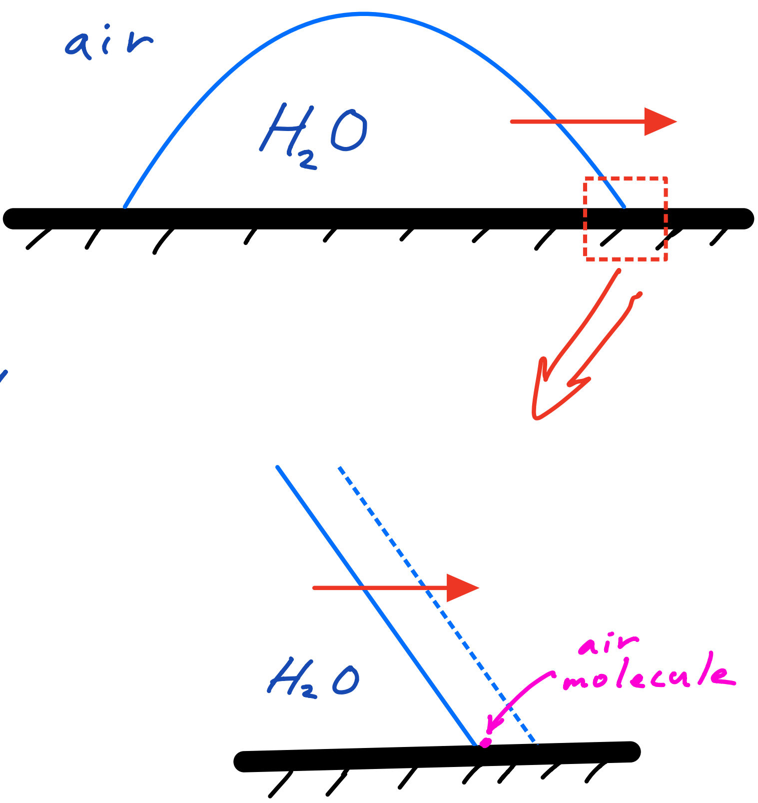
Lecture 18. Contact line motion; spreading

Contact Line Motion: The Lore / Paradox

- contact line motion violates the NO-SLIP boundary condition in vicinity of contact line

$\Rightarrow$ drops can't move!

- air molecule adjoining boundary ahead of advancing line displaced by liquid



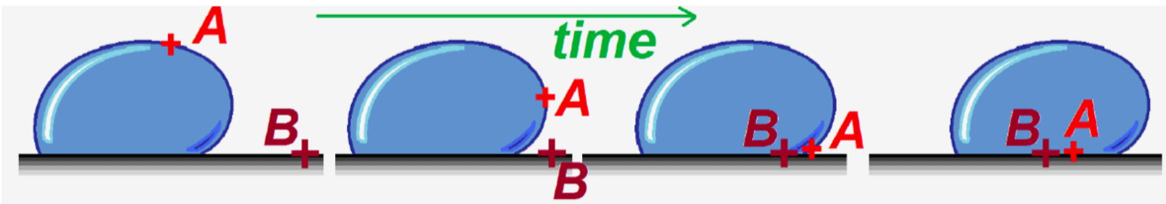
Resolution

Recall: continuum hypothesis valid to ~10 molecules

$\Rightarrow$ expect it to fail on molecular scale, scale of contact line

A few key observations

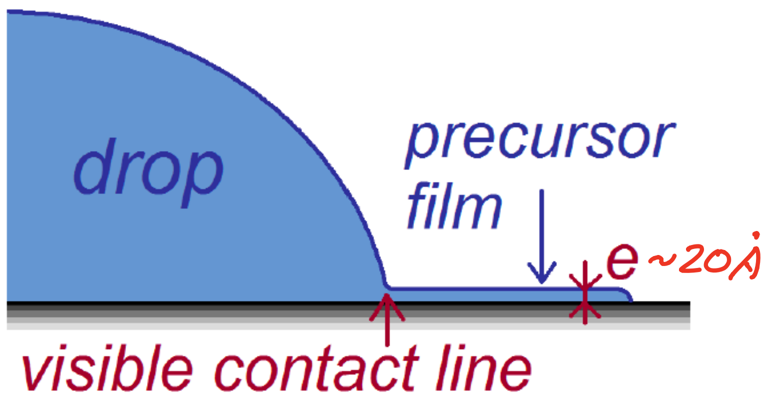
I. Rolling Drops (E. Dussan 1977)



- contact line advances like a tractor tread

II. Precursor Films (Hardy 1919)

- molecular-scale films may adjoin drop, depending on surface chemistry

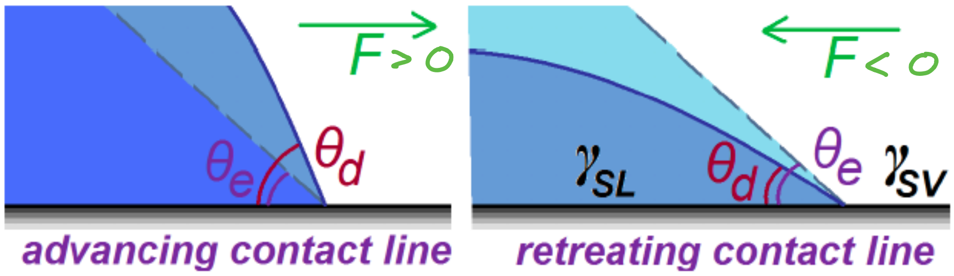


- circumvents the moving-contact-line problem on the drop scale

- shifts problem to the molecular scale

  ⟹ a matter of surface chemistry rather than fluid mechanics

The Moving Contact Line

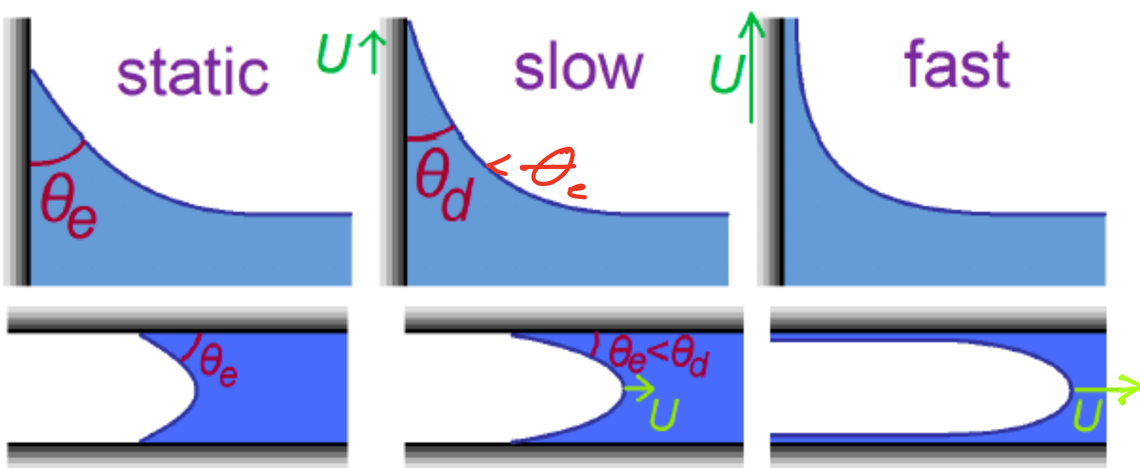


Force of Traction:

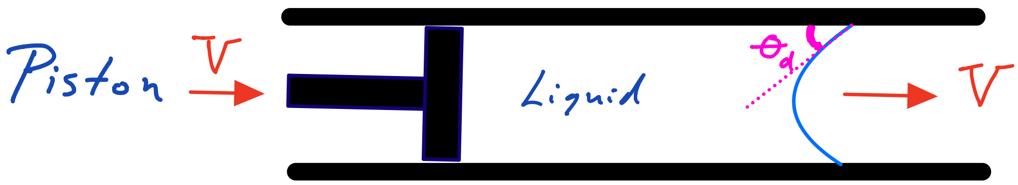
$$F(\theta_d) = \gamma_{SV} - \gamma_{SL} - \gamma\cos\theta_d = \gamma(\cos\theta_e - \cos\theta_d)$$

How does F depend on V? i.e. what is $\theta_d(V)$?

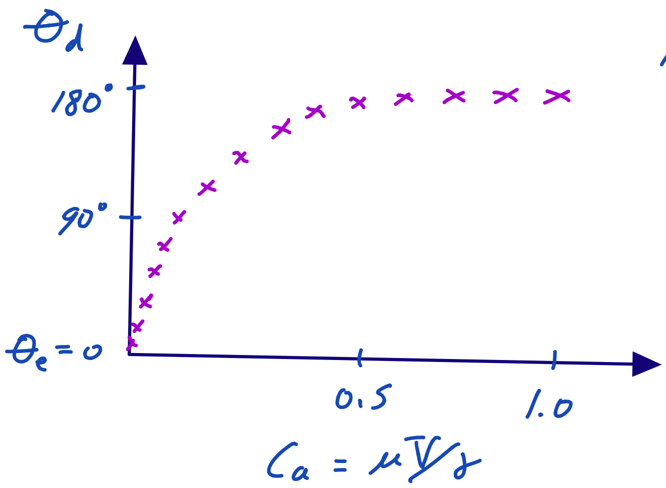
- retreating contact line ($F<0$) can be examined with plate retraction expts, or by pushing air through a tube.

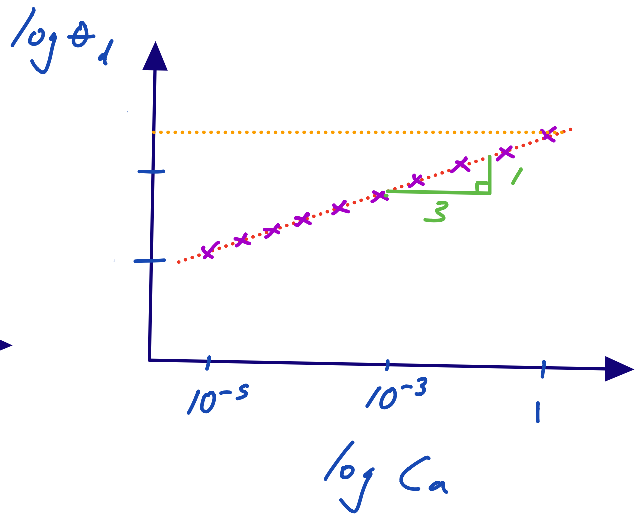


- case of advancing contact line ($F > 0$) examined by Hoffman (1975) for the case of $\theta_e = 0$



Observations





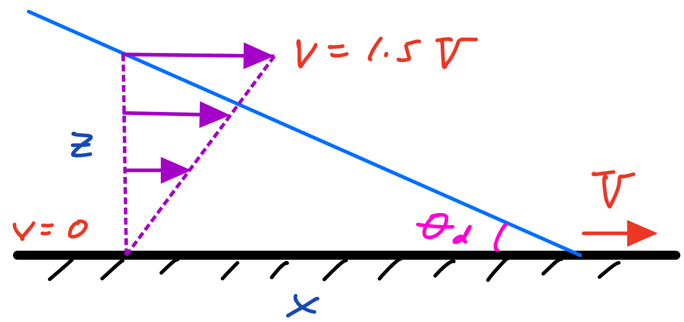
$$\Rightarrow \boxed{\theta_d \sim V^{\frac{1}{3}} \sim Ca^{\frac{1}{3}}}$$ Tanner's Law

How can we rationalize this dependence?

Flow near contact line of spreading liquid ($\theta_d > \theta_e$)

Consider $\theta_d \ll 1$, so that $\tan\theta_d = \frac{z}{x} \simeq \theta_d$

$\Rightarrow \quad z = \theta_d x$



Velocity gradient: $\frac{dV}{dz} \simeq \frac{V}{\theta_d x}$

Rate of viscous dissipation in corner:

$$\Phi = \iint_{\text{CORNER}} \mu \left(\frac{dV}{dz}\right)^2 dV = \mu \int_0^\infty dx \int_0^{z_{max} = \theta_d x} \frac{V^2}{\theta_d x^2} \, dx$$

$$= 3\mu \int_0^\infty \frac{V^2}{\theta_d^2 x^2} \theta_d x \, dx = \frac{3\mu V^2}{\theta_d} \int_0^\infty \frac{dx}{x}$$

Dodgy Bit (de Gennes):

$$\int_0^\infty \frac{dx}{x} \approx \int_a^L \frac{dx}{x} = \ln\frac{L}{a} \equiv \ell_D$$

$L \rightarrow$ drop size

a ← molecular scale

where in expts, $15 < \ell_D < 20$

Energetics:

$$F V = \Phi = \frac{3\mu l_D}{\theta_d} V^2 \quad \boxtimes$$

Rate of work done by surface forces

Dissipation rate

Recall: $F = \gamma(\cos\theta_e - \cos\theta_d)$

In limit $\theta_e < \theta_d \ll 1$, $\cos\theta_{e/d} \simeq 1 - \frac{\theta_{e/d}^2}{2}$

$$\Rightarrow F = \frac{\gamma}{2}(\theta_d^2 - \theta_e^2) \Rightarrow \text{sub into } \boxtimes$$

$$\Rightarrow \frac{\gamma}{2}(\theta_d^2 - \theta_e^2) V = \frac{3\mu l_D}{\theta_d} V^2$$

Contact Line Speed:

$$\boxed{V(\theta_d) = \frac{V^*}{6 l_D} \theta_d (\theta_d^2 - \theta_e^2)}$$

where $V^* \equiv \gamma/\mu \simeq 30 \text{ m/s}$

Note: 1. rationalizes Hoffman's data (obtained for $\theta_e = 0$)

$$\Longrightarrow V \sim \theta_d^3, \quad \theta_d \sim V^{\frac{1}{3}}$$

2. $V = 0$ for $\theta_d = \theta_e \Rightarrow$ static equilib

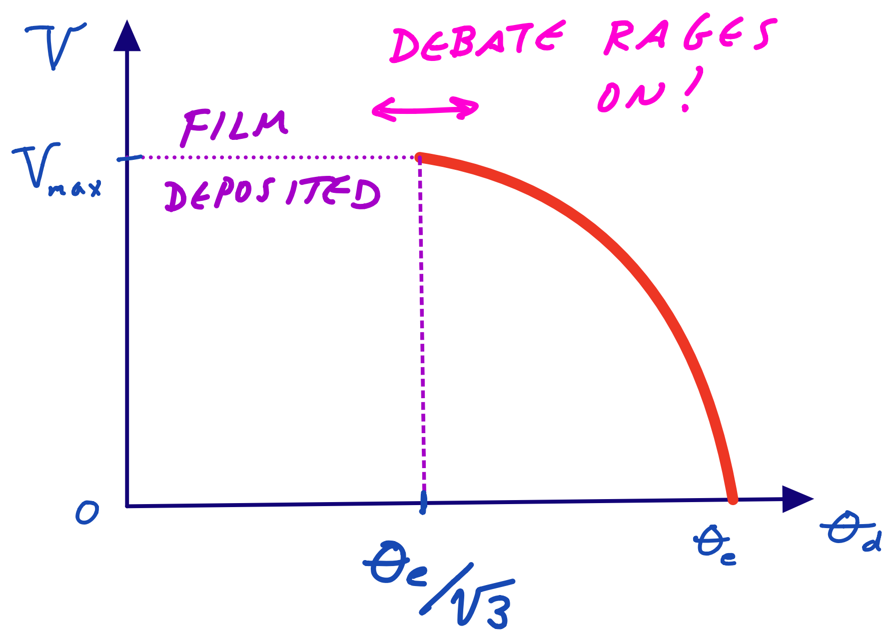
3. $V \to 0$ as $\theta_d \to 0$: dissipation enhanced in a sharp corner

4. $V(\theta_d)$ has a MAX value when

$$\frac{dV}{d\theta_d} = \frac{V^*}{6\ell_D}\left(3\theta_d^2 - \theta_e^2\right) = 0$$

$$\Rightarrow \theta_d = \frac{\theta_e}{\sqrt{3}}$$

$$\Rightarrow V_{MAX} = \frac{V^*}{6\ell_D}\,\frac{2}{3\sqrt{3}}\,\theta_e^3$$



Eg. In water $V^* = 70$ m/s, with $\theta_e = 0.1$ rad and $\ell_D = 20$, we deduce

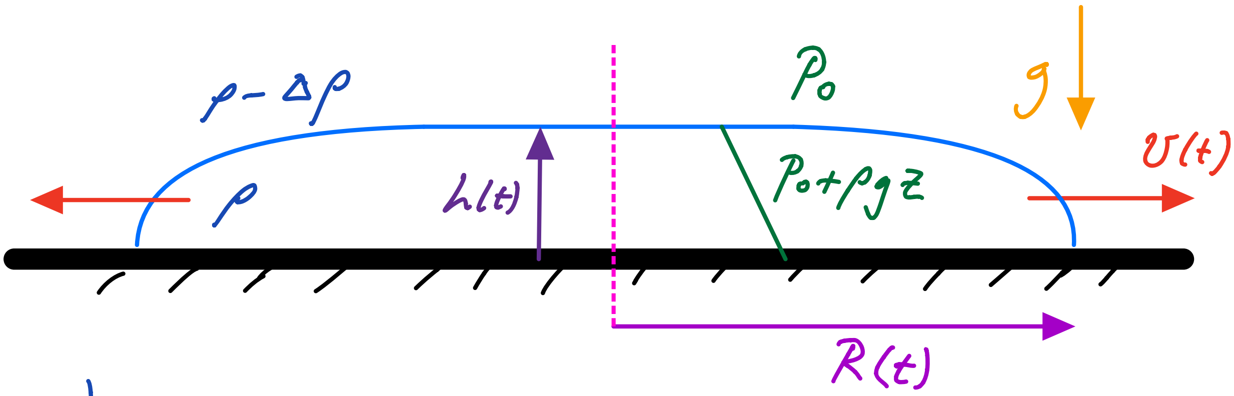
$$V_{MAX} = 0.2 \text{ mm/s}$$

$\Rightarrow$ sets lower bound on extraction speed for water coating

Spreading

Recall: Gravity Currents

Stage I : $Re \gg 1$



Scaling : $\frac{\partial p}{\partial z} \sim \rho g \Rightarrow P_c \sim \rho g H$

$$\frac{\partial p}{\partial r} \sim \rho \underline{u} \cdot \underline{\nabla u} \sim \rho U^2/r$$

- flow forced by gravity, resisted by inertia :

$$\frac{\Delta \rho g h}{r} \sim \rho U^2/r$$

$$\Rightarrow U \sim \sqrt{g' h} \quad \text{where } g' = g \frac{\Delta\rho}{\rho} \text{ is reduced gravity}$$

Continuity : $V = \pi R^2(t)\, h(t) = \text{const volume}$

$$\Rightarrow h(t) = \frac{V}{\pi R^2(t)} \sim V/R^2(t)$$

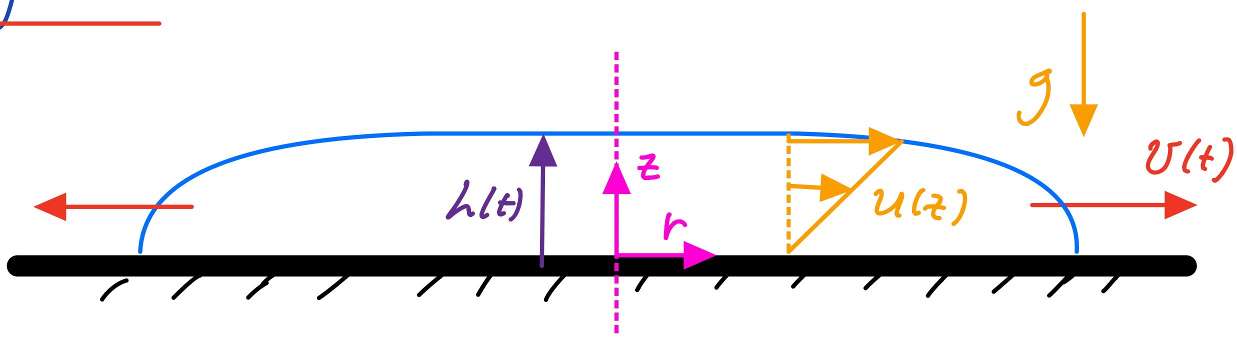
$$\Rightarrow U = \frac{dR}{dt} \sim \sqrt{g' V} \frac{1}{R}$$

$$\Rightarrow R\, dR \sim \sqrt{g' V}\, dt$$

$$\Rightarrow \boxed{R(t) \sim (g' V)^{\frac{1}{4}} t^{\frac{1}{2}}}$$

Note: $v \sim \sqrt{g'h}$ decreases until $Re = \frac{vh}{\nu} \lesssim 1$, where we reach...

Stage II : $Re \ll 1$



- flow forced by gravity, resisted by viscosity

$$\frac{\partial p}{\partial z} \sim \rho g \Rightarrow p_c \sim \rho g H \text{ is horizontal pressure diff.}$$

$$\frac{\partial p}{\partial p} \sim \nu \frac{\partial u^2}{\partial z^2} \Rightarrow \frac{\rho g H}{R} \sim \nu \frac{v}{H^2}$$

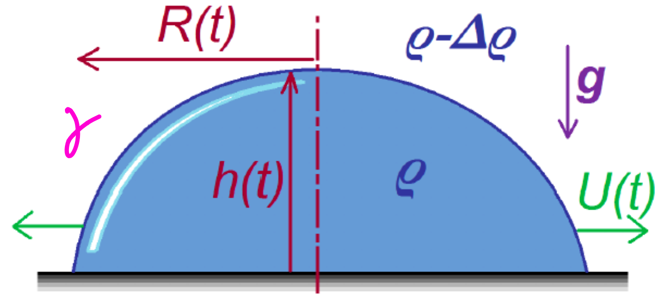
$\Rightarrow$ Eliminate $H \sim V/R^2$ from continuity

$$\Rightarrow v = \frac{dR}{dt} \sim \frac{\rho g' V^3}{\nu R^7}$$

$$\Rightarrow \boxed{R \sim \left(\frac{\rho g' V^3}{\nu}\right)^{\frac{1}{8}} t^{\frac{1}{8}}}$$

The Spreading of Small Drops on Solids

- driven by both gravity and curvature pressures



Gravity: $\underset{\sim}{\nabla} p_g \sim \frac{\rho g h}{R}$

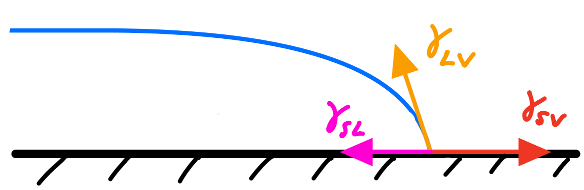
Curvature: $\underset{\sim}{\nabla} p_c \sim \gamma \frac{h}{R^3}$

Which dominates? $\frac{\nabla p_g}{\nabla p_c} \sim \frac{\rho g R^2}{\gamma}$ = Bond #

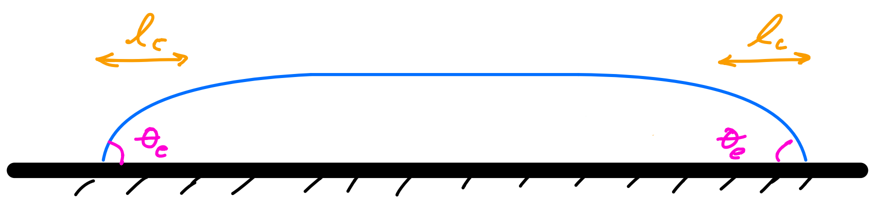
$\Rightarrow Bo = \frac{\rho g R^2}{\gamma} = \frac{\rho g V}{\gamma h} \sim \frac{1}{h}$ $\Rightarrow$ gravity becomes progressively more important

Recall! drop behaviour depends on spreading parameter:

$$S = \gamma_{SV} - \gamma_{SL} - \gamma$$

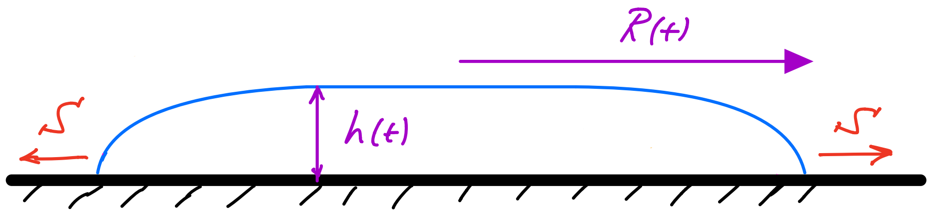


When $S < 0$: Partial Wetting



- spreading arises until a puddle obtains, with a flat central portion matching onto menisci that meet the boundary at θ_e

When $S > 0$: Complete Wetting



- here, one expects spreading forced by tension at contact line

$$\underbrace{\frac{\mu V}{h} \cdot \pi R^2}_{\text{Viscous resistance}} = \underbrace{S \cdot 2\pi R}_{\text{Spreading force applied at contact line}}$$

$$\Rightarrow R \frac{dR}{dt} \sim \frac{S}{\mu} h \sim \frac{S}{\mu} \frac{V}{R^2}$$

$$\Rightarrow R^3 \frac{dR}{dt} \sim SV/\mu$$

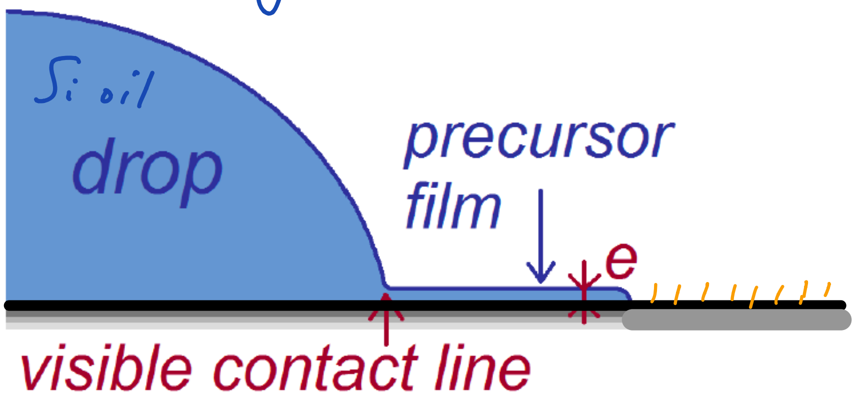
$$\Rightarrow \boxed{R(t) \sim \left(\frac{SV}{\mu}\right)^{\frac{1}{4}} t^{\frac{1}{4}}}$$

- but this is rarely observed
- instead, one sees $R \sim t^{\frac{1}{10}}$. Why?

Hardy (1919) : observed a precursor film

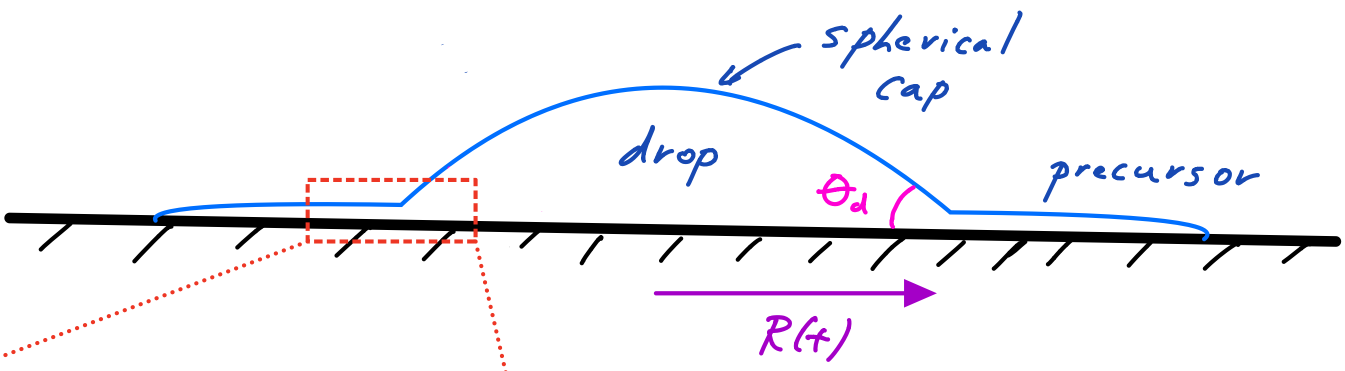
- placed dust/pillars ahead of spreading drop : precursor film knocks them down

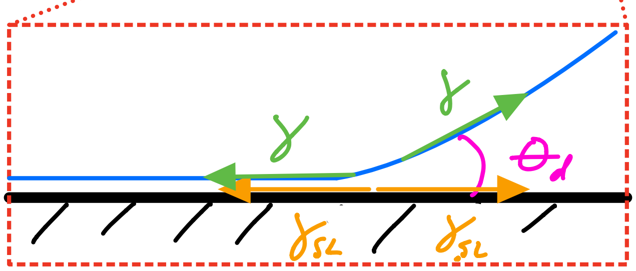
- the film is otherwise invisible, $e \sim 20\,\text{Å}$
- its origins lie in the force imbalance at the contact line ($S > 0$).



- the stability of this precursor film results from interaction between fluid and solid (e.g. van der Waals forces)

Physical Picture





- force balance at contact line changed dramatically by precursor film

Force at Contact Line:

$$F = \gamma + \cancel{\gamma_{SL}} - \gamma\cos\theta_d - \cancel{\gamma_{SL}} = \gamma(1-\cos\theta_d)$$

$$\approx \gamma\,\theta_d^2/2 \quad \text{for small } \theta_d$$

<u>Note</u>: $\tilde{F} \ll S$ owing to precursor film

Now recall: $F\mathcal{V} = \dfrac{3\mu l_D}{\theta_d}\mathcal{V}^2$

defined speed of contact line $\mathcal{V}(\theta_d)$

Letting $F \to \tilde{F} = \dfrac{\gamma\theta_d^2}{2}$, we find

$$\mathcal{V} = \frac{dR}{dt} = \frac{\theta_d}{3l_D\mu}\tilde{F} = \frac{V^*}{6l_D}\theta_d^3 \quad ✩$$

If drop is small, it is a section of a sphere, so $V = \frac{\pi}{4}R^3\theta_d$ *, so that $\frac{dV}{dt} = 0$ gives

$$\frac{3}{R}\frac{dR}{dt} = -\frac{1}{\theta_d}\frac{d\theta_d}{dt}.$$

Sub in for $\frac{dR}{dt}$ from ✩:

$$\frac{1}{\theta_d}\frac{d\theta_d}{dt} = -\frac{V^*}{R}\theta_d^3$$

Sub $R = L\,\theta_d^{-\frac{1}{3}} = \left(\dfrac{V}{\theta_d}\right)^{\frac{1}{3}}$ from *

where $L \equiv V^{\frac{1}{3}}$

$$\Rightarrow \frac{d\theta_d}{dt} = -\frac{V^*}{L}\theta_d^{13/3}$$

$$\Rightarrow \boxed{\theta_d \simeq \left(\frac{L}{V^* t}\right)^{3/10}}$$

Thus, via $*$, we have

$$\boxed{R \sim L\left(\frac{\nu^* t}{L}\right)^{\frac{1}{10}}}$$

where $\nu^* = \frac{\sigma}{\mu}$

$L = V^{\frac{1}{3}}$