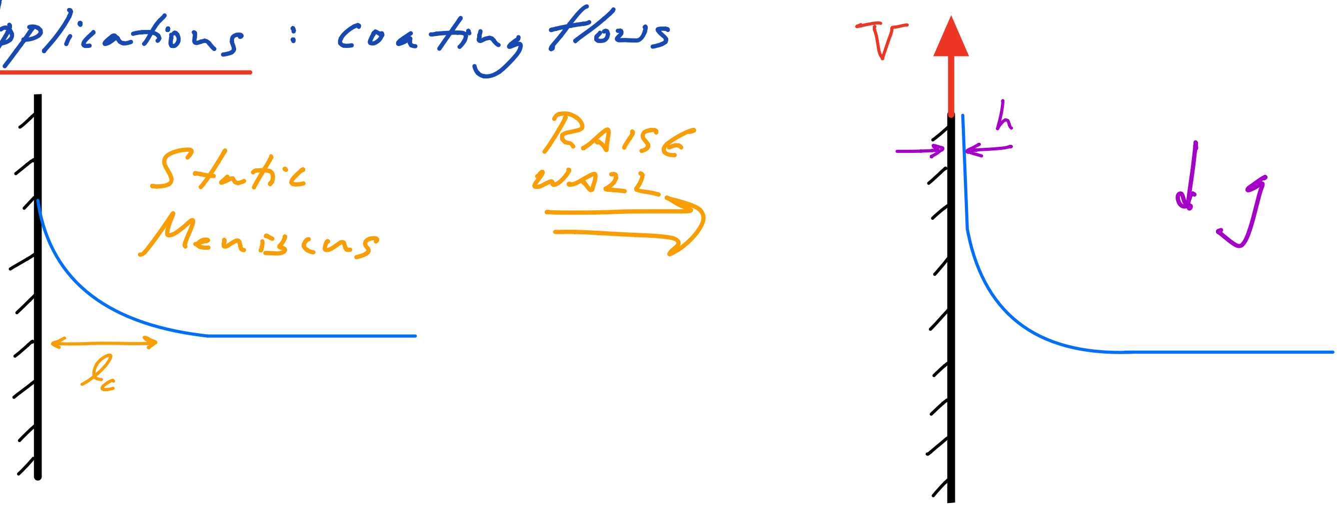
Lecture 17. Coating flows

Landau-Levich-Derjaguin ("LLD") Problem

- withdraw a plate from a viscous liquid
  $\Rightarrow$ how thick is the film?

Applications: coating flows



One might expect: $\mu \frac{V}{h^2} \sim \rho g$ (VISCOSITY vs GRAVITY)

$$\Rightarrow \quad h \sim \left(\frac{\mu V}{\rho g}\right)^{\frac{1}{2}} \sim \underbrace{\ell_c \, Ca^{\frac{1}{2}}}_{\text{indep of } \sigma} \qquad \text{(Derjaguin 1943)}$$

where $\ell_c = \left(\frac{\sigma}{\rho g}\right)^{\frac{1}{2}}$

and $Ca = \frac{\mu V}{\sigma} = \frac{\mu V / \cancel{a}}{\sigma / \cancel{a}} = \frac{\text{VISCOUS}}{\text{CURVATURE}} = \text{Capillary \#}$

But this scaling is only observed as $Ca \to 1$.
At low Ca, the coating is resisted by curvature pressure rather than gravity.

Recall 2D static meniscus from Lec. 6.

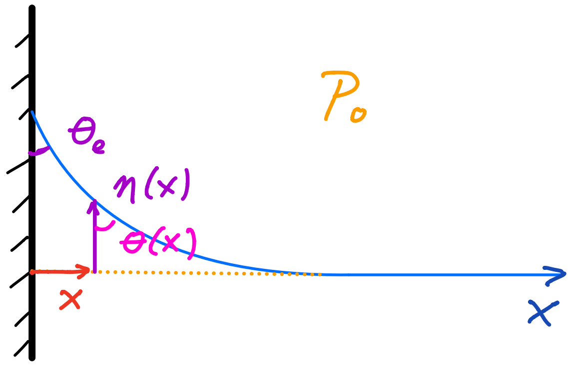
$$\eta(x) = \sqrt{2}\, \ell_c \left[1 - \sin\theta(x) \right]$$

and internal pressure

$$p(x) = p_o - \rho g \eta(x) .$$

As $x \to 0$, $\eta(x) \to \sqrt{2}\, \ell_c$

and $p(0) \to p_o \underbrace{- \sqrt{2}\, \rho g \ell_c}_{\text{suction}}$



- it is this capillary suction pressure inside the meniscus that resists rise when $Ca \ll 1$.

Define Inner + Outer Zones

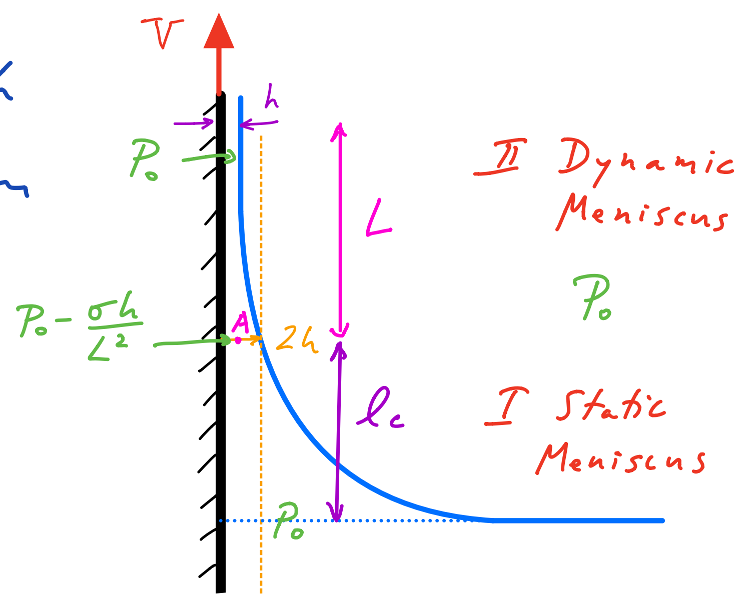
I. Static Meniscus (outer zone)

Balance gravity + curvature: $\rho g \eta \sim \sigma \underline{\nabla} \cdot n$

$\Rightarrow$ curvature $\underline{\nabla} \cdot \underline{n} \sim \dfrac{1}{\ell_c}$

II. Dynamic Meniscus: Coating / Inner Zone

- balance viscous stresses and curvature pressures



- define as zone over which film thickness decreases from 2h to h $\Rightarrow$ vertical extent L to be specified by pressure matching

In Zone II, $\underline{\nabla} \cdot \underline{n} \sim \dfrac{h}{L^2}$

Pressure matching at A: $P_0 - \sigma \frac{h}{L^2} \sim P_0 - \rho g l_c$

$$\Rightarrow L^2 \sim \frac{\sigma h}{\rho g l_c} \sim l_c h \Rightarrow L = \sqrt{l_c h} \quad *$$

is geometric mean of l_c, h.

Force balance in Zone II: viscous stresses vs curvature pressure

$$\frac{\mu V}{h^2} \sim \frac{\Delta P}{L} \sim \sigma \frac{h}{L^2} \frac{1}{L}$$

Sub for L $*$: $h^3 \sim \frac{\mu V}{\sigma} L^3 \sim C_a l_c^{3/2} h^{3/2}$

$$\Rightarrow \boxed{h \sim l_c C_a^{2/3}} \quad \text{where } l_c = \left(\frac{\sigma}{\rho g}\right)^{\frac{1}{2}}, \quad C_a = \mu V/\sigma$$

Eg. Jump out of swimming pool at 1 m/s:

$$C_a \sim \frac{10^{-3} \cdot 10^{0}}{10^{-1}} \sim 10^{-2}$$

$\Rightarrow h \sim 100\,\mu m \Rightarrow$ multiply by your surface area $A \sim 2m^2$

$\Rightarrow 300g$ of H_2O

A detailed solution requires matched asymptotics and yields $h \sim 0.94\, l_c C_a^{2/3}$

$\Rightarrow$ long live scaling arguments

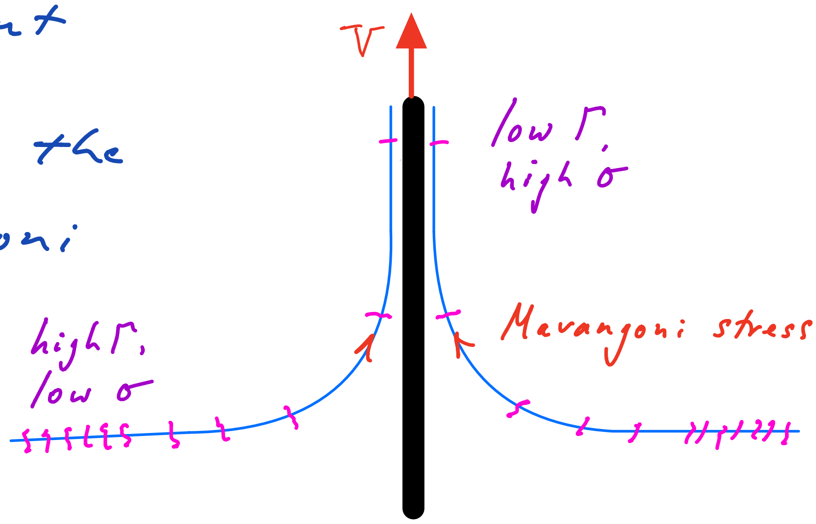
Implicit in above scaling: $h \ll L \Rightarrow C_a^{\frac{1}{3}} \ll 1$,

$L \ll l_c \Rightarrow C_a^{1/3} \ll 1$, $\rho g \ll \frac{\sigma h}{L^3} \Rightarrow C_a^{\frac{1}{3}} \ll 1$.

The Influence of Surfactant

- surfactant decreases σ which slightly alters h, but

$\Rightarrow$ principle effect is the generation of Marangoni stresses that encourage entrainment

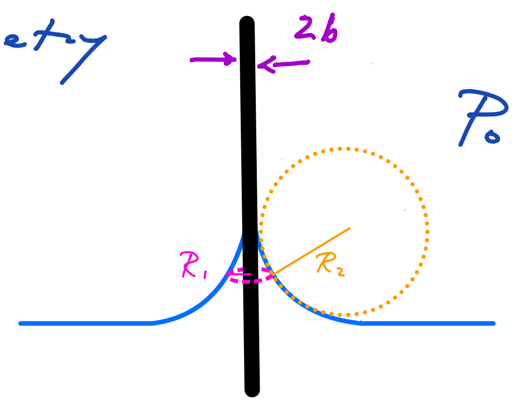


$\Rightarrow$ typically h doubles.

Different Geometries

I. Fiber Coating

- similar dynamics, but geometry is different



Normal Force Balance:

$$\cancel{P_0} - \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \cancel{P_0} - \rho g z$$

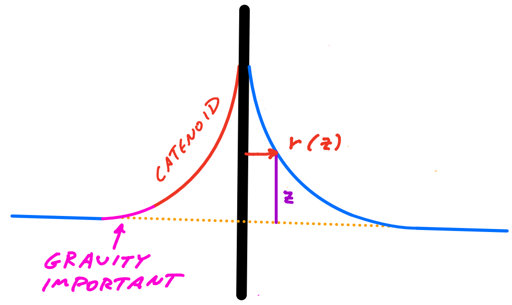
If thread radius small, $b \ll \ell_c$,

$\frac{1}{R_1} \sim \frac{1}{b} \Rightarrow$ curvature pressures dominate, can't be balanced by gravity

$\Rightarrow$ interface must take the form of a catenoid

$$\frac{1}{R_1} + \frac{1}{R_2} = 0$$

For the total wetting case, $\theta_e = 0$, one finds



$$r(z) = b \cosh\left(\frac{z-h}{b}\right)$$

where $h = b \ln\left(\frac{2 l_c}{b}\right)$

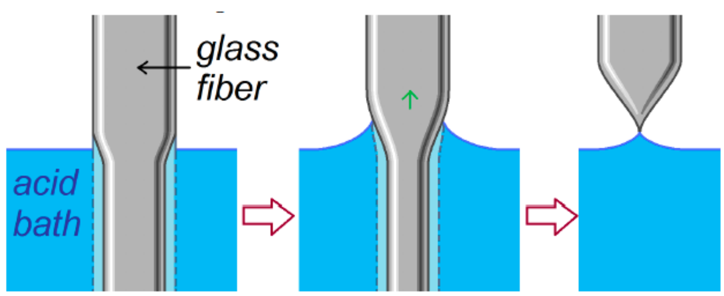
Note: 1. gravity prevents meniscus from extending to ∞

$\Rightarrow$ h deduced by cutting it off at l_c

2. h is just a few times b ($h \ll l_c$)

$\Rightarrow$ lateral extent of meniscus greatly exceeds its height

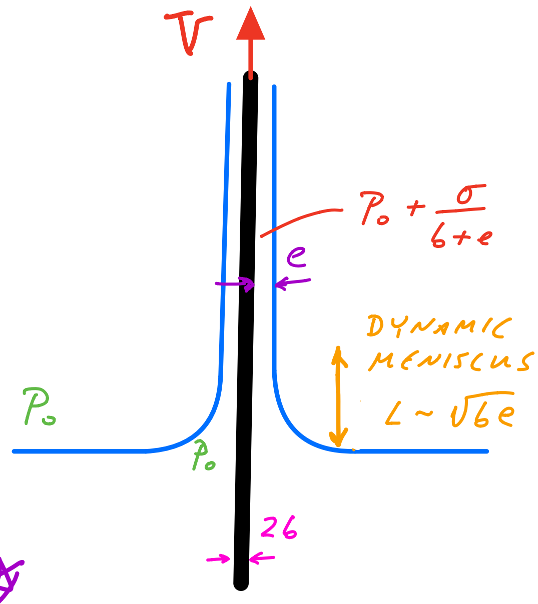
Application: etching of microtips of Atomic Force Microscopes



Forced Wetting of Fibers

e.g. optical fiber coating

- there is no longer suction into meniscus, where $P \sim P_0$
- however, $P_{film} \sim P_0 + \frac{\sigma}{b}$, so there is still a ΔP that resists entrainment



Force balance:

$$\frac{\mu V}{e^2} \sim \frac{\Delta P}{L} = \frac{1}{L}\frac{\sigma}{b} \quad \star$$

VISCOUS CURVATURE

Pressure match of static and dynamic meniscus:

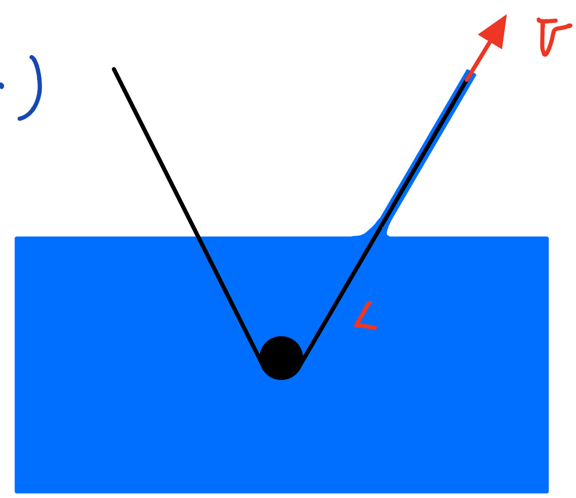
$$\frac{e}{L^2} \sim \frac{1}{b} \Longrightarrow L = \sqrt{be}$$ sub into $\star$

Bretherton's Law:

$$e = b\, Ca^{2/3}$$

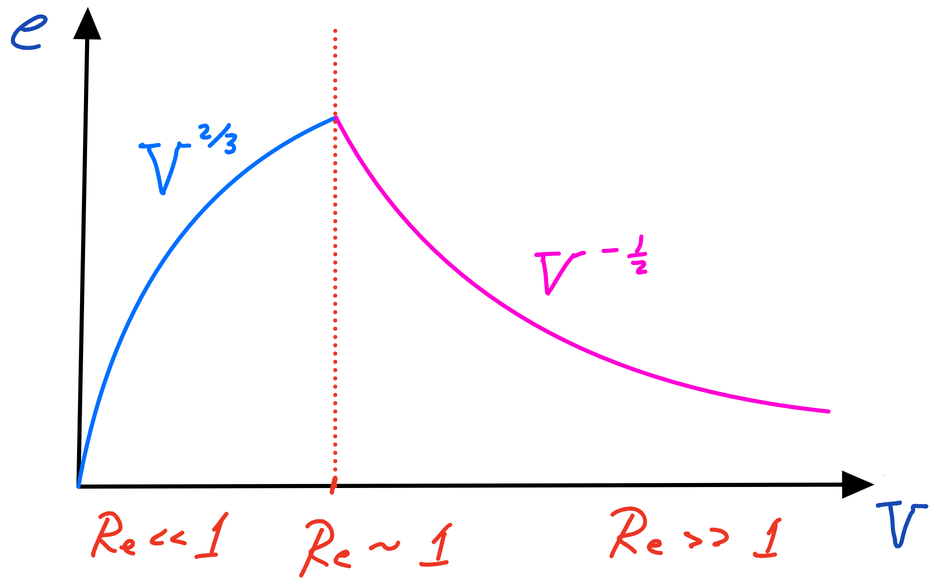
valid when $e \ll b$ i.e. $Ca^{2/3} \ll 1$

At higher Ca (pulling speeds) film is just the viscous boundary layer that develops as thread is pulled through the reservoir:



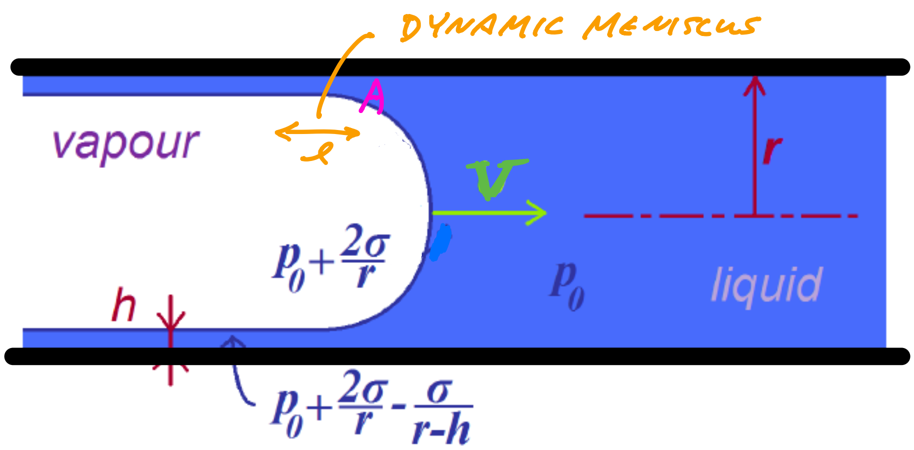
$\delta \sim \left(\frac{\mu}{\rho} \frac{L}{V} \right)^{\frac{1}{2}}$ where L is the submerged path

Summary



Displacement of an interface in a tube

e.g. pumping oil out of porous rock
(typically 40% of oil left behind)



In limit of $h \ll r$ ($C_a \ll 1$), the pressure gradient in the dynamic meniscus region $\propto$

$$\nabla p \sim \frac{\sigma}{r}\frac{1}{\ell}$$ where ℓ is the extent of the dynamic meniscus

As on the fiber, pressure matching across the dynamic meniscus:

$$P_0 + \frac{2\sigma}{r} - \cancel{\frac{\sigma}{r-h}} \sim P_0 + \frac{\sigma h}{\ell^2}$$

$$\Rightarrow \ell \sim (hr)^{\frac{1}{2}} \text{ when } h \ll r$$

Force balance: $$\frac{\mu V}{h^2} \sim \frac{\sigma}{r\ell} \sim \frac{\sigma}{r(hr)^{\frac{1}{2}}}$$

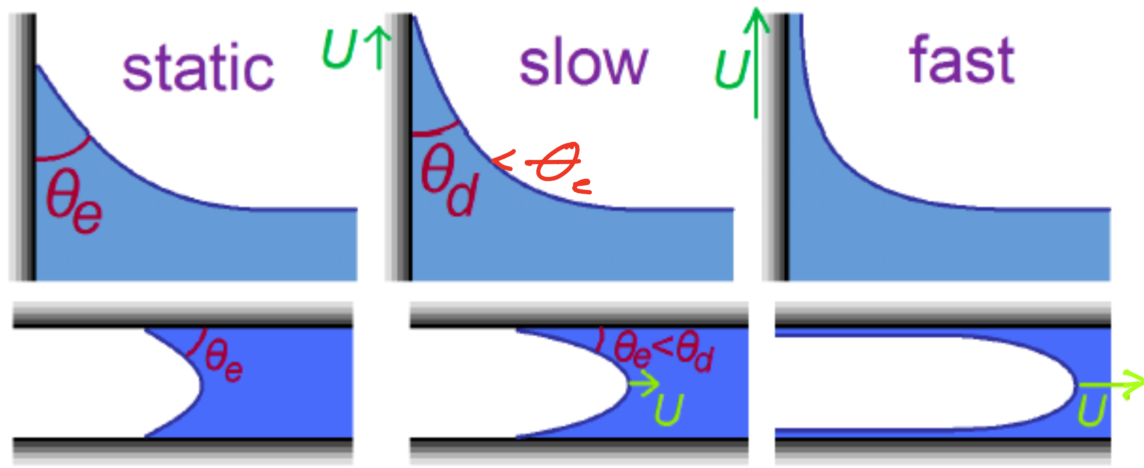
$$\Rightarrow \boxed{h \sim r\, C_a^{2/3}}$$ Bretherton (1961)

where $C_a = \mu V/\sigma$ (same as on fiber)

Contact Line Dynamics

RETRACTION

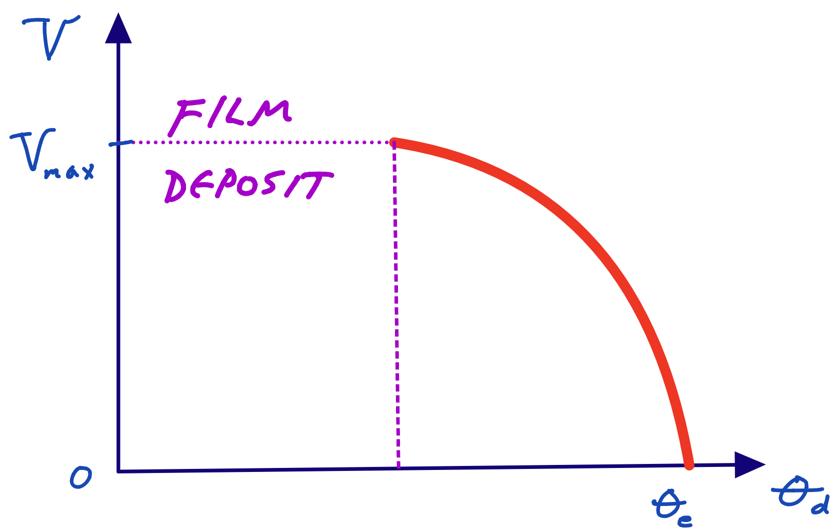
DISPLACEMENT



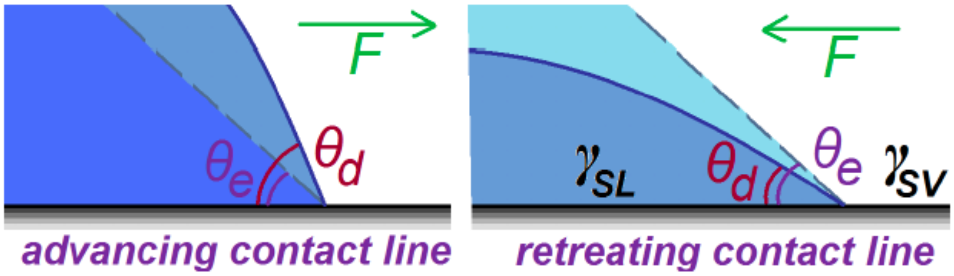
- at low speeds V, the contact line advances, and the contact angle $\theta_d < \theta_e$ is called the DYNAMIC CONTACT ANGLE
- at sufficiently high speed, the contact line cannot keep up with V

$\Longrightarrow$ a film is entrained onto the solid

Observation



Consider a clean system free of hysteresis



Force of traction pulling liquid toward dry region:

$$F(\theta_d) = \gamma_{SV} - \gamma_{SL} - \gamma \cos\theta_d$$
$$= \gamma(\cos\theta_e - \cos\theta_d)$$

Note: $F(\theta_e) = 0$ in equilibrium

How does F depend on V?

i.e. what is $\theta_d(V)$?