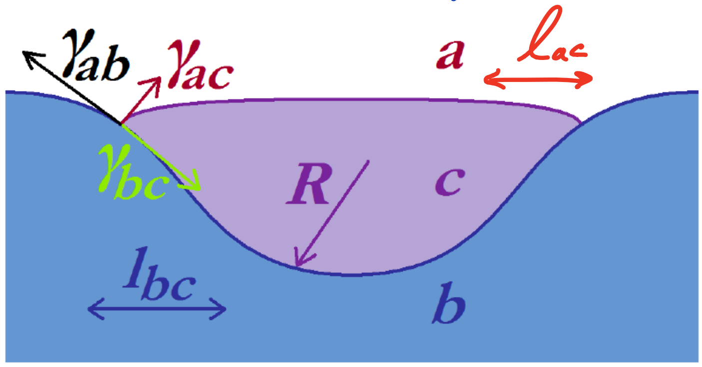
Lecture 19: Oil on water; Surface waves

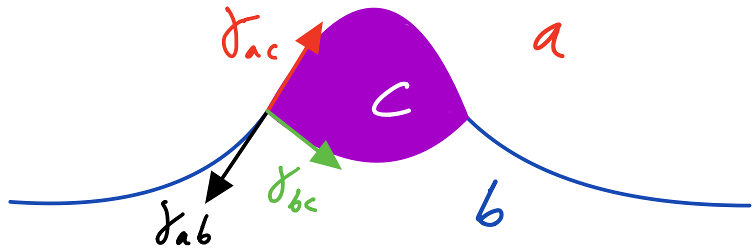
Immiscible Drops at an Interface

Pendant Lenses

$\rho_a < \rho_b < \rho_c$



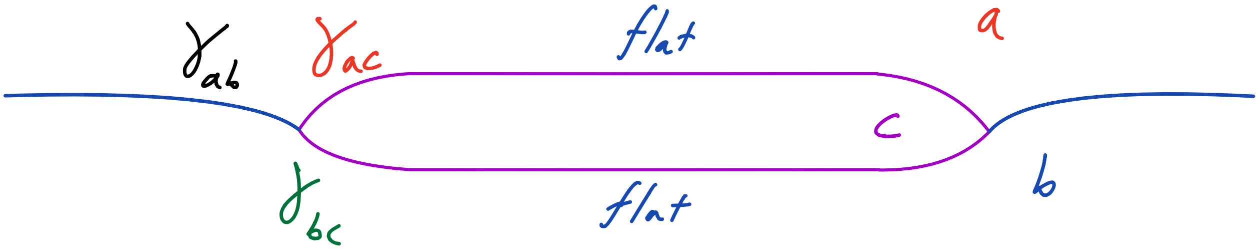
$\rho_c < \rho_a < \rho_b$



- stable only for drops small w.r.t. capillary length

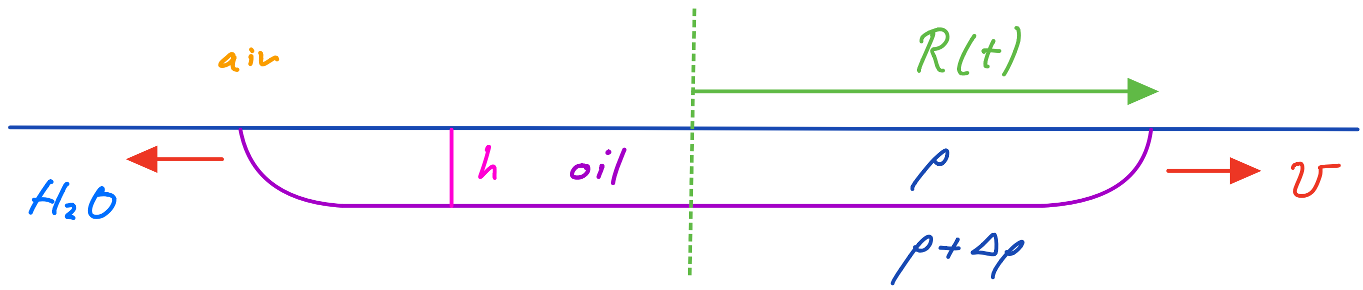
$$l_{ac} \sim \sqrt{\frac{\gamma_{ac}}{(\rho_a-\rho_c)g}} \quad , \quad l_{bc} \sim \sqrt{\frac{\gamma_{bc}}{(\rho_b-\rho_c)g}}$$

Sessile Lens: $\rho_a < \rho_c < \rho_b$ e.g. oil on water



Oil Spill: 4 Distinct Phases

Phase I : Inertia vs Gravity



As previously, $v \sim (g' h)^{\frac{1}{2}}$

$$R(t) \sim (g' V_0)^{\frac{1}{4}} \, t^{\frac{1}{2}} \qquad \text{where } g' = g \frac{\Delta\rho}{\rho}$$

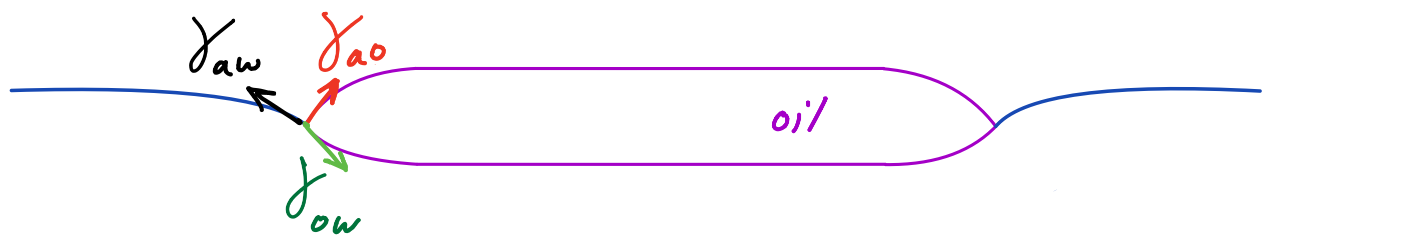
Phase II : Gravity vs Viscosity

- as previously (similar scaling though less Φ)

$$\Rightarrow R \sim \left(\frac{\rho g' V_0^3}{\nu} \right)^{\frac{1}{8}} t^{\frac{1}{8}}$$

Phase III : Line tension vs. Viscosity

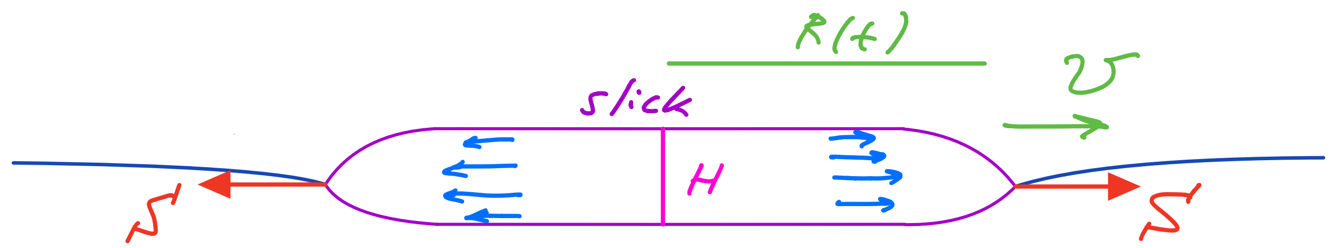
- behaviour depends on magnitude of spreading parameter $S = \gamma_{aw} - \gamma_{oa} - \gamma_{ow}$



For $S<0$: an equilibrium configuration arises; the drop assumes the form of a sessile lens

For $S>0$: the oil will completely cover the water, spreading to a layer of molecular thickness

Stage III A : viscous dissipation within spreading slick is dominant



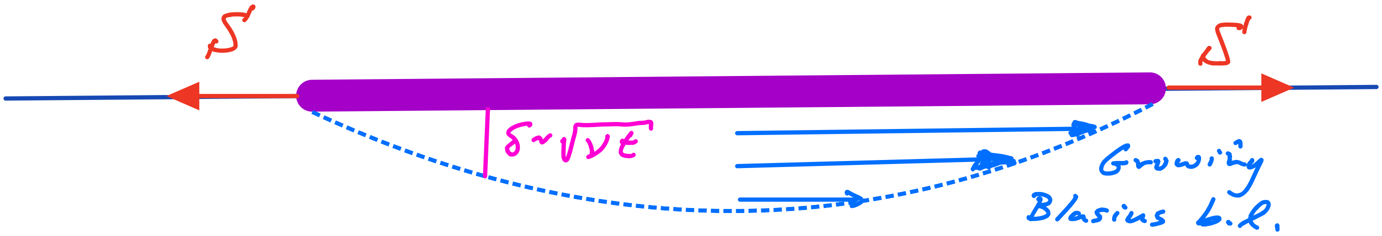
Scaling : $\mu \frac{U}{H} \pi R^2 \sim S \cdot 2\pi R$

$$\Rightarrow UR \sim \frac{S}{\mu} H \sim \frac{S}{\mu} \frac{V}{R^2} \quad \text{where } H \sim \frac{V}{R^2}$$

$$\Rightarrow R^3 \frac{dR}{dt} \sim \frac{SV}{\mu}$$

$$\Rightarrow \boxed{R \sim \left(\frac{SV}{\mu}\right)^{\frac{1}{4}} t^{\frac{1}{4}}} \quad \text{as previously}$$

Stage III B : viscous dissipation in underlying water dominant



Scaling :

$$\mu \frac{v}{\delta} \cdot \pi R^2 \sim \mathcal{S} \cdot 2\pi R \quad \text{where } \delta \sim \sqrt{\nu t}$$

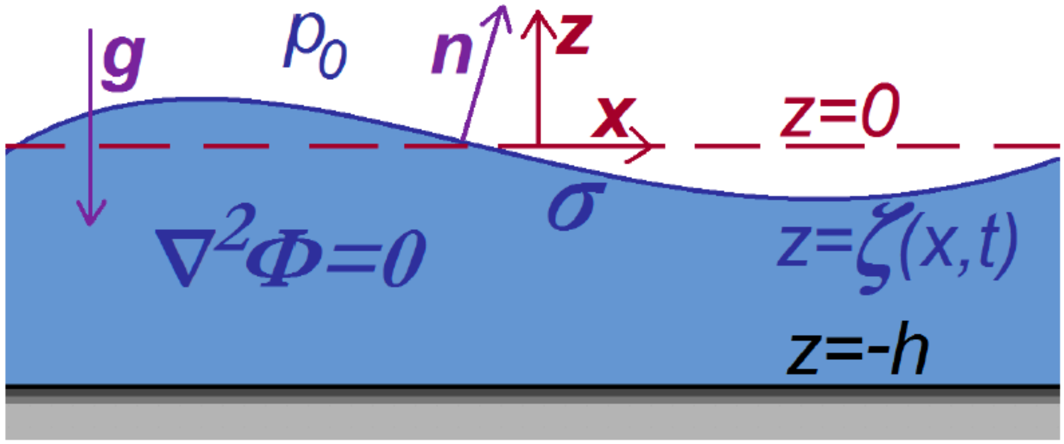
$$\Rightarrow R \frac{dR}{dt} \sim \frac{\mathcal{S}}{\mu} \sqrt{\nu}\, t^{\frac{1}{2}} \quad \text{where } \mu = \rho \nu$$

$$\Rightarrow \boxed{R \sim \left(\frac{\mathcal{S}}{\mu}\right)^{\frac{1}{2}} \nu^{\frac{1}{4}} t^{3/4}}$$

Reference : James Fay, Oil on the Sea

- Jensen, JFM (1995)

Water Waves



Assume $Re \gg 1$, so that ...

- motion of fluid may be described to leading order as inviscid and irrotational.
- must deduce a sol'n for the velocity potential ϕ satisfying (where $\underline{u} = \underline{\nabla}\phi$):

$$\boxed{\nabla^2 \phi = 0}$$ subject to kinematic + dynamic boundary conditions

<u>B.C.s</u> 1. $\dfrac{\partial \phi}{\partial z} = 0$ on $z = -h$

2. Kinematic BC: $\dfrac{D\zeta}{Dt} = u_z$ on $z = \zeta$

$$\Rightarrow \frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} = \frac{\partial \phi}{\partial z} \quad \text{on } z = \zeta$$

3. Dynamic B.C. (Time-dep Bernoulli applied at free surface)

$$\rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho |\underline{\nabla}\phi|^2 + \rho g \zeta + P_s = \underbrace{f(t)}_{\text{indep of } x}$$

where $P_s = P_0 + \sigma \underline{\nabla} \cdot \underline{n}$

Here $\underline{n} = \dfrac{(-\zeta_x, 1)}{(1+\zeta_x^2)^{\frac{1}{2}}}$ is the unit normal

$$\underline{\nabla}\cdot\underline{n} = \frac{-\zeta_{xx}}{(1+\zeta_x^2)^{3/2}} \quad \text{is the curvature}$$

$$P_s = p_0 - \sigma\frac{\zeta_{xx}}{(1+\zeta_x^2)^{3/2}}$$

Now consider small-amplitude waves and linearize the system of eqns and BCs (i.e. assume ϕ, ζ small, so neglect any terms involving ϕ^2, ζ^2, $\phi\zeta$ or their derivatives)

$$\nabla^2\phi = 0 \quad \text{in} \quad -h \le z \le 0$$

B.C.s $\underline{1}$. $\dfrac{\partial\phi}{\partial z} = 0$ on $z=-h$

$\underline{2}$. $\dfrac{\partial\zeta}{\partial t} = \dfrac{\partial\phi}{\partial z}$ on $z=0$

$\underline{3}$. $\rho\dfrac{\partial\phi}{\partial t} + \rho g\zeta + p_0 - \sigma\zeta_{xx} = f(t)$ on $z=0$

Seek normal modes, sol'ns of the form:

$$\left.\begin{aligned}\zeta &= \hat{\zeta}\, e^{ik(x-ct)} \\ \phi &= \hat{\phi}_z(z)\, e^{ik(x-ct)}\end{aligned}\right\}$$

travelling waves in x-direction with phase speed $c = \omega/k$ and wavelength $\lambda = 2\pi/k$.

Subing into harmonic equation:

$$\hat{\phi}_{zz} - k^2\hat{\phi} = 0$$

$\Rightarrow$ solns are $\hat{\phi}(z) = e^{kz}, e^{-kz}$ <u>or</u> $\sinh kz, \cosh kz$

One may satisfy B.C. 1, $\dfrac{\partial\hat{\phi}}{\partial z} = 0$ on $z = -h$ by choosing:

$$\hat{\phi}(z) = A\cosh k(z+h)$$

< a constant

Now, <u>B.C. 2</u> $\Rightarrow$ $-ikc\hat{\zeta} = Ak\sinh kh$ ☆

<u>B.C. 3</u>. $\Rightarrow$ $\left(-ikc\rho A\cosh kh + \rho g\hat{\zeta} + k^2\sigma\hat{\zeta}\right)e^{ik(x-ct)}$

$= f(t)$ indep of x

i.e. $-ikc\rho A\cosh kh + \rho g\hat{\zeta} + k^2\sigma\hat{\zeta} = 0$ ⊠

☆ $\Rightarrow$ $A = \dfrac{-ic\hat{\zeta}}{\sinh kh}$ $\Rightarrow$ sub into ⊠

$$\boxed{c^2 = \left(\frac{g}{k} + \frac{\sigma k}{\rho}\right)\tanh kh}$$

where $c = \frac{\omega}{k}$ is the phase speed

Since $c = \omega/k$, this yields the

<u>Dispersion Relation</u>:

$$\boxed{\omega^2 = \left(gk + \frac{\sigma k^3}{\rho}\right)\tanh kh}$$

Note: as $h \to \infty$, $\tanh h \to 1$ and we obtain the deep-water dispersion relation $\omega^2 = gk + \frac{\sigma k^3}{\rho}$

Physical Interpretation

- the relative importance of surface and gravity is prescribed by the Bond number

$$B_o = \frac{\rho g}{\sigma k^2} = \frac{\rho g \lambda^2}{4\pi^2 \sigma} = \frac{\lambda^2}{\lambda_c^2}$$

- for air-water, $B_o \sim 1$ for $\lambda_c \sim 1.7\,\mathrm{cm}$ CAP LENGTH
- for $B_o \gg 1$ $(\lambda \gg \lambda_c)$, surface tension negligible
  $\Rightarrow$ GRAVITY WAVES
- for $B_o \ll 1$ $(\lambda \ll \lambda_c)$, gravity negligible
  $\Rightarrow$ CAPILLARY WAVES

Special Cases

Recall: Shallow $(kh \ll 1)$: $\tanh kh \simeq kh - \frac{1}{3}k^3h^3 + \dots$
Deep $(kh \gg 1)$: $\tanh kh \simeq 1$

$\boxed{A.}$ Gravity Waves: $B_o \gg 1$, $c^2 = \frac{g}{k}\tanh kh$

a.) Shallow Water $(kh \ll 1) \Rightarrow c = \sqrt{gh}$

- all wavelengths travel at same speed
  i.e. NON-DISPERSIVE

$\Rightarrow$ one can only surf in shallow water

b.) Deep Water $(kh \gg 1) \Rightarrow c = \sqrt{g/k}$

- long waves travel fastest

e.g. drop large stone in a pond

$\boxed{B.}$ Capillary Waves: $B_o \ll 1$, $c^2 = \frac{\sigma k}{\rho} \tanh kh$

a.) Deep water: $kh \gg 1 \Rightarrow c = \sqrt{\frac{\sigma k}{\rho}}$

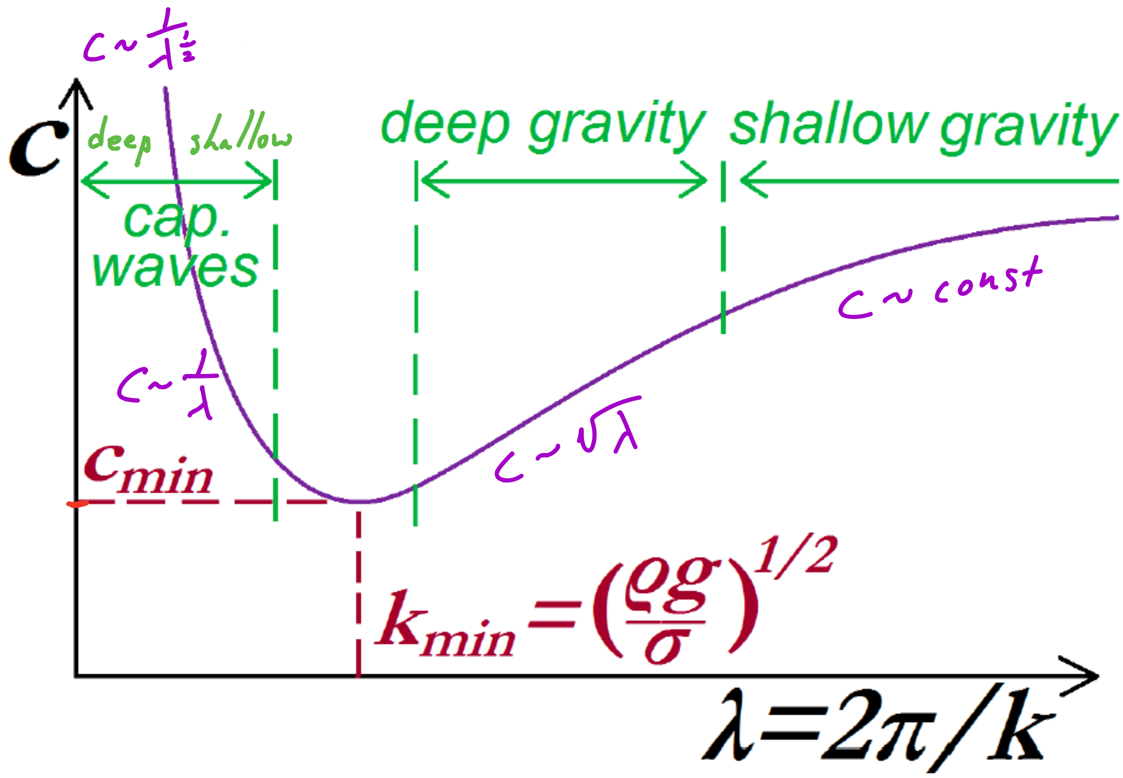
- short waves travel fastest

e.g. raindrop hits a pond

b.) Shallow water: $kh \ll 1$

$\Rightarrow c = \sqrt{\frac{\sigma h k^2}{\rho}}$

Summary



Note: 1. Four distinct scalings for $c(\lambda)$.

2. When $\frac{dc}{dk} = 0$, we have

$$c_{min} = \left(\frac{4 g \sigma}{\rho}\right)^{\frac{1}{4}} \quad \text{for} \quad k = \left(\frac{\rho g}{\sigma}\right)^{\frac{1}{2}}$$

3. Group Velocity: when $c = c(\lambda)$, a wave is called DISPERSIVE since the different (Fourier) wave components (corresponding to different λ, k) separate or disperse

e.g. deep-water gravity: $c \sim \sqrt{\lambda}$

- in a dispersive system, the energy of a wave component does not propagate at the phase speed $c = \omega/k$, but rather at the...

GROUP VELOCITY : $c_g = \frac{d\omega}{dk} = \frac{d}{dk}(ck)$

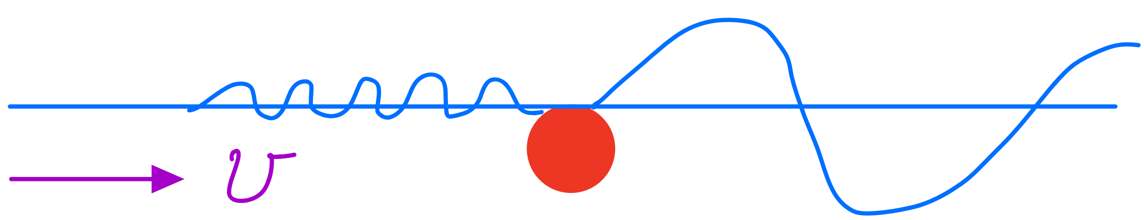
e.g. DEEP GRAVITY WAVES : $c_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}c$

DEEP CAPILLARY WAVES : $c_g = \frac{3}{2}c$

4. Flow past an obstacle

Note c_{min} : if $U < c_{min}$, no steady waves generated by the obstacle

- if $U > c_{min}$: there are 2 k-values for which $c = U$



i) the smaller represents a gravity wave with $c_g = c/2 < c$ $\Rightarrow$ energy swept downstream

ii) the larger k represent a capillary wave with $c_g = \frac{3}{2}c > c$ $\Rightarrow$ energy swept upstream (but quickly dissipated due to small λ)

e.g. waves generated by a fishing line