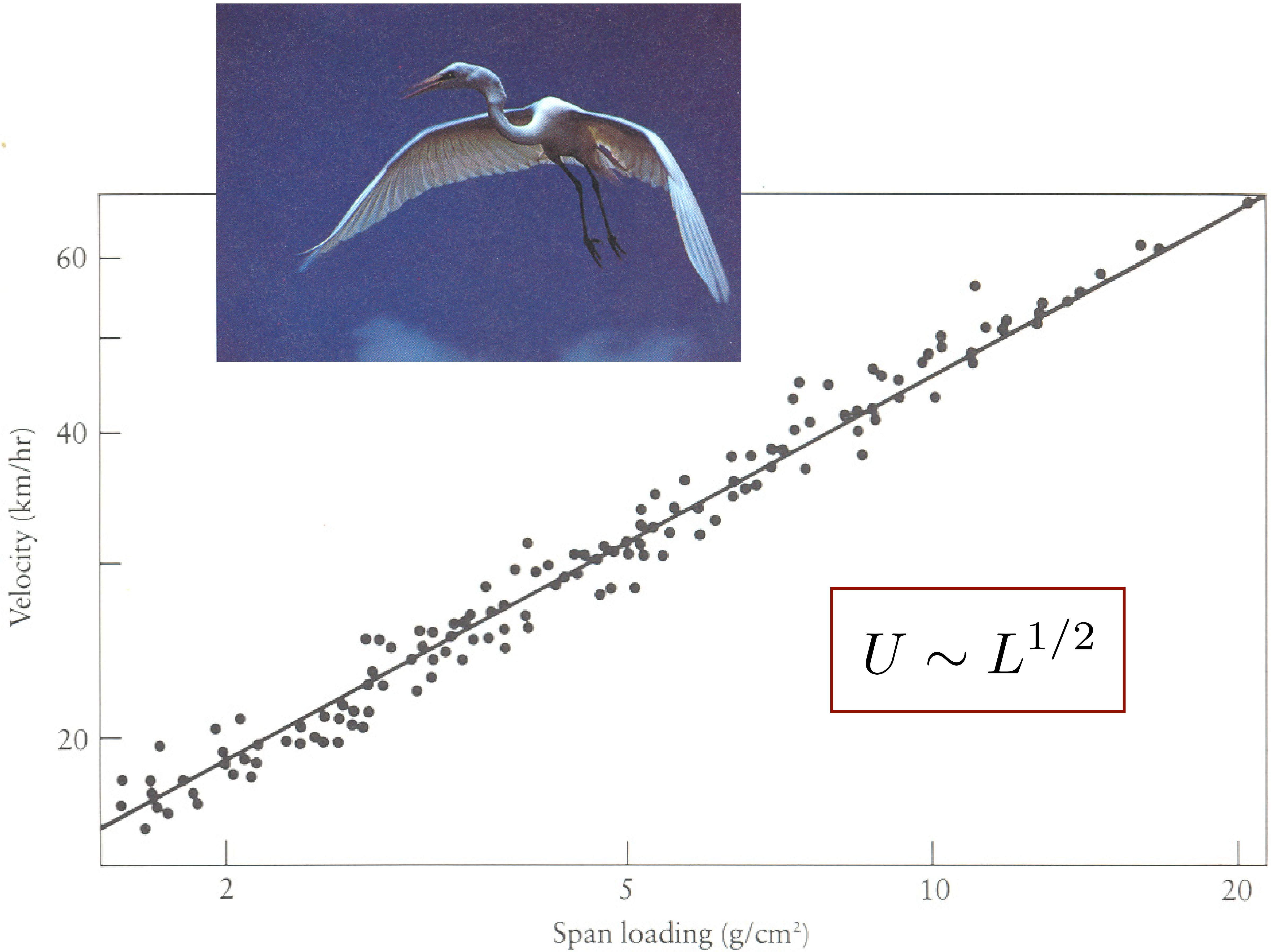


Drag and lift both proportional to $\rho\, U^2 L^2$

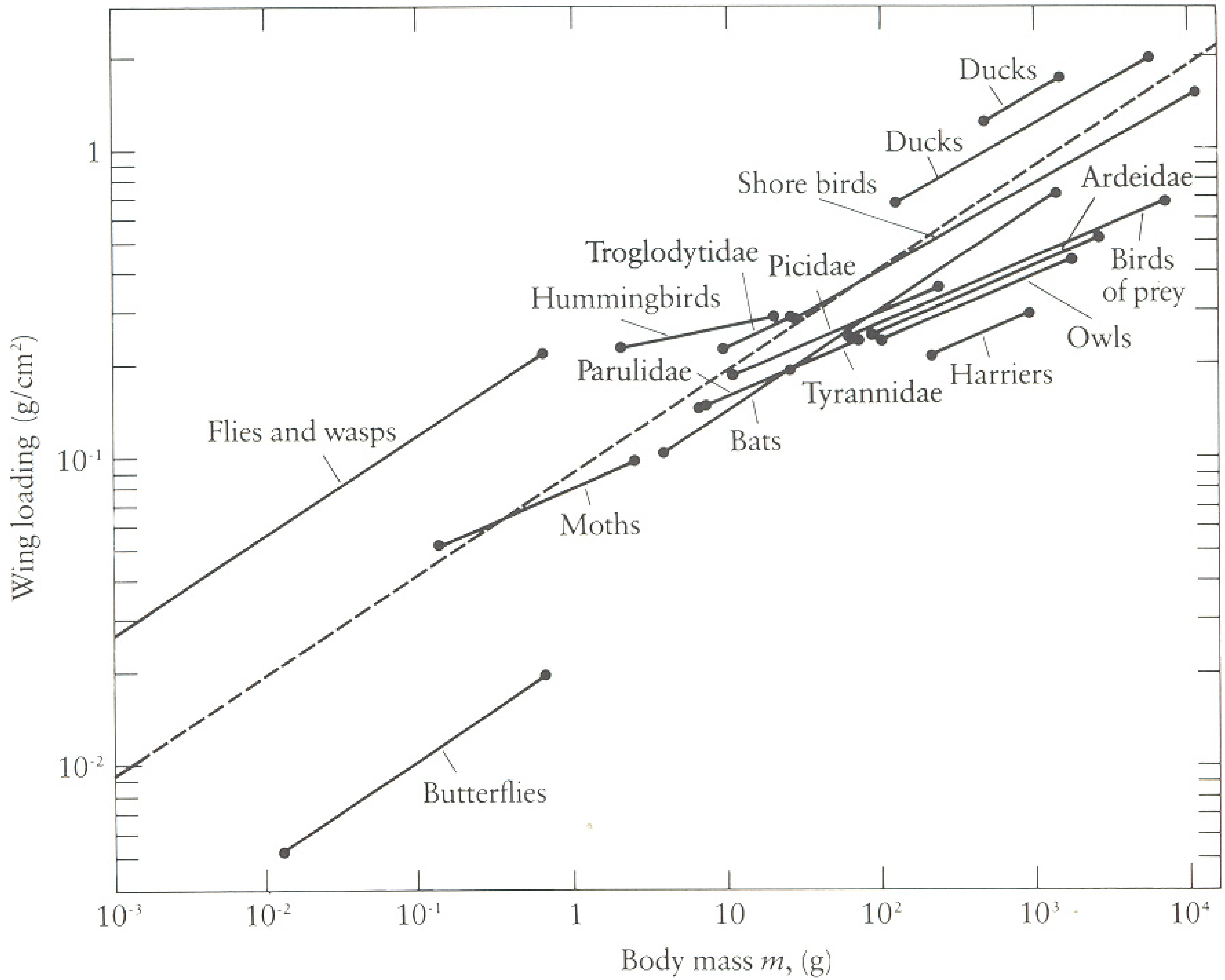
Vertical force balance requires: $Mg \sim \rho\, U^2 L^2$

Since $M \sim L^3$, we have the scaling $U \sim L^{1/2}$

or alternatively $U \sim L^{1/2} \sim \sqrt{\frac{M}{L^2}} \sim (span\ loading)^{1/2}$

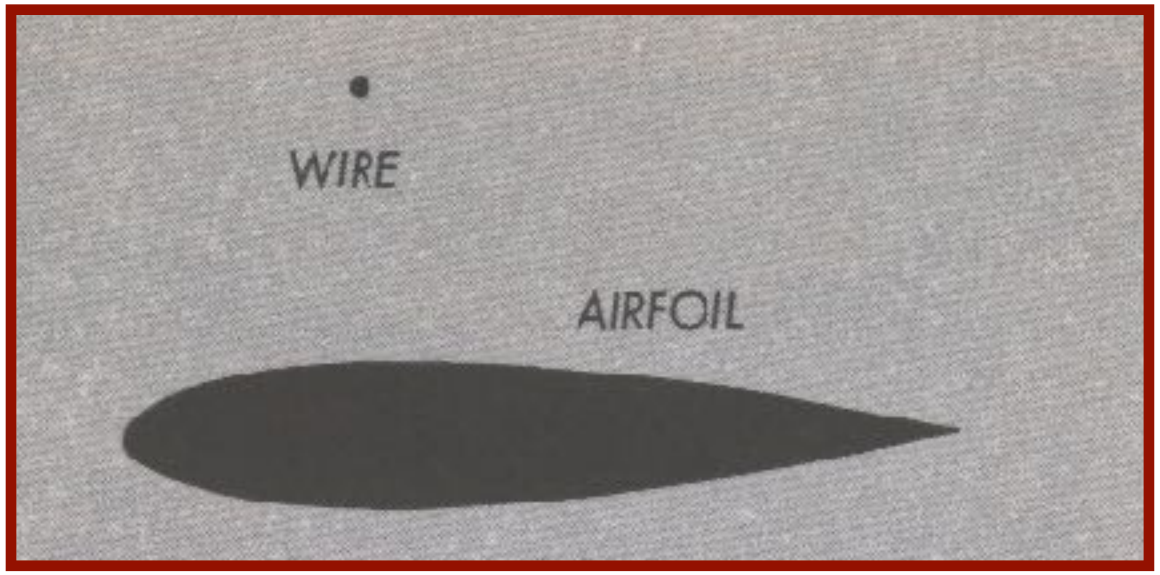


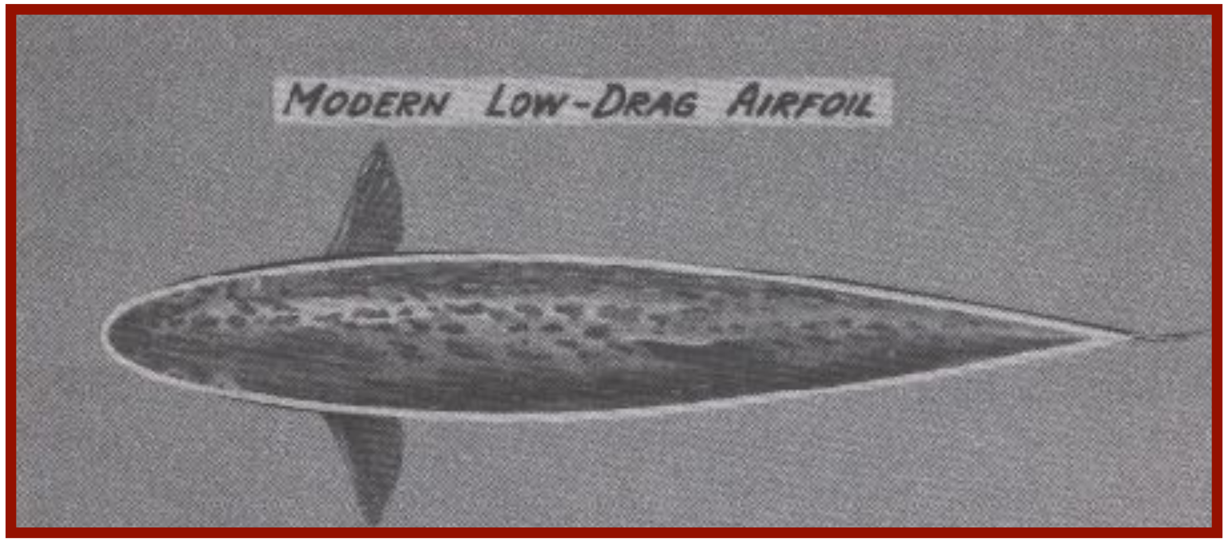
$$U \sim L^{1/2}$$



Ducks
Ducks
Shore birds
Ardeidae
Troglodytidae
Picidae
Hummingbirds
Birds
of prey
Owls
Parulidae
Harriers
Tyrannidae
Bats
Flies and wasps
Moths
Butterflies
Wing loading (g/cm²)
1
10⁻¹
10⁻²
10⁻³
10⁻²
10⁻¹
1
10
10²
10³
10⁴
Body mass m, (g)

How does the speed of a fish depend on its size?





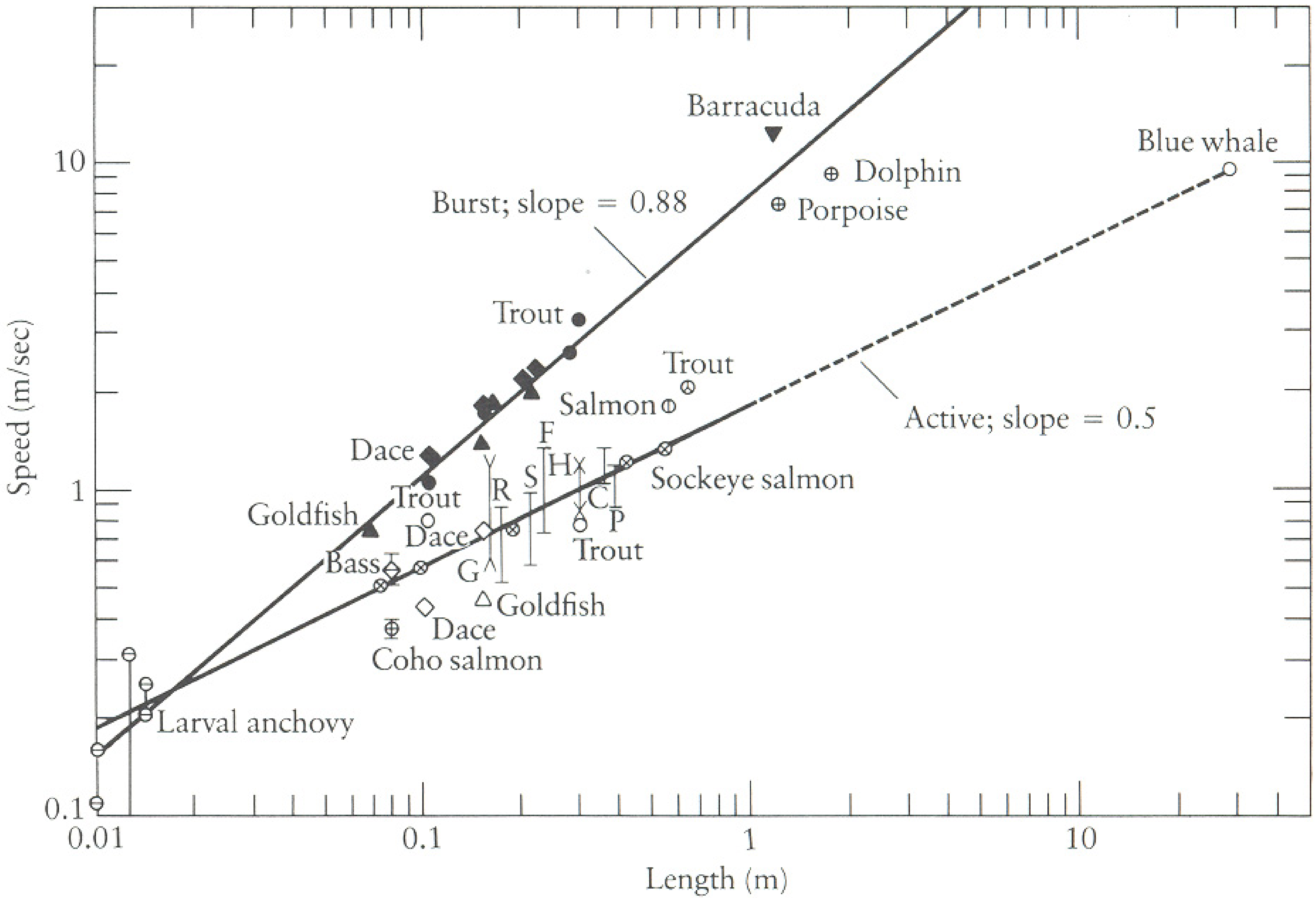
Propulsive force: $F \sim L^2$

Hydrodynamic resistance: skin friction $$D \sim \mu \frac{U}{\delta} L^2$$

where $\delta \sim L \; Re^{-1/2}$ is the boundary layer thickness

$Re = \dfrac{UL}{\nu}$ is the Reynolds number

Force balance (F=D) indicates $$U \sim L^{1/3}$$



Discrepancy: variance from isometry?

reducing drag through body flexion?

How does the speed of a rowboat depend on the number of rowers?

Hydrodynamic drag

$$D = \rho\, U^2 L^2\ F(Re, shape)$$

where $F(Re, shape) \rightarrow const$ as $Re \rightarrow \infty$

assuming isometric boats

Power required at speed U

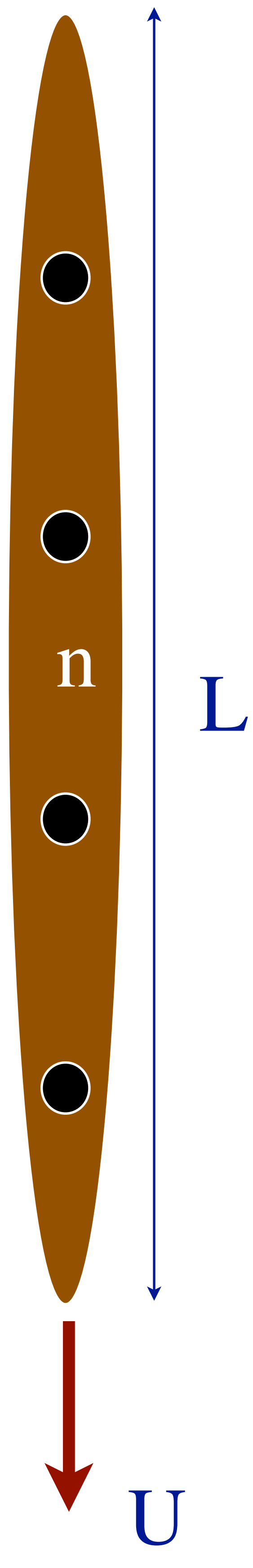
$$P \sim D\ U \sim \rho\, U^3 L^2\ \sim\ n$$

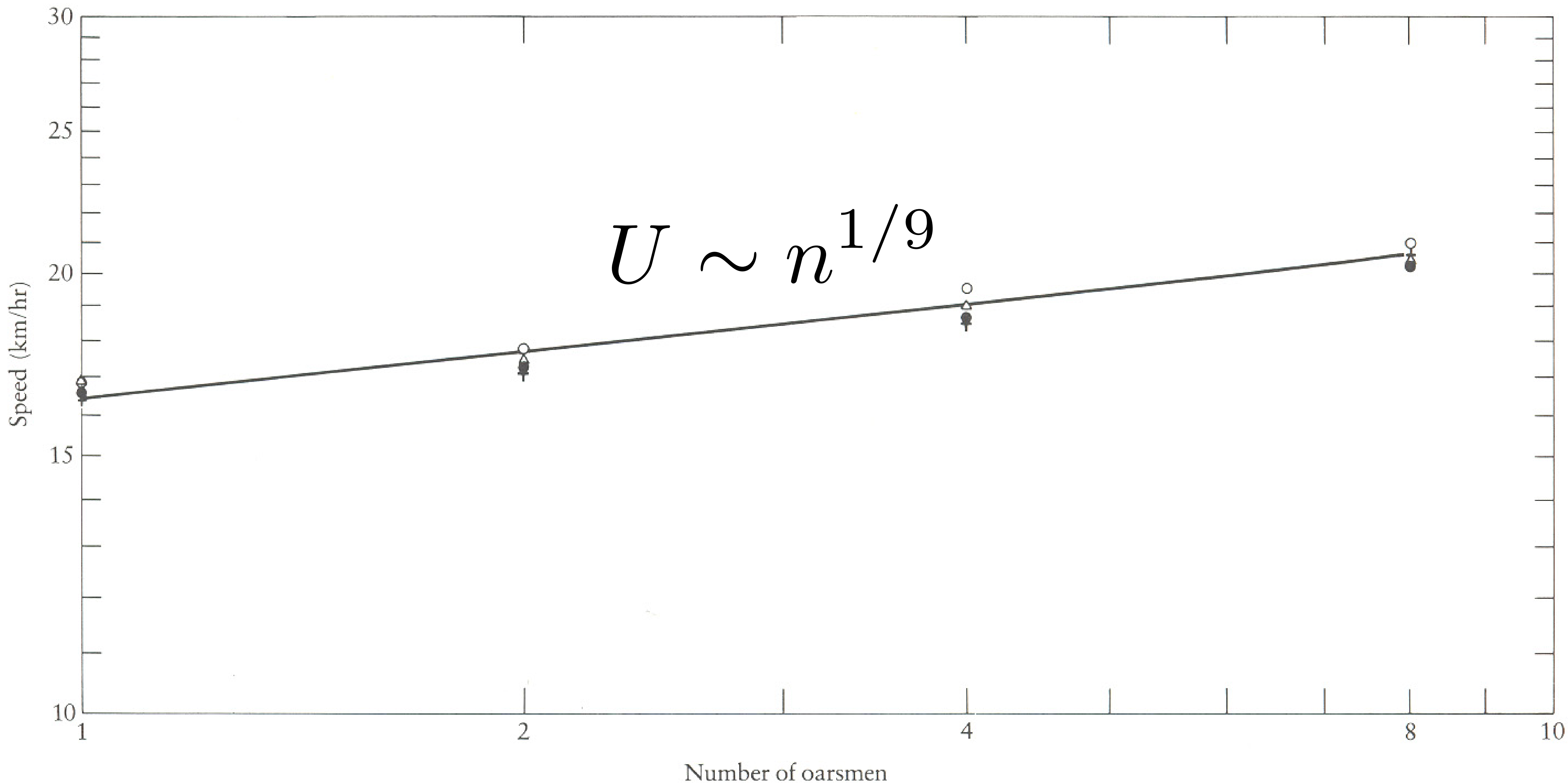
Archimedes: $L^3 \sim n$

weight of water displaced = total weight supported

Inferred scaling:

$$\boxed{U \sim\ n^{1/9}}$$





30
25
20
15
10
Speed (km/hr)
$U \sim n^{1/9}$
1
2
4
8
10
Number of oarsmen

And what if n > 8?

The wonderful world of fluid mechanics

Fluid flows governed by Navier Stokes Equations

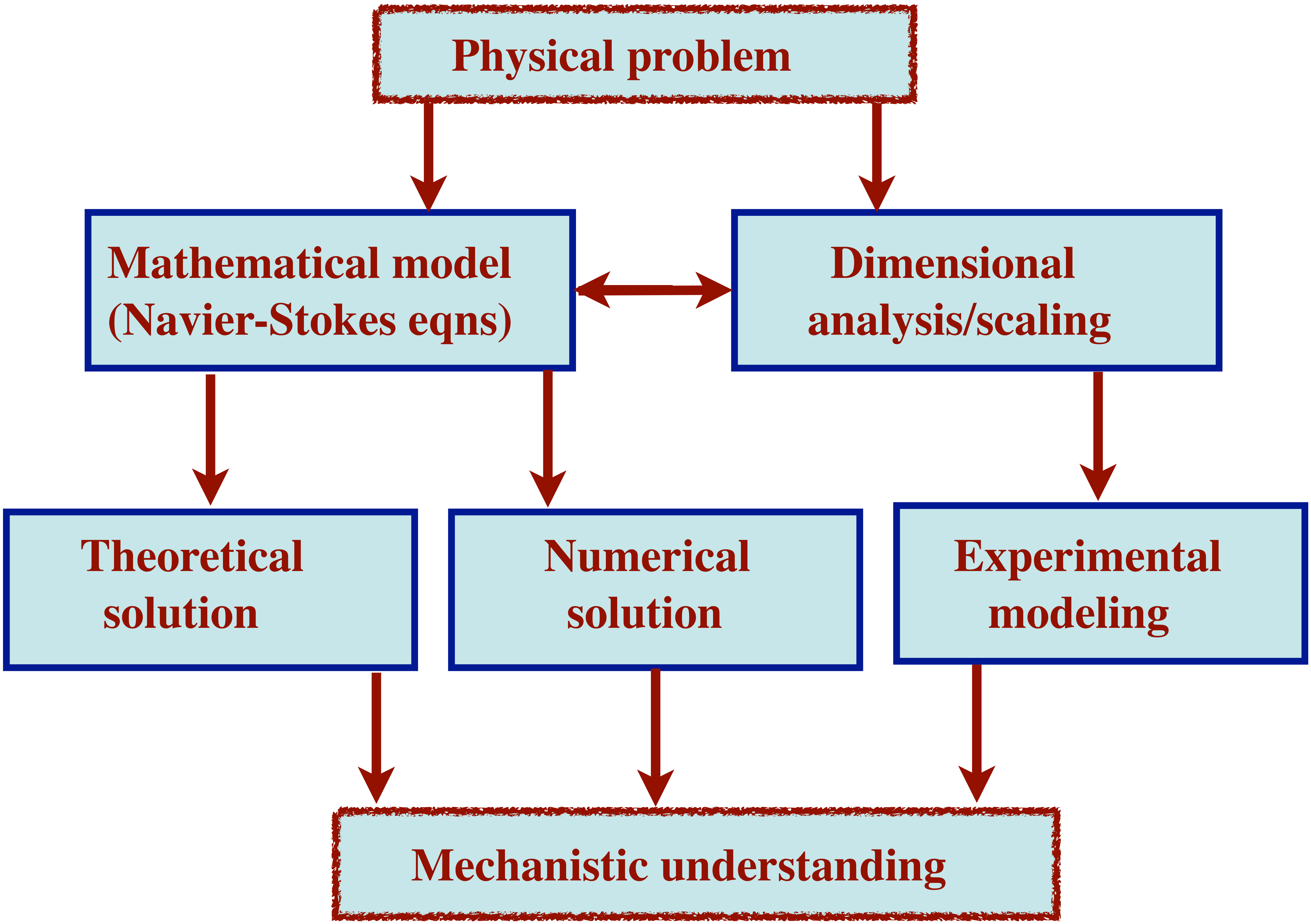
$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{u} \quad , \qquad \nabla\cdot\mathbf{u} = 0$$

MOMENTUM **CONTINUITY**

- for an incompressible fluid with density ρ and viscosity $\mu = \rho\nu$ in the presence of gravity $\mathbf{g}$, this represents a systems of 4 equations in 4 unknowns $(\mathbf{u}, p)$
- a continuum statement of Newton's laws, these equation prescribe the evolution of the fluid velocity $\mathbf{u}$ and pressure p
- an infinitesimal fluid blob may accelerate in response to pressure gradients, gravity and viscous stresses
- these must be solved subject to appropriate boundary conditions: no slip conditions on rigid boundaries, continuity of velocity and stress boundary conditions at fluid interfaces
- describe an enormous class of physical problems spanning a vast range of scales

Celestial Dynamics	Planetary flows	Solar convection	Interstellar medium	Galactic dynamics	Intergalactic jets	FD of the Universe
Geophysical Fluids	Atmospheric dynamics	Oceanography	Mantle convection, plate tectonics	Dynamo theory	Magnetospheric physics	Geological fluids
Environmental fluid mechanics	Groundwater flows	Waste management	Disease transmission	Natural ventilation	Natural hazards	Climate change
Aerodynamics	Sports balls	Ballistics	Drag reduction in sport	Aeronautics: planes and jets	Astronautics: shuttles and rockets	Skydiving, parachuting
Engineering fluid dynamics	Microfluidics	Oil recovery	Wind and water loading on structures	Materials processing	Centrifugation	Engine design, turbo-machinery
Biological fluid dynamics	Swimming, flying	Bioconvection	Blood flow	Evolutionary biology	Molecular machines	Cell division
Household fluid dynamics	Soap films, bubbles, foams	Convection (boiling)	NonNewtonian/ multiphase flows	Culinary arts	Smoke rings	Alcoholic beverages

Problem solving in fluid mechanics



Nondimensionalization

- the dimensionless groups found by Buckingham's Theorem represent the relative magnitudes of terms in NS equations

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{u} \quad , \qquad \nabla\cdot\mathbf{u} = 0$$

Select characteristic scales: $\mathbf{u}' \sim \mathbf{u}/U$, $\mathbf{x}' \sim \mathbf{x}/L$, $t' \sim t/(L/U)$

Case 1: inertial effects >> viscous effects, choose $p' \sim p/(\rho U^2)$

$$\frac{D\mathbf{u}}{Dt} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u} \longrightarrow \frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p \ , \ \nabla\cdot\mathbf{u} = 0$$

- reduces to Euler's equations in the limit of $Re = \dfrac{Ua}{\nu} \gg 1$

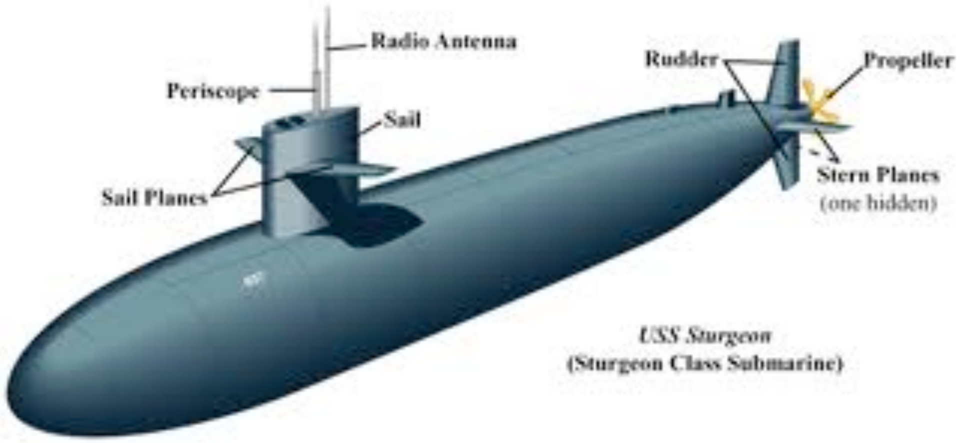
Case 2: inertial effects << viscous effects, choose $p' \sim p/(\mu U/a)$

$$Re\,\frac{D\mathbf{u}}{Dt} = -\nabla p + \nabla^2\mathbf{u} \longrightarrow 0 = -\nabla p + \nu\nabla^2\mathbf{u} \ , \ \nabla\cdot\mathbf{u} = 0$$

- reduces to Stokes equations in the limit of $Re = \dfrac{Ua}{\nu} \ll 1$

Dynamic similarity

- arises between two fluid systems when those systems are characterized by identical dimensionless parameters
- allows one to model (impractically large or small) physical systems in a laboratory setting



Eg. 1 Submarine building

Dimensional analysis yields $$\frac{D}{\rho U^2 a^2} = F(Re, shape)$$

- $F(Re, shape)$ may be deduced from experimental modeling with miniature subs (and high U to match Re)
- results can be scaled up to deduce drag D and power required DU for a submarine to cruise submerged at a given speed

Eg. 2 Wind-tunnel testing

- one wants to know wind-induced pressure distribution around a building (*e.g*. the John Hancock tower)

$$\frac{P}{\rho U^2} = F(Re)$$ deduced from lab experiments

- must match Re for dynamic similarity: miniature models require the use of high laboratory wind speeds U

Eg. 3 Low Reynolds number swimming in water

- bacteria swim with $Re = Ua/\nu \sim 10^{-5}$
  since $a \sim 1\mu m$

- dynamic similarity can be achieved in a laboratory-scale setting (a ~ 5cm) by increasing fluid viscosity

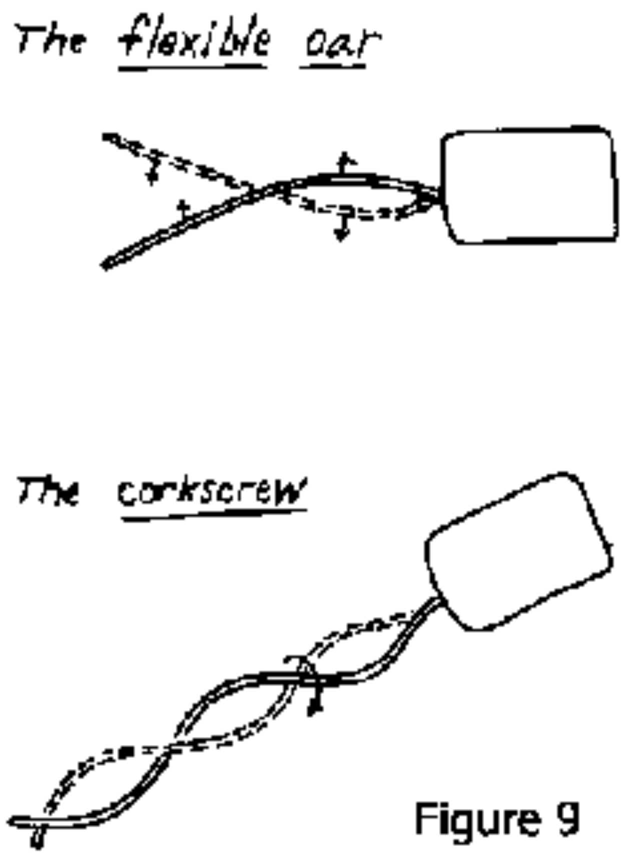


Figure 9

Eg. 4 Rotating flows

Consider a frame of reference rotating with constant angular velocity $\boldsymbol{\Omega}$

The velocity in the rotating frame, $\mathbf{v} = \mathbf{u} - \boldsymbol{\Omega} \wedge \mathbf{r}$, satisfies

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p_d \; + \; \nu\nabla^2\mathbf{v} - 2\,\boldsymbol{\Omega} \wedge \mathbf{v}$$

where the dynamic pressure $p_d = p - \rho\mathbf{g}\cdot\mathbf{r} - \frac{1}{2}\rho\Omega^2 r^2$

Nondimensionalize:

$$\boldsymbol{\Omega} \sim \Omega\hat{\mathbf{k}} \;,\; \mathbf{r} \sim a\mathbf{r}' \;,\; \mathbf{v} \sim U\mathbf{v}' \;,\; \nabla \sim \frac{1}{a}\nabla' \;,\; p \sim \rho\Omega U a\; p'$$

Governing equations:

$$\boxed{R_o\frac{D\mathbf{v}}{Dt} + 2\hat{\mathbf{k}} \wedge \mathbf{v} = -\nabla p + E_k\nabla^2\mathbf{v}}$$

where Rossby number $R_o = \frac{U}{\Omega a} = \frac{Inertia}{Coriolis}$, Ekman number $E_k = \frac{\nu}{\Omega a^2} = \frac{Viscous}{Coriolis}$

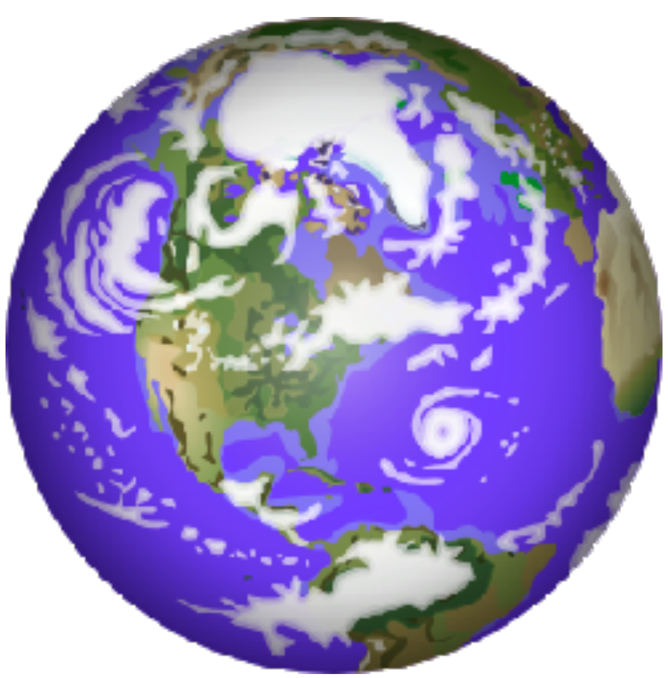
Rapid rotation limit $(R_o, E_k \ll 1)$: Taylor-Proudman constraint $\boldsymbol{\Omega}\cdot\nabla\mathbf{u} = 0$

Eg. 4 Geophysical flows

- motion of atmosphere and oceans influenced by earth's rotation Ω
- dimensionless groups are the Rossby and Ekman numbers:

$$R_o = \frac{U}{\Omega a} \quad , \quad E_k = \frac{\nu}{\Omega a^2}$$

- dynamic similarity in lab experiments (a ~1m) requires high $\Omega \sim 1/s$



Means of comparison and degrees of similitude

I. Metaphor

II. Physical analogy

III. Dynamic similarity

IV. Mathematical equivalence

- arises when two systems have precisely the same mathematical description
- one system may be understood directly in terms of the other
- arises when dynamic similarity is achieved in fluid systems, but also arises more broadly…

Gravitoelectromagnetism

- in limit of weak spacetime curvature (weak gravitational fields)

GEM equations	Maxwell's equations
$\nabla \cdot \mathbf{E}_g = -4\pi G \rho_g$	$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$
$\nabla \cdot \mathbf{B}_g = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{B}_g = -\frac{4\pi G}{c^2}\mathbf{J}_g + \frac{1}{c^2}\frac{\partial \mathbf{E}_g}{\partial t}$	$\nabla \times \mathbf{B} = \frac{1}{\epsilon_0 c^2}\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$

Lorentz force

GEM equation	EM equation
$\mathbf{F_g} = m\left(\mathbf{E}_g + \mathbf{v} \times 4\mathbf{B}_g\right)$	$\mathbf{F_e} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$

Fluid Mechanics: Governing Equations

Navier-Stokes equations:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{u} \quad , \qquad \nabla\cdot\mathbf{u} = 0$$

Vorticity: $\omega = \nabla \wedge \mathbf{u} = 0$ is a measure of the flow's swirl

Taking curl of momentum equation eliminates p and gravity, yielding

$$\frac{D\omega}{Dt} = \omega\cdot\nabla\mathbf{u} + \nu\nabla^2\omega$$

advection *vortex stretching* *diffusion*

Inviscid, irrotational flows | **Electrostatics**

$$\omega = \nabla \wedge \mathbf{u} = 0$$

$$\nabla^2 \Phi = 0$$

$$\mathbf{u} = \nabla \Phi$$

$$\mathbf{E} = \nabla \Phi$$

- **source-sink flows**
- **fields from point charges**

- **flows near walls identical to electric fields near insulating boundaries**

$$\mathbf{n} \cdot \nabla \Phi = 0$$

Mathematical methods directly transferable

- **method of images, superposition, Gauss's Law, complex potentials**

Static heat equation

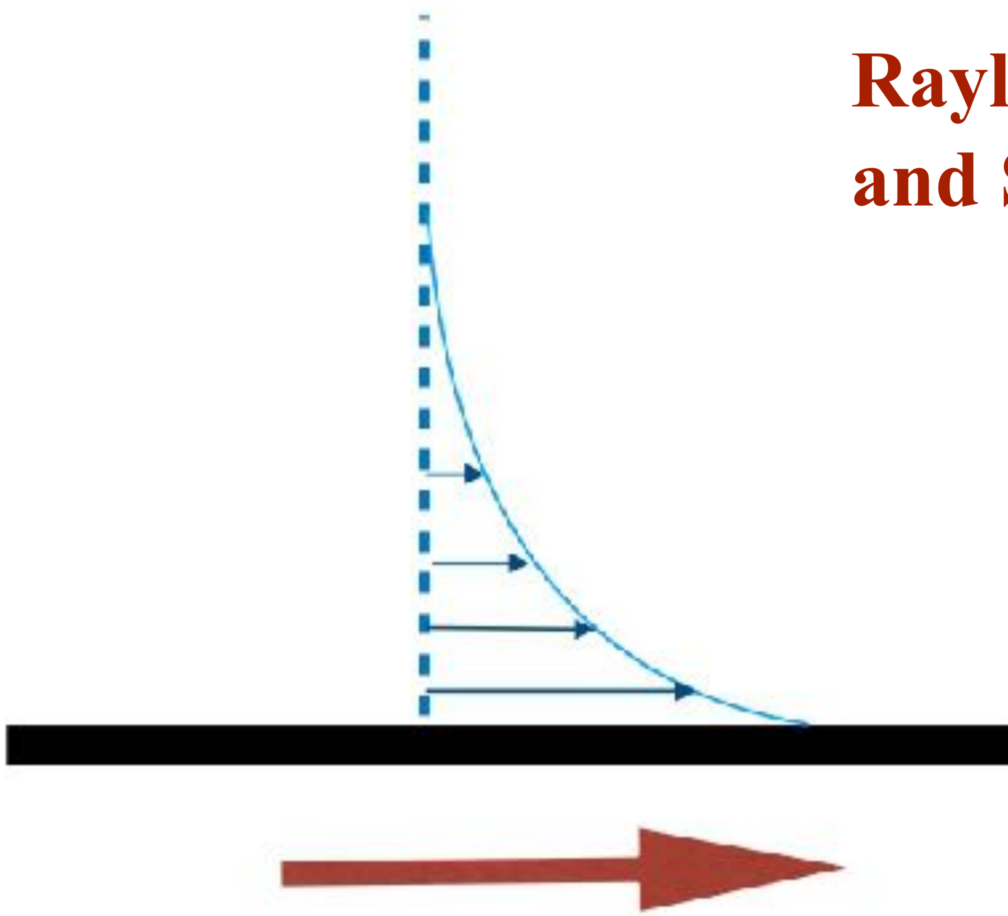
$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T$$

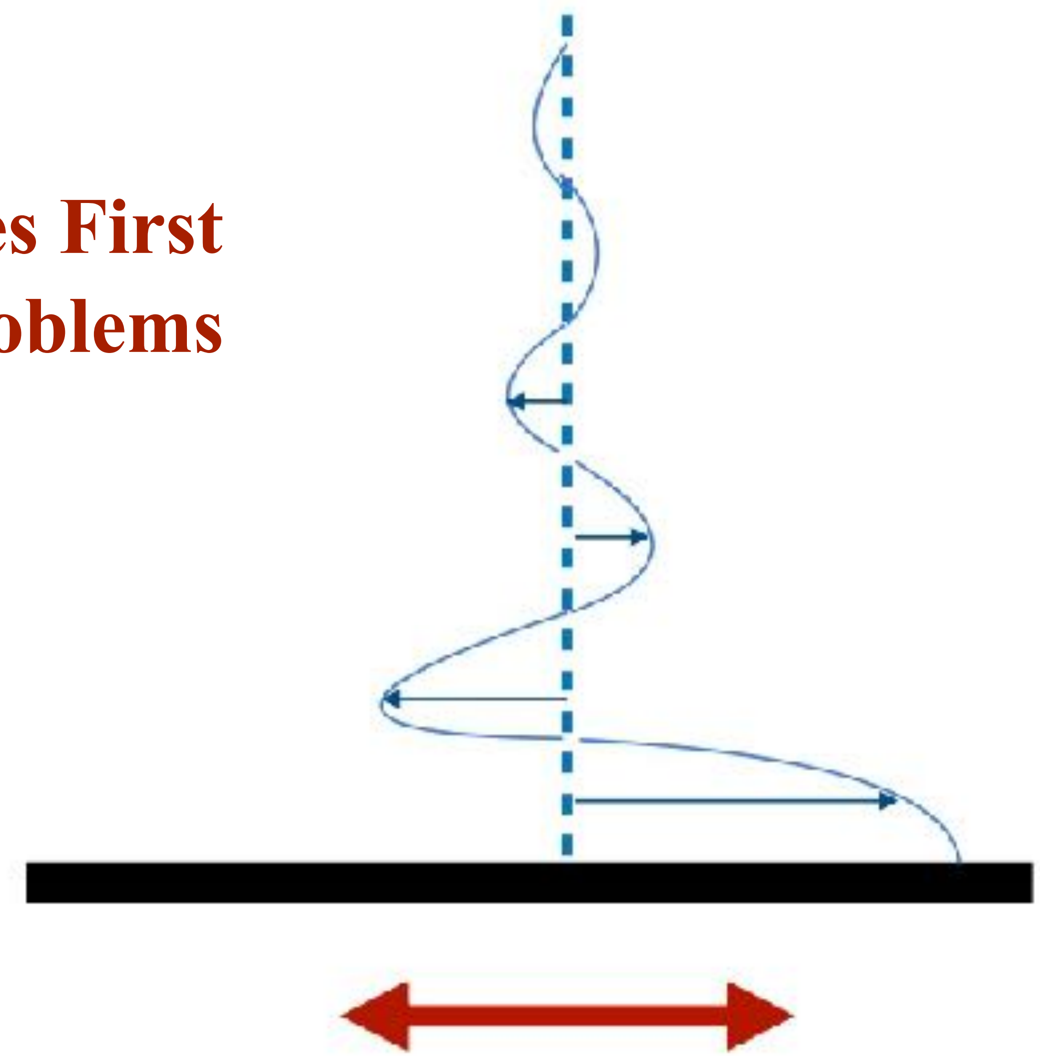
Vorticity equation for unidirectional flows

$$\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega$$

- **vorticity evolves as a passive scalar**

Rayleigh-Stokes First and Second Problems





Vortex dynamics

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = (\omega + \mathbf{\Omega}) \cdot \nabla \mathbf{u} + \nu \nabla^2 \omega$$

Rapidly rotating flows

$$\mathbf{\Omega} \cdot \nabla \mathbf{u} = 0$$

Taylor-Proudman Thm

- flow constrained to be 2D

Magnetic induction equation

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = (\mathbf{B} + \mathbf{B_0}) \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{B}$$

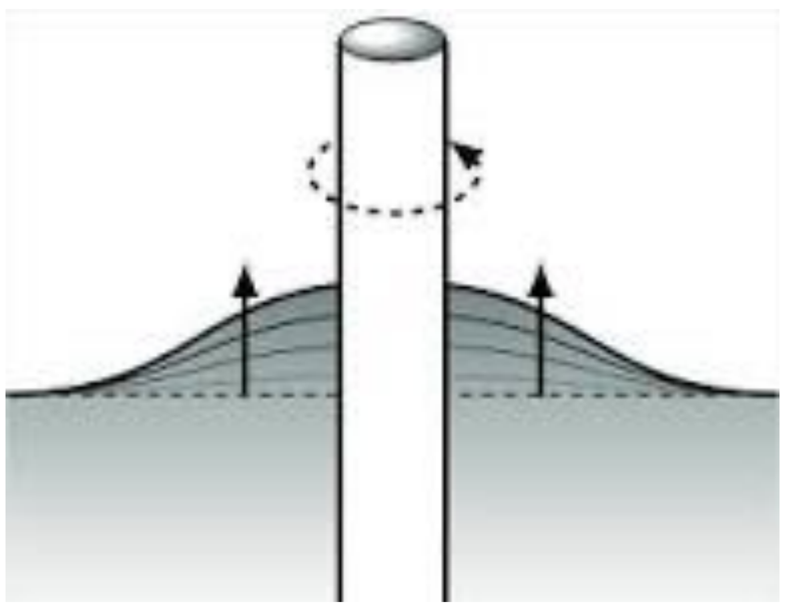
Strong background field

$$\mathbf{B_0} \cdot \nabla \mathbf{u} = 0$$

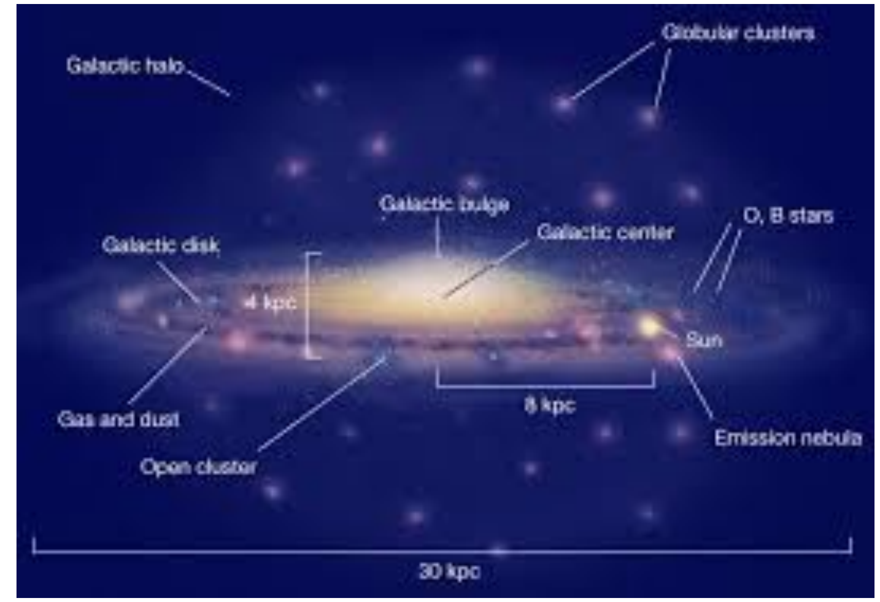
Taylor column

Magnetic column

The Weissenberg effect



The Galactic bulge



- exploit the analogy between elastic fluids and electrically conducting fluids
- might the Galactic bulge be an analog of the Weissenberg effect?