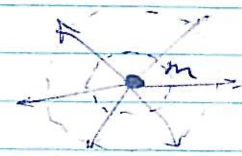
Re: $\ln z = \ln r e^{i\theta}$
$= \ln r + \ln e^{i\theta}$
$= \ln r + i\theta$

source strength

C. Sources and Sinks : $w(z) = \frac{m}{2\pi} \ln(z)$



$\Rightarrow \phi = \frac{m}{2\pi} \ln r \quad , \quad \psi = \frac{m}{2\pi}\theta$

$u_r = \frac{m}{2\pi r} \quad , \quad u_\theta = 0$

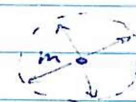
Pot Flow	Electrostatics
m	Q
$\underline{u}$	$\underline{E}$
ϕ	ϕ_{elec}

Note : direct analogy with electrostatics

$\Rightarrow$ can apply methods of E+M to solve potential flow problems

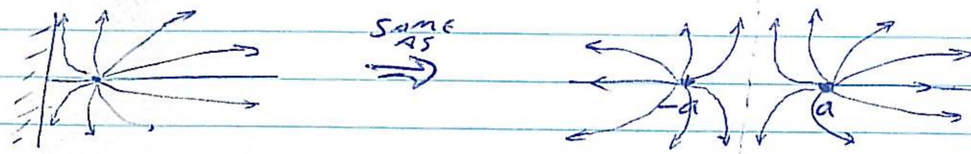
Eg. 1 Gauss' Law to get $\underline{u}$ from m

$u(r) \cdot 2\pi r = m$



Eg. 2 Superposition to deduce $w(z)$, flow field for multiple source/sinks

Eg. 3. Method of Images $\Rightarrow$ used to look at stagnation pt. flow
$\Rightarrow$ try source near a wall

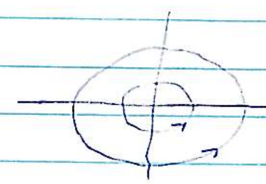


$\Rightarrow w(z) = \frac{m}{2\pi} \ln(z-a) + \frac{m}{2\pi} \ln(z+a)$

[Aside : analogies also exist between vorticity dynamics and magnetostatics
since $\underline{j} \sim \underline{\nabla} \wedge \underline{B}$ and $\underline{\omega} = \underline{\nabla} \wedge \underline{u}$
eg. Biot-Savart Law , Gauss Law]

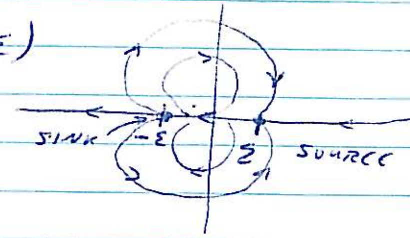
D. Irrotational Vortex : $w(z) = \frac{-i\Gamma}{2\pi} \ln z$

$\Rightarrow u_r = 0 \quad , \quad u_\theta = \frac{\Gamma}{2\pi r}$



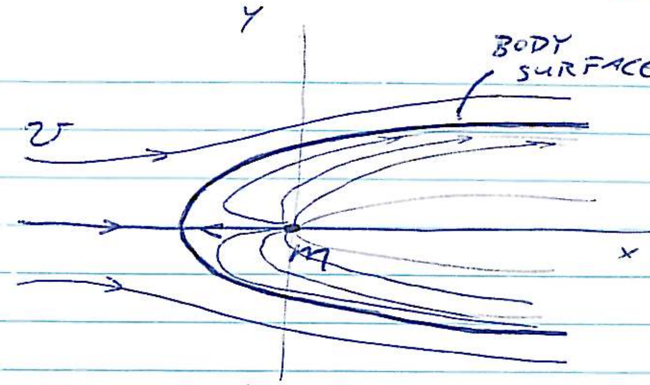
E. Dipole : $w(z) = \frac{m}{2\pi} \ln(z+\varepsilon) - \frac{m}{2\pi} \ln(z-\varepsilon)$

Point dipole : $w(z) \rightarrow \frac{m\varepsilon}{\pi z} = \frac{\mu}{z}$ as $\varepsilon \rightarrow 0$



IMPORTANT POINT : one may describe more complex potential flows by superposing these basic components.

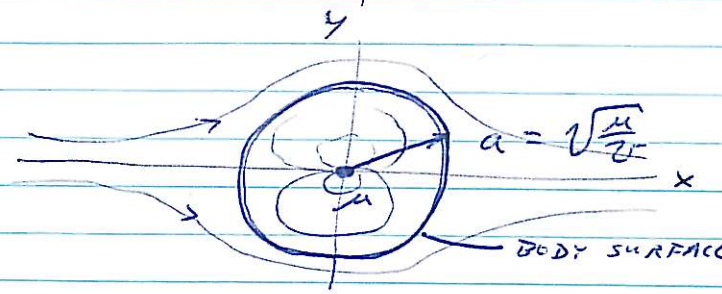
Eg. 1 Flow Past a Half-Body



$$w(z) = \underbrace{\mathcal{V} z}_{\text{uniform flow in } \hat{x}} + \underbrace{\frac{m}{2\pi} \ln z}_{\text{pt source}}$$

⇒ choose $\mathcal{V}$, m such that body surface corresponds to a streamline

Eg. 2 Flow Past a Cylinder

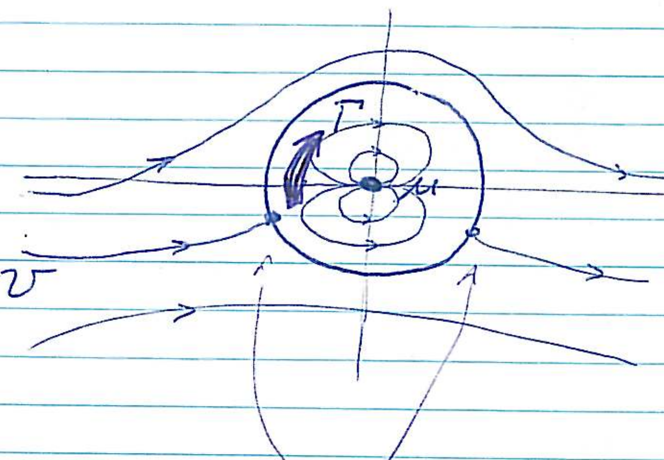


$$w(z) = \underbrace{\mathcal{V} z}_{\text{uniform flow in } \hat{x} \text{ direction}} + \underbrace{\frac{\mu}{z}}_{\text{point dipole of strength } \mu \text{ at origin}}$$

⇒ body surface $a = \sqrt{\mu/\mathcal{V}}$

Eg. 3 Flow Past a cylinder with circulation (see PS #5)

$$w(z) = \underbrace{\mathcal{V} z}_{\text{uniform flow in x-direction}} + \underbrace{\frac{\mu}{z}}_{\text{point dipole}} + \underbrace{\frac{i\Gamma}{2\pi} \ln z}_{\text{irrotational vortex}}$$



NOTE: high P stagnation pts

⇒ LIFT!

~~Some Comments on the Relevance of~~ Shortcomings of Potential (Inviscid irrotational) Flow in describing high Re flows

Recall: we began by nondimensionalizing N-S eqns to obtain

$$\frac{\partial \underline{u}^*}{\partial t^*} + \underline{u}^* \cdot \underline{\nabla}^* \underline{u}^* = -\underline{\nabla}^* p^* + \frac{1}{Re} \nabla^{*2} \underline{u}^*$$

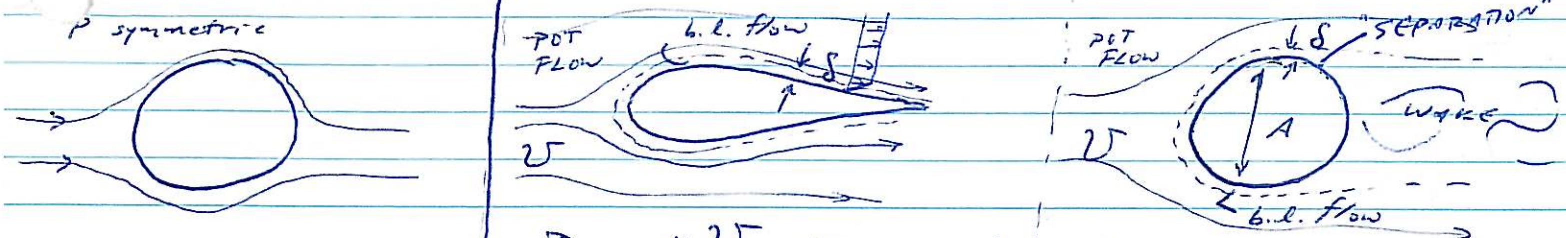
In limit of large Re, we assumed viscous effects were negligible

$\Rightarrow$ Euler's Eqn: $\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p$

- solving this eqn required that we drop NO-SLIP B.C.
- this shortcoming of the potential flow description $\Rightarrow$ D'Alembert's Paradox

We have also seen that vorticity cannot be generated in an inviscid fluid (Kelvin's Circulation Thm); however, it must be generated at rigid bdries by the no-slip condition in ANY REAL FLUID.

POTENTIAL FLOW | REAL FLUIDS: STREAMLINED BODY ; BLUFF BODY



$D = 0$ | $D \sim \mu \frac{U}{\delta} \cdot$ (Surface Area) SKIN FRICTION | $D \sim \rho U^2$ (Cross-sectional Area) FORM DRAG

IMPORTANT POINT: viscous effects important within viscous b.l. adjoining body surface

Thickness of Boundary layer: within b.l., viscous forces comparable to inertial forces

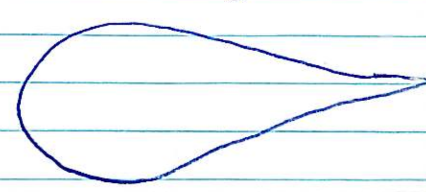
Scaling $\Rightarrow$ $\underline{u} \cdot \underline{\nabla}\, \underline{u} \sim \nu \nabla^2 \underline{u}$

$$\Rightarrow \frac{U^2}{a} \sim \nu \frac{U}{\delta^2} \Rightarrow \boxed{\delta \sim \frac{a}{Re^{\frac{1}{2}}}}$$ thins as $Re \to \infty$

i.e.: $\frac{\text{FORM DRAG}}{\text{SKIN FRICTION}} \sim \frac{\rho U^2 \cancel{a^2}}{\mu U/\delta \, \cancel{a^2}} \sim \frac{U\delta}{\nu} \sim Re^{\frac{1}{2}} \gg 1$ at high Re

$\Rightarrow$ importance of streamlining in sport

SAME DRAG:



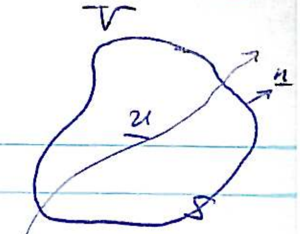
To solve high Re flows correctly, one must consider the boundary layer on the body surface.

Nevertheless, the irrotational inviscid flow description is useful in

i.) defining flow upstream of the body (i.e. outside b.l. + wake)

ii.) providing "outer solution" which b.l. must match onto.

$\Rightarrow$ we shall return to b.l. flows at a later date



The Momentum Integral (for steady inviscid flow)

$$\rho\, \underset{\sim}{u} \cdot \underset{\sim}{\nabla}\, \underset{\sim}{u} = - \underset{\sim}{\nabla}(p + \chi)$$

Now use $\quad \underset{\sim}{\nabla} \cdot (\rho\, \underset{\sim}{u}\, \underset{\sim}{u}) = \rho \cancel{(\underset{\sim}{\nabla} \cdot \underset{\sim}{u})}^{\,0}\, \underset{\sim}{u} + \rho\, \underset{\sim}{u} \cdot \underset{\sim}{\nabla}\, \underset{\sim}{u}$ PROVE BY INDEX NOTATION

$$\Rightarrow \int_V \left[\underset{\sim}{\nabla} \cdot \rho\, \underset{\sim}{u}\, \underset{\sim}{u} + \underset{\sim}{\nabla}(p + \chi) \right] dV = 0$$

and via the Div Thm :

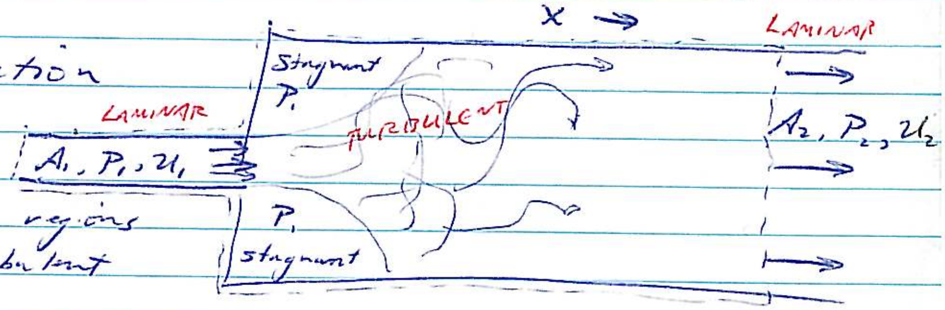
$$\int_S \Big\{ \underbrace{\rho (\underset{\sim}{u} \cdot \underset{\sim}{n})\, \underset{\sim}{u}}_{\text{MOMENTUM FLUX}} + \underbrace{(p + \chi)\, \underset{\sim}{n}}_{\text{BOUNDARY FORCES ASSOCIATED WITH P and BODY FORCE}} \Big\} dS = 0$$

$\Rightarrow$ This is an expression of the conservation of momentum and may be used even if the intermediate flow is turbulent

e.g. Expanding Pipe Junction

What is ΔP?

- jet emerges from narrow inlet leaving stagnant corner regions
- can't use Bernoulli $\because$ turbulent



Momentum Integral in downstream ($\hat{x}$) direction :

$$-\rho u_1^2 A_1 - P_1 A_2 + P_2 A_2 + \rho u_2^2 A_2 = 0$$

Mass Conservation : $\quad A_1 u_1 = A_2 u_2$

$$\Rightarrow \quad P_2 - P_1 = \rho u_1^2 \frac{A_1}{A_2}\left(1 - \frac{A_1}{A_2}\right)$$

Note : 1. this expression is only valid for $A_2 > A_1$ and gives the pressure drop due to the sudden widening (i.e. that due to the resulting TURBULENCE).

2. if $A_1 = A_2$; this inviscid model predicts $P_2 - P_1 = 0$

+ Maxworthy's bubble expt.

LECTURE 18

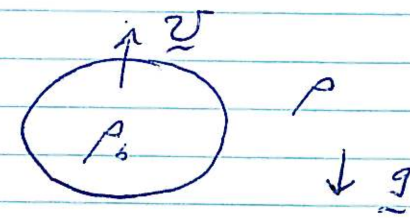
Nov. 13

Today:

1. Collect PS 4
2. PS #5 $\Rightarrow$ Handout
3. DEMO for PS #4
4. Translation of Body in 3D Irrotational Flow

$\Rightarrow$ Concept of Added Mass

Preliminary Comment:



Release bubble from rest.
Q: what is its initial acceleration?

$$\underbrace{\rho_b \mathcal{V} \dot{\underset{\sim}{U}}}_{\text{"Ma"}} = \underbrace{\rho_b \mathcal{V} \underset{\sim}{g}}_{\text{Weight}} - \underbrace{\rho \underset{\sim}{g}}_{\text{Buoyancy}} \mathcal{V} \quad \Leftarrow \text{NAIVE GUESS}$$

$$\Rightarrow \dot{\underset{\sim}{U}} = \frac{\rho_b - \rho}{\rho_b} \underset{\sim}{g} \approx -10^3 \underset{\sim}{g}$$

ie. in 1 sec, should rise $d = \frac{1}{2} 10^3 g t^2 \sim 10^6 \text{cm} = 10 \text{ km}$

$\Rightarrow$ MUST BE WRONG

Lecture 20 : 2001

Today : 1 Reminder PS #5 due M

2. Water strider video

3. Added Mass

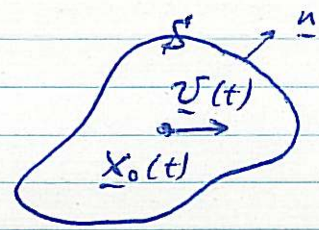
4. Stream Fn + Complex Potentials

Translation of a body in 3-D Irrotational Flow ... (continued)

$\Rightarrow$ recall the bubble paradox

We were seeking a sol'n

$$\phi(\underline{x}, t) = \underline{U} \cdot \underbrace{\underline{\Phi}(\underline{x} - \underline{x}_0(t))}_{\text{DEPENDS ONLY ON BODY SHAPE}}$$



We found that the KE of the entire fluid may be simply expressed:

$$T = \tfrac{1}{2} \alpha_{ij} U_i U_j$$

where $\alpha_{ij} = -\rho \int_{\text{BODY}} \Phi_i n_j \, dS$ is the ADDED MASS TENSOR

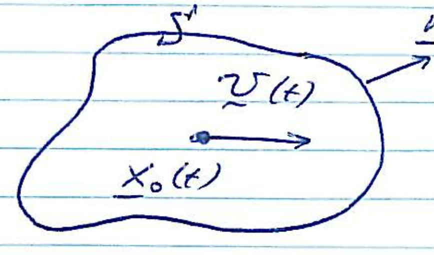
$\rightarrow$ WE SHALL SEE HOW THIS ARISES IN A FORCE ON THE BODY

Eg. For a sphere, ...

S_∞

Translation of a Body in 3-D Irrotational Flow

$$\nabla^2 \phi = 0$$



B.C.s 1. $\underline{n} \cdot \nabla \phi = \underline{n} \cdot \underline{U}$ on body

2. $\nabla \phi \to 0$ as $|\underline{x}| \to \infty$

Note: problem is linear in $\underline{U}(t)$ and position only occurs as position relative to body. We thus seek a sol'n

$$\phi(\underline{x}, t) = \underline{U} \cdot \underline{\Phi}(\underline{x} - \underline{x}_0(t))$$

The vector quantity $\underline{\Phi}$ necessarily satisfies

$\nabla^2 \underline{\Phi} = 0$ B.C.s 1. $\underline{n} \cdot \nabla \underline{\Phi} = \underline{n}$ on body

2. $\nabla \underline{\Phi} \to 0$ as $|\underline{x}| \to \infty$

$\Rightarrow$ $\underline{\Phi}$ is a function that depends only on the geometry of the body and not on its motion

Kinetic Energy of Fluid

$$T = \int_{\text{fluid}} \tfrac{1}{2} \rho u^2 \, dV = \int_{\text{fluid}} \frac{\rho}{2} \nabla \cdot (\phi \nabla \phi) \, dV$$

$$= \frac{1}{2} \rho \int_{S + S_\infty} \phi (\hat{\underline{n}} \cdot \nabla \phi) \, dS$$ by Div Thm, where $\hat{\underline{n}}$ is outward normal to fluid

$$= - \int_{S_{body}} \frac{\rho}{2} \phi \frac{\partial \phi}{\partial n} \, dS$$ where $\underline{n}$ is outward normal to body

Note: integral over S_∞ vanishes since $\phi, \frac{\partial \phi}{\partial n} \to 0$ there

Formally, $\phi \sim \frac{1}{r^2}$ (dipole as in sphere case previously considered

$\frac{d\phi}{dr} \sim \frac{1}{r^3}$

$$\Rightarrow \int_{S_\infty} \phi \frac{\partial \phi}{\partial n} \, dS \sim \int_{S_\infty} \frac{1}{r^5} \, dS \sim \lim_{R \to \infty} \frac{R^2}{R^5} \to 0$$

$$\Rightarrow T = -\frac{1}{2}\rho\int_{BODY} \overbrace{\underline{V}\cdot\underline{\underline{\Phi}}}^{\phi}\ \overbrace{\underline{V}\cdot\underline{n}}^{\frac{\partial\phi}{\partial n}\text{ by B.C. I}}\, dS$$

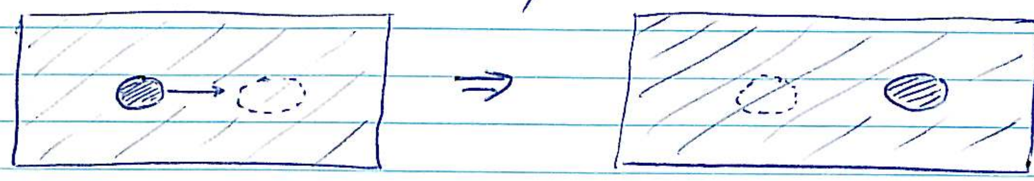
$$= \frac{1}{2}\alpha_{ij}V_iV_j \quad = -\frac{1}{2}\rho\,\underline{V}\cdot\left[\int_{BODY}\underline{\underline{\Phi}}\,\underline{n}\,dS\right]\cdot\underline{V}$$

where $\boxed{\alpha_{ij} = -\rho\int_{BODY}\Phi_i n_j\,dS}$ "ADDED MASS" or "VIRTUAL MASS" OF THE BODY

⇒ FN OF BODY FORM (shape, orientation) ONLY

⇒ in order to accelerate the body, we must supply KE to the fluid as well as the body

i.e.



Eg. for a sphere, we have seen $\phi = -\frac{1}{2}\frac{a^3}{r^3}\underline{V}\cdot\underline{x}$,

so that $\underline{\underline{\Phi}} = -\frac{1}{2}\frac{a^3}{r^3}\underline{x}$, and

$$\alpha_{ij} = \frac{\rho}{2}\int_{sphere}\cancel{\frac{a^3}{r^3}}^{\,1}\underline{n}\,\underline{x}\,dS = \frac{\rho}{2}\int_{sphere}\overbrace{\underline{\nabla}\,\underline{x}}^{\underline{\underline{I}}}\,dV \quad \text{by Gen D.N Thm}$$

$$\underline{\underline{\alpha}} = \frac{\rho}{2}\left(\frac{4\pi a^3}{3}\right)\underline{\underline{I}}$$

which is an isotropic tensor with magnitude equal to one half of the mass of the displaced fluid

Force on the Body

Recall $\underline{F} = \int_S \underline{\underline{T}}\cdot\underline{n}\,dS = -\int_S p\,\underline{n}\,dS$ for inviscid flow

We deduce p from time-dep Bernoulli:

$$p + \rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho u^2 - \rho\,\underline{g}\cdot\underline{x} = \text{Const} = 0 \ (\text{say})$$

Recall $\phi = \underline{V}(t)\cdot\underline{\underline{\Phi}}(\underline{x}-\underline{x}_0(t))$

$$\Rightarrow \frac{\partial\phi}{\partial t} = \dot{\underline{V}}\cdot\underline{\underline{\Phi}} - (\underset{=\underline{V}}{\dot{\underline{x}}_0}\cdot\underbrace{\underline{\nabla}\,\underline{\underline{\Phi}})\cdot\underline{V}}_{\nabla\phi} = \dot{\underline{V}}\cdot\underline{\underline{\Phi}} - \underline{V}\cdot\underline{\nabla}\phi$$

$$= \dot{\underline{V}}\cdot\underline{\underline{\Phi}} - \underline{V}\cdot\underline{u}$$

$$\frac{\partial\phi}{\partial t} = \dot{\underline{U}}\cdot\underline{\Phi} - \underline{U}\cdot\underline{u}$$

so $$-p = \rho\left[\dot{\underline{U}}\cdot\underline{\Phi} - \underline{U}\cdot\underline{u}\right] + \tfrac{1}{2}\rho u^2 - \rho\underline{g}\cdot\underline{x}$$

and $$\underline{F} = \int_S \rho\Big[\underbrace{\dot{\underline{U}}\cdot\underline{\Phi}}_{\text{ACCELERATION OF FLUID}} - \underbrace{\underline{U}\cdot\underline{u} + \tfrac{1}{2}u^2}_{\text{STEADY FORCE}} - \underbrace{\underline{g}\cdot\underline{x}}_{\text{BUOYANCY}}\Big]\underline{n}\,dS$$

Consider each force component in turn:

Buoyancy Force

$$\underline{F}_b = \rho\int_S -(\underline{g}\cdot\underline{x})\,\underline{n}\,dS = \rho\int -\underline{\nabla}(\underline{g}\cdot\underline{x})\,dV \quad \text{by Div Thm}$$

$$= -\rho\underline{g}\int_V dV = -\rho V\underline{g}$$

Archimedes: buoyancy force = weight of displaced fld

Steady Force: $\underline{F}_s$:

Note: $\underline{u} \neq \underline{U}$ necessarily on bdry since no-slip does not apply in inviscid flows, but $\underline{n}\cdot\underline{u} = \underline{n}\cdot\underline{U}$ ✩

Consider $$\int_S \tfrac{1}{2}u^2\underline{n}\,dS = \int_{\substack{\text{FLUID}\\ \text{VOL}\ V}} \underline{\nabla}\left(\frac{u^2}{2}\right)dV$$

$$= \int_V \underline{u}\cdot\underline{\nabla}\underline{u} + \underline{u}\wedge\cancelto{0}{(\underline{\nabla}\wedge\underline{u})}\,dV$$

$$= \int_V \underline{\nabla}\cdot(\underline{u}\,\underline{u}) - \underline{u}\,\cancelto{0}{\underline{\nabla}\cdot\underline{u}}\,dV$$

$$= \int_S \underline{n}\cdot\underline{u}\,\underline{u}\,dS = \int_S \underline{n}\cdot\underline{U}\,\underline{u}\,dS \quad \text{by ✩}$$

$$\therefore\ \underline{F}_s = \rho\int_S (\underline{n}\cdot\underline{U})\underline{u} - (\underline{U}\cdot\underline{u})\underline{n}\,dS$$

$$\left[F_i^s = \rho\int_S u_i n_j U_j - u_j U_j n_i\,dS\right]$$

$$\underline{F}_s = \rho \int_S \underline{n} \cdot \underline{V}\, \underline{u} - \underline{n}(\underline{V} \cdot \underline{u})\, dS$$

$$= \rho \int_{V_{FLUID}} \underline{\nabla} \cdot (\underline{V}\, \underline{u}) - \underline{\nabla}(\underline{V} \cdot \underline{u})\, dV$$

$$= \rho \int_V \cancelto{0}{(\underline{\nabla} \cdot \underline{V})}\underline{u} + \underline{V} \cdot \underline{\nabla}\, \underline{u} - \underline{\nabla}(\underline{V} \cdot \underline{u})\, dV$$

$$= \rho \int_V V_j \frac{\partial u_i}{\partial x_j} \hat{e}_i - \hat{e}_i \frac{\partial}{\partial x_i}(V_j u_j)\, dV$$

$$= \rho \int_V V_j \hat{e}_i \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) dV$$

$$= 0 \quad \text{since} \quad \underline{\nabla} \wedge \underline{u} = 0$$

$\Rightarrow$ Most general form of D'Alembert's Paradox:

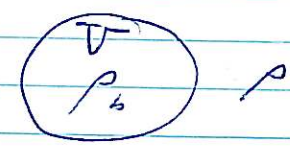
STEADY FORCE VANISHES IN IRROTATIONAL FLOW (even for unsteady motion)

Acceleration Force: $\underline{F}^a = \int_S \rho\, \underline{\dot{V}} \cdot \underline{\Phi}\, \underline{n}\, dS$

$$F_i^a = \rho \int_S \dot{V}_j \Phi_j n_i\, dS = \dot{V}_j \left(\rho \int_S \Phi_j n_i\, dS \right)$$

$$= -\alpha_{ij} \dot{V}_j$$

i.e. $\underline{F}^a = -\underline{\underline{\alpha}} \cdot \underline{\dot{V}}$



Example: Rising spherical bubble $\left(\underline{\underline{\alpha}} = \frac{1}{2}\rho V \underline{\underline{I}}\right)$

$$\underbrace{\rho_b V \underline{\dot{V}}}_{\text{"ma"}} = \underbrace{\rho_b V \underline{g}}_{\text{BUBBLE WEIGHT}} + \Big[\underbrace{-\tfrac{1}{2}\rho V \underline{\dot{V}}}_{\text{ACCELERATION "ADDED MASS"}} \underbrace{- \rho \underline{g} V}_{\text{BUOYANCY FORCE}} \Big]$$

$$\Rightarrow \underline{\dot{V}} = \left(\frac{\rho_b - \rho}{\rho_b + \frac{1}{2}\rho} \right) \underline{g} \simeq -2\underline{g} \quad \text{for } \rho_b \ll \rho$$

$\Rightarrow$ Gas bubble accelerates at twice acceleration of gravity!

Lecture 22 2001

$\Rightarrow$ Sports Balls

Lecture 23

Today: Handout PS #6 $\rightarrow$ due

1. Lubrication Theory

Lecture 26 2002

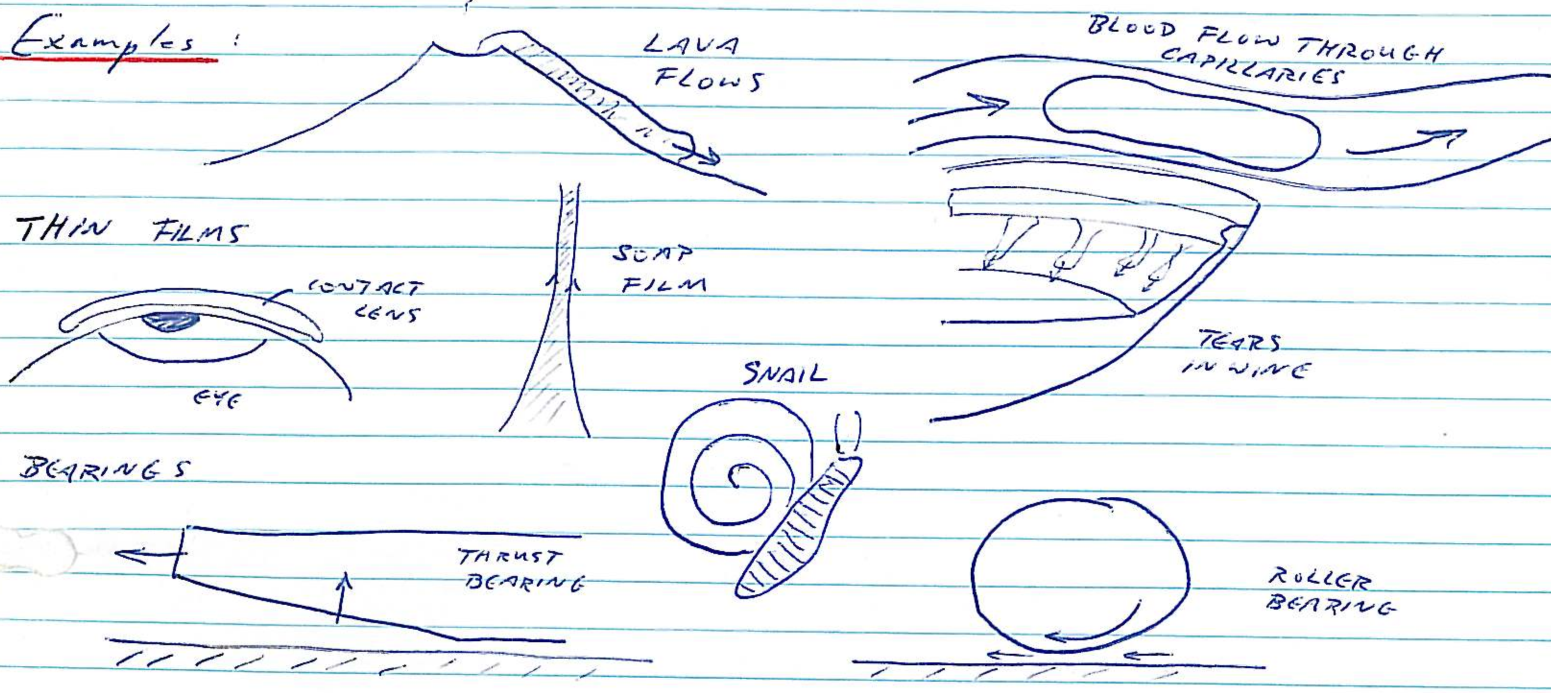
Prelims: PS #6, Return PS #5

Today: Lub Theory

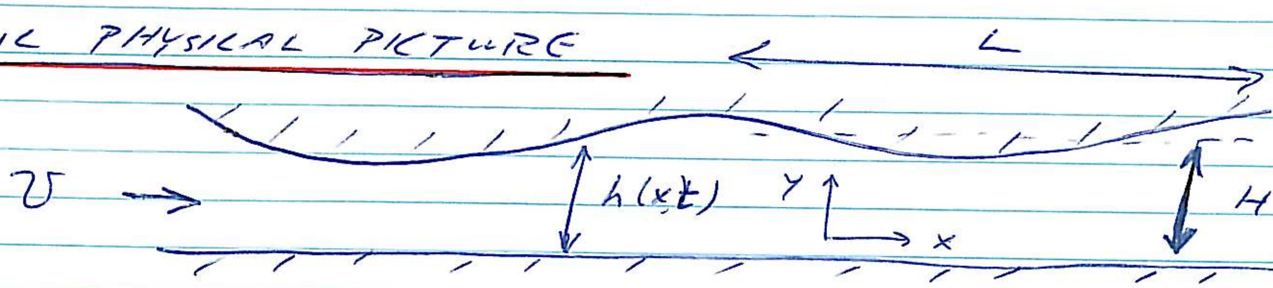
LUBRICATION THEORY (nearly unidirectional flows)

- useful for many flow situations where the lengthscale in the direction of the flow greatly exceeds that across the flow

Examples:



GENERIC PHYSICAL PICTURE



Width of fluid gap $h(x,t)$ must vary slowly with x.
Write $\underline{u} = (u, v)$.

Nondimensionalize N-S Eqns:

$$u' = \frac{u}{U}, \quad v' = \frac{v}{V}, \quad t' = \frac{t}{L/U}$$

$$x' = \frac{x}{L}, \quad y' = \frac{y}{H}, \quad p' = \frac{p}{P_0}$$

where H, L are characteristic distances in y, x-directions
U, V " " velocities in x, y- "
where $L \gg H$

Continuity :

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{U}{L}\frac{\partial u'}{\partial x'} + \frac{V}{H}\frac{\partial v'}{\partial y'} = 0$$

$$\Rightarrow \frac{U}{L} \sim \frac{V}{H} \quad \underline{\text{ie.}} \quad V \sim \frac{H}{L} U \ll U$$

$\Rightarrow$ cross-stream velocity is small : flow is almost unidirectional

Momentum Equations

$$\hat{x}: \quad \rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho \frac{U^2}{L}\frac{Du'}{Dt'} = -\frac{P_0}{L}\frac{\partial p'}{\partial x'} + \mu\Big(\underbrace{\frac{U}{L^2}}_{\text{SMALL}}\frac{\partial^2 u'}{\partial x'^2} + \underbrace{\frac{U}{H^2}}_{\text{DOMINANT VISCOUS TERM}}\frac{\partial^2 u'}{\partial y'^2}\Big)$$

Scaling :

$$\frac{\text{INERTIA}}{\text{VISCOSITY}} \sim \left(\frac{\mu U/H^2}{\rho U^2/L}\right)^{-1} = \frac{UH}{\nu}\frac{H}{L} = Re\,\frac{H}{L}$$

Inertial terms are thus small relative to viscous terms provided $Re\,\frac{H}{L} \ll 1$. Note Re itself need not be small.

In this limit, the viscous stresses must balance the pressure force:

$$\underline{\text{ie.}} \quad 0 = -\frac{P_0}{L}\frac{\partial p'}{\partial x'} + \frac{\mu U}{H^2}\frac{\partial^2 u'}{\partial y'^2}$$

$$\Rightarrow P_0 = \frac{\mu L U}{H^2} \quad \text{is the characteristic pressure} \left(\gg \frac{\mu U}{H}\right)$$

The redimensionalized $\hat{x}$-momentum eqn is thus (to leading order) :

$$\boxed{\mu \frac{\partial^2 u}{\partial y^2} = + \frac{\partial p}{\partial x}}$$

Note : same as for unidirectional flow, but $\frac{\partial p}{\partial x}$ is not necessarily constant.

$$\hat{y}\text{-mom}: \quad \rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\rho\frac{UV}{L}\frac{Dv'}{Dt'} = -\frac{\mu L U}{H^3}\frac{\partial p'}{\partial y'} + \mu\left(\frac{V}{L^2}\frac{\partial^2 v'}{\partial x'^2} + \frac{V}{H^2}\frac{\partial^2 v'}{\partial y'^2}\right)$$

Use $V = \frac{H}{L}U$

$$\Rightarrow \underbrace{\frac{UH}{\nu}\frac{H^3}{L^3}}_{\text{SMALL}}\frac{Dv'}{Dt'} = -\frac{\partial p'}{\partial y'} + \underbrace{\frac{H^4}{L^4}}_{\text{SMALL}}\frac{\partial^2 v'}{\partial x'^2} + \underbrace{\frac{H^2}{L^2}}_{\text{SMALL}}\frac{\partial^2 v'}{\partial y'^2}$$

Leibnitz Formula:

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(y,x)\,dy = \int_{g(x)}^{h(x)} \frac{\partial f(y,x)}{\partial x}\,dy + f[h(x),x]\frac{dh(x)}{dx} - f[g(x),x]\frac{dg(x)}{dx}$$

To leading order in $\varepsilon = \frac{H}{L} \ll 1$, the y-momentum eqn thus yields

$$\frac{\partial p}{\partial y} = 0 \Longrightarrow \boxed{P = P(x,t)} \text{ only}$$

We may thus integrate the $\hat{x}$-mom eqn: $\frac{\partial^2 u'}{\partial y'^2} = -\frac{\partial p'}{\partial x'}$

$$\Rightarrow u'(x',y',t') = \frac{1}{2}\left(\frac{\partial p'}{\partial x'}\right)y'^2 + C_1(x')\,y' + C_2(x')$$

NEXT PG

$\Rightarrow$ at any x', this is a parabolic velocity profile determined by $\frac{dp'}{dx'}$ and the appropriate BCs applied at upper & lower B.C.s

Determine $C_i(x')$ by applying BC.s

eg. B.C. 1. $u'(x',y',t') = 0$ at $y' = 0$

2. $u'(x',y',t') = U'(x',t)$ at $y' = h'(x',t')$

where $U' = \frac{u_{bdry}}{U} = \frac{U^*}{U}$, $h' = \frac{h(x,t)}{H}$

ve: $u'(x',y',t') = \frac{1}{2}\left(\frac{\partial p'}{\partial x'}\right)y'(y'-h') + U'\frac{y'}{h'}$

Conservation of Mass

- conserve mass over lengthscales comparable to $L \gg H$
Recall: $h(x)$ slowly varying.

Volume flux: $Q = \int_0^{h(x,t)} u(x,y,t)\,dy$

$$\Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\int_0^{h(x,t)} u(x,y,t)\,dy \quad \Rightarrow \text{evaluate via Leibnitz' Rule}$$

$$= \int_0^h \frac{\partial u}{\partial x}\,dy + \frac{\partial h}{\partial x}\cdot u(x,h,t)$$

By continuity, $\int_0^h \frac{\partial u}{\partial x}\,dy = \int_0^h -\frac{\partial v}{\partial y}\,dy = -v\Big|_0^{h(t)} = -V(h)$

SPEED OF UPPER BDDY

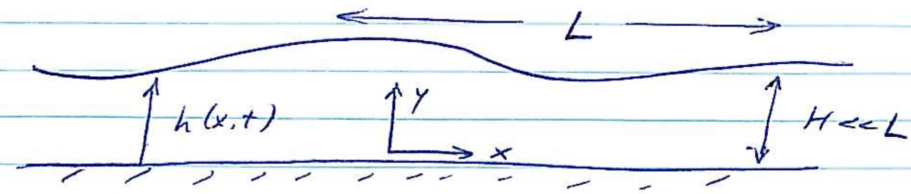
We thus have

$$\frac{\partial Q}{\partial x} = -V^* + U^*\frac{\partial h}{\partial x}$$

where U^*, V^* are $\hat{x}$- and $\hat{y}$-velocities of upper bdry

$$\Rightarrow \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = U^*\frac{\partial h}{\partial x} \quad \text{since } V^*_{(x)} = \frac{\partial h(x)}{\partial t}$$

Recap: LUBRICATION THEORY



$$\underset{\sim}{u} = (u, v) \sim (U, V)$$

We deduced $V \sim \frac{H}{L} U << U$

$$P_0 \sim \frac{\mu U L}{H^2}$$

$\hat{x}$-mom: $\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}$ ☆ $\Rightarrow$ PROCEED

$\hat{y}$-mom: $\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$

$$\rho \frac{UV}{L} \frac{Dv'}{Dt'} = -\frac{\mu L U}{H^3} \frac{\partial p'}{\partial y'} + \mu \left(\frac{V}{L^2} \frac{\partial^2 v'}{\partial x'^2} + \frac{V}{H^2} \frac{\partial^2 v'}{\partial y'^2} \right)$$

Use $V = U\frac{H}{L} \Rightarrow \underbrace{\frac{UH}{\nu}\left(\frac{H}{L}\right)^3 \frac{Dv'}{Dt'}}_{\text{small}} = \underbrace{-\frac{\partial p'}{\partial y'}}_{O(1)} + \underbrace{\left(\frac{H}{L}\right)^4 \frac{\partial^2 v'}{\partial x'^2}}_{\text{small}} + \underbrace{\left(\frac{H}{L}\right)^2 \frac{\partial^2 v'}{\partial y'^2}}_{\text{small}}$

To leading order in $\varepsilon = \frac{H}{L} << 1$, the y-momentum eqn thus yields

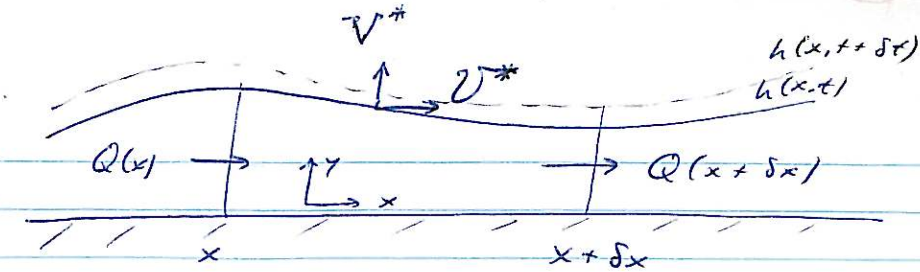
☆ $\frac{\partial p}{\partial y} = 0 \Rightarrow p = p(x,t)$ only

We may thus integrate the $\hat{x}$-mom eqn ☆ to find

$$u(x,y,t) = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + C_1(x,t)\, y + C_2(x,t)$$

$\Rightarrow$ at any x, this is a parabolic velocity profile determined by $\frac{\partial p}{\partial x}$ and the appropriate B.C.s applied at upper + lower bodies

We consider here the special case where the upper body motion is prescribed, in order to get a useful expression for continuity.



Conservation of Mass

Volume Flux: $$Q = \int_0^{h(x,t)} u(x,y,t)\,dy$$

$$\Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\int_0^{h(x,t)} u(x,y,t)\,dy$$

Recall Leibnitz: $$\frac{d}{dx}\int_{g(x)}^{h(x)} f(y,x)\,dy = \int_{g(x)}^{h(x)} \frac{\partial f(y,x)}{\partial x}\,dy + f[h(x),x]\frac{dh}{dx} - f[g(x),x]\frac{dg}{dx}$$

$$\Rightarrow \frac{\partial Q}{\partial x} = \int_0^h \frac{\partial u}{\partial x}\,dy + \frac{\partial h}{\partial x}\cdot \underbrace{u(x,h,t)}_{\longrightarrow\ U^* \text{ (body speed)}}$$

By continuity: $$\int_0^h \frac{\partial u}{\partial x}\,dy = \int_0^h -\frac{\partial v}{\partial y}\,dy = -v\Big|_0^{h(t)} = -V^*$$

We thus have $$\frac{\partial Q}{\partial x} = -V^* + U^*\frac{\partial h}{\partial x}$$

where U^*, V^* are the upper body speeds. Since $V^* = \frac{\partial h}{\partial t}$, we may now write

$$\boxed{\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = U^*\frac{\partial h}{\partial x}} \qquad \odot$$

When $h(x)$ is slowly varying, this reduces to

$$\boxed{\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0}$$

Note: $\odot$ may also be derived by writing

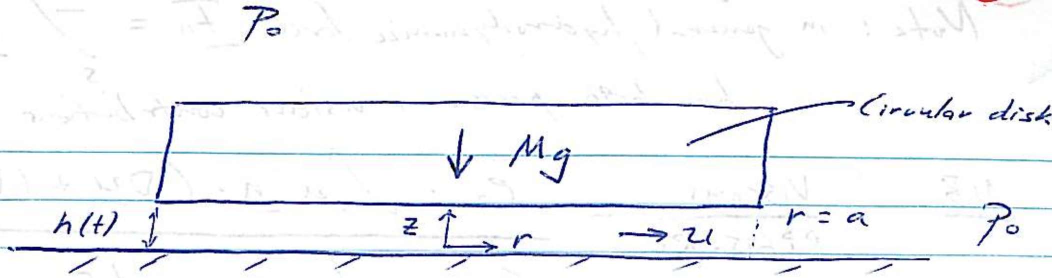
$$\frac{Dh}{Dt} = \frac{\partial h}{\partial t} - U^*\frac{\partial h}{\partial x} = \frac{\partial Q}{\partial x}$$

MATERIAL DERIVATIVE IN FRAME MOVING WITH UPPER BODY

UNITS : $\dfrac{\frac{L^2}{\cancel{T}}\frac{L^2}{T^2}}{\frac{L}{\cancel{T^2}}\cancel{T}} \sim \dfrac{L^3}{T^2}$

$$L \sim \left(\frac{g\bar{V}^3}{L}\right)^{\frac{1}{5}} t^{\frac{1}{5}}$$

$$\sim \left(\frac{\frac{L}{T^2}\frac{L^3}{T^3}}{L^2/T}\right)^{\frac{1}{5}} T^{\frac{1}{5}} =$$



Squeeze Film

In the lubrication limit, the Navier Stokes equations yield

$$\frac{\partial p}{\partial z} = 0 \quad \text{and} \quad 0 = -\frac{\partial p}{\partial r} + \mu \frac{\partial^2 u}{\partial z^2}$$

$$\Rightarrow p = p(r) \text{ only}$$

B.C.s
1. $u = 0$ on $z = 0$
2. $u = 0$ on $z = h(t)$

Velocity Profile : $u = -\frac{1}{2\mu}\frac{dp}{dr} z(h - z)$

Vol Flux : $Q(r) = \int_0^h u(r) \cdot 2\pi r \, dz$

$$\Rightarrow Q(r) = -\frac{\pi r}{\mu}\frac{dp}{dr}\int_0^h zh - z^2 \, dz = -\frac{\pi r}{\mu}\frac{dp}{dr}\left[\frac{h^3}{2} - \frac{h^3}{3}\right]$$

$$Q(r) = -\frac{\pi r}{6\mu}\frac{dp}{dr} h^3$$

Conservation of Mass : $\pi r^2 \dot{h} = Q(r) = \frac{\pi r h^3}{6\mu}\frac{dp}{dr}$

$$\Rightarrow \frac{\partial p}{\partial r} = 6\mu \frac{\dot{h}}{h^3} r \Rightarrow p(r) = \frac{3\mu \dot{h}}{h^3}(r^2 - a^2) + P_o$$

Force Balance : (note again viscous contribution to F_r is negligible) * see over

$$Mg = \int_0^a 2\pi r (p(r) - P_o) \, dr = \frac{6\pi \mu \dot{h}}{h^3}\int_0^a (r^3 - a^2 r) dr$$

$$= \frac{6\pi\mu\dot{h}}{h^3}\left(\frac{a^4}{4} - \frac{a^4}{2}\right) = -\frac{3\pi\mu\dot{h}}{2h^3} a^4$$

$$\int_{h_o}^{h} \frac{dh}{h^3}\left(\frac{-3\pi\mu a^4}{2Mg}\right) = \int_0^t dt \Rightarrow t = \frac{3\pi\mu a^4}{4Mg}\left(\frac{1}{h^2} - \frac{1}{h_o^2}\right)$$

is time for disc to move from h_o to h

$$\Rightarrow \boxed{h(t) = \left(\frac{1}{h_o^2} + \frac{4}{3}\frac{Mgt}{\pi\mu a^4}\right)^{-\frac{1}{2}}}$$

Note: plate takes an infinite time for $h \to 0$

Note: in general, hydrodynamic force $\underline{F}_H = -\int_S \underline{n}\cdot\underline{\underline{T}}\,dS$ has both pressure + viscous contributions

NB.
$$\frac{\text{VISCOUS}}{\text{PRESSURE}} = \frac{\hat{e}_z \cdot \int_S \mu\, \underline{n}\cdot\left(\underline{\nabla u} + (\underline{\nabla u})^T\right) dS}{\hat{e}_z \cdot \int_S -p\,\underline{n}\,dS}$$

$$\simeq \frac{\mu \frac{V}{H} L}{\mu \frac{VL}{H^2} L} = \frac{H^2}{L^2} = \varepsilon^2 \ll 1$$

$\Rightarrow$ pressure component dominant

2001

Lecture 24 → Taylor Film
26 /2002

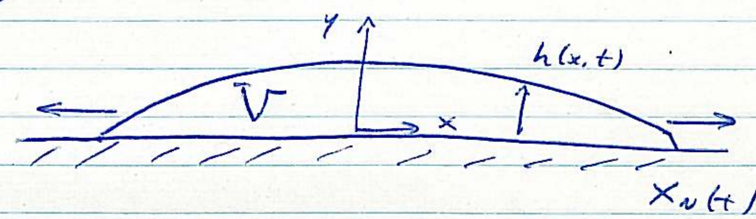
Lecture 25

Today : 0. Any Q's → film?

1. 2D Gravity Current ⇒ Simpson's book

2. Squeeze film

Last time, we were considering a constant volume release of lava from a crack.



Simple scaling suggested

$$L \sim \left(\frac{g V^3 T}{\nu}\right)^{\frac{1}{5}} \sim t^{\frac{1}{5}}$$

$$H \sim \left(\frac{\nu V^2}{g T}\right)^{\frac{1}{5}} \sim t^{-\frac{1}{5}}$$

By using, ...

$$u_x + w_z = 0$$

$$h_t + \underbrace{u_s}_{\text{surface}} h_x = w \quad \Rightarrow \quad \text{Kinematic Condition}$$

$$\int_0^h u_x \, dz + w_s = 0$$

$$\int_0^h u_x \, dz + h_t + u h_x = 0$$

$$\Rightarrow \quad \frac{\partial Q}{\partial x} + h_t = 0$$

$$Q = \int_0^h u \, dz$$

$$\frac{\partial Q}{\partial x} = \int_0^h \frac{\partial u}{\partial x} \, dz + u_s h_x$$

Continuity : last time

2-135

Lecture ~~26~~ 27 : 2001/2

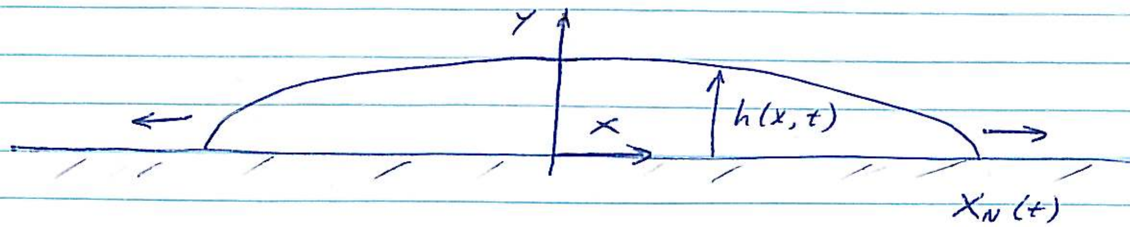
Prelims 1. Reschedule → 2nd class next week : T/W ?
→ Following week W/R/F ?

2. Course Projects

Today: 1. a final example of LWB THEORY

Viscous Gravity Current : 2D geometry

- constant volume lava release from a crack



vol V = const
$\sim LH$

Note: we can extract some information from scaling.
In the lubrication limit, viscous terms dominate inertia :

$\hat{y}$-mom : $\dfrac{\partial p}{\partial y} \sim -\rho g \Rightarrow P \sim \rho g H$

$\hat{x}$-mom : $\dfrac{\partial p}{\partial x} \sim \rho \nu \nabla^2 u \Rightarrow \dfrac{\rho g H}{L} \sim \rho \nu \dfrac{u}{H^2}$

We note that $u \sim \dfrac{L}{T}$ and $HL \sim V$, so that

$$uL \sim \frac{L^2}{T} \sim \frac{\cancel{\rho} g H^3}{\cancel{\rho} \nu} \sim \frac{\cancel{\rho} g}{\cancel{\rho} \nu} \frac{V^3}{L^3}$$

$$\Rightarrow L \sim \left(\frac{\cancel{\rho} g V^3}{\nu}\right)^{\frac{1}{5}} t^{\frac{1}{5}} \qquad \text{i.e. } L \sim t^{\frac{1}{5}} \,!$$

$$\text{and } H \sim \left(\frac{\nu V^2}{g}\right)^{\frac{1}{5}} t^{-\frac{1}{5}} \qquad H \sim t^{-\frac{1}{5}}$$

By using the eqns from lubrication, we may prove this predicted scaling, and deduce a soln for the gravity current shape.

$\hat{y}$-mom : $\dfrac{\partial p}{\partial y} = -\rho g \Rightarrow p(x,y,t) = p_0 - \rho g (y - h(x,t))$

so that $\dfrac{\partial p}{\partial x} = \rho g \dfrac{\partial h}{\partial x}$

$\hat{x}$-mom : $\mu \dfrac{\partial^2 u}{\partial y^2} = + \dfrac{\partial p}{\partial x} = + \rho g \dfrac{\partial h}{\partial x}$

integrating : $u = \dfrac{1}{2\mu} \dfrac{\partial p}{\partial x} y^2 + C_1(x,t)\, y + C_2(x,t)$

B.C.s : 1. $u(y=0) = 0 \Rightarrow C_2 = 0$

2. Vanishing tangential stress at surface : $\dfrac{\partial u}{\partial y} \approx 0$ at $y = h$

$$\Rightarrow \quad 0 = \frac{1}{\mu}\frac{\partial p}{\partial x} h + c_1(x,t) \quad \Rightarrow \quad c_1(x,t) = -\frac{1}{\mu}\frac{\partial p}{\partial x} h(x,t)$$

so that $\quad u(x,y,t) = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\, y\,(2h - y)$

Vol. Flux $\quad Q = \int_0^h u(x,y,t)\, dy = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\int_0^h (2yh - y^2)\, dy$

$$\Rightarrow \quad Q = -\frac{1}{3\mu}\frac{\partial p}{\partial x} h^3 = -\frac{\rho g}{3\mu} h^3 \frac{\partial h}{\partial x}$$

Mass Conservation: $\quad \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad$ (valid when $\frac{\partial h}{\partial x} << 1$)

i.e. $\quad \boxed{\frac{\partial h}{\partial t} - \frac{g}{3\nu}\frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right) = 0} \quad ☆$

must solve subject to $\quad \int_0^{x_N(t)} h(x,t)\, dx = V \quad$ (const)

and $\quad h(x_N(t)) = 0.$

We proceed by seeking a similarity solution with a form suggested by our scaling argument:

$$h = \left(\frac{\nu V^2}{g}\right)^{\frac{1}{5}} t^{-\frac{1}{5}} f(\eta) \qquad \text{i.e.} \quad \frac{h}{H} = f(\eta) = f\left(\frac{x}{L}\right)$$

where $\quad \eta = \frac{x}{L} = \left(\frac{\nu}{gV^3}\right)^{\frac{1}{5}} \frac{x}{t^{\frac{1}{5}}} \qquad$ so we need only determine $f(\eta)$

Define $\quad c_1 = \left(\frac{\nu V^2}{g}\right)^{\frac{1}{5}}, \quad c_2 = \left(\frac{\nu}{gV^3}\right)^{\frac{1}{5}} = \frac{c_1}{V}$ and sub h into ☆

Note: $\quad h = c_1 t^{-\frac{1}{5}} f(\eta), \quad \eta = \frac{c_1}{V} x\, t^{-\frac{1}{5}}$

$$\Rightarrow \frac{\partial \eta}{\partial t} = \frac{c_1}{V} x\left(-\frac{1}{5} t^{-6/5}\right) = -\frac{1}{5}\,\eta/t \quad , \quad \frac{\partial \eta}{\partial x} = \frac{c_1}{V} t^{-\frac{1}{5}}$$

$$\Rightarrow \frac{\partial h}{\partial t} = -\frac{1}{5} c_1 t^{-6/5} f(\eta) + c_1 t^{-\frac{1}{5}} f'(\eta) \frac{\partial \eta}{\partial t}$$

$$= -\frac{1}{5} c_1 t^{-6/5} f(\eta) - \frac{1}{5} c_1 t^{-6/5} \eta f'(\eta)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right) = c_1^4 t^{-4/5} \frac{\partial}{\partial x}\left(f^3 \frac{\partial f}{\partial x}\right)$$

Note: $\quad \frac{\partial}{\partial x} f(\eta) = \frac{\partial \eta}{\partial x}\frac{\partial f}{\partial \eta} = \frac{c_1}{V} t^{-\frac{1}{5}} f'(\eta)$

14.6

$$\Rightarrow \frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right) = \frac{c_1^6}{V^2} t^{-\frac{6}{5}} \frac{\partial}{\partial \eta}\left(f^3 \frac{\partial f}{\partial \eta}\right)$$

Subing into ✩ :

$$-\frac{1}{5} c_1 \cancel{t^{-6/5}} f - \frac{1}{5} c_1 \cancel{t^{-6/5}} \eta f' - \frac{9}{3V} \cancel{t^{-6/5}} c_1 \frac{c_1^5}{V^2} (f^3 f')' = 0$$

$$\Rightarrow \boxed{-\frac{1}{5}(\eta f)' - \frac{1}{3}(f^3 f')' = 0} \quad *$$

subject to $f(\eta_N) = 0$ (and $f'(0) = 0$), and

Volume Constraint : $\int_0^{x_N(t)} h(x,t)\, dx = V$

ie. $\int_0^{\eta_N} c_1 t^{-\frac{1}{5}} f(\eta) \frac{dx}{d\eta} d\eta = V$ but $\frac{d\eta}{dx} = \frac{c_1}{V} t^{-\frac{1}{5}}$

$$\Rightarrow \int_0^{\eta_N} f(\eta)\, d\eta = 1 \quad \odot$$

integrating $*$: $\frac{1}{5}\eta f + \frac{1}{3} f^{\cancel{3}\,2} f' = \cancel{\text{const}} \rightarrow 0$ since $f(\eta_N) = 0$

$$\Rightarrow \frac{d}{d\eta}\left(\frac{1}{10}\eta^2 + \frac{1}{9} f^3\right) = 0$$

$$\Rightarrow f^3 + \frac{9}{10}\eta^2 = \text{const}$$

Now $f(\eta_N) = 0 \Rightarrow 0 + \frac{9}{10}\eta_N^2 = \text{const}$

$$\Rightarrow f(\eta) = \left(\frac{9}{10}\right)^{\frac{1}{3}} (\eta_N^2 - \eta^2)^{\frac{1}{3}} = \left(\frac{9}{10}\right)^{\frac{1}{3}} \eta_N^{\frac{2}{3}} \left[1 - \left(\frac{\eta}{\eta_N}\right)^2\right]^{\frac{1}{3}}$$

where η_N is determined from the volume constraint $\odot$

$$\Rightarrow 1 = \left(\frac{9}{10}\right)^{\frac{1}{3}} \eta_N^{2/3} \int_0^{\eta_N} \left[1 - \left(\frac{\eta}{\eta_N}\right)^2\right]^{\frac{1}{3}} d\eta$$

Let $y = \frac{\eta}{\eta_N} \Rightarrow d\eta = \eta_N\, dy$

$$1 = \left(\frac{9}{10}\right)^{\frac{1}{3}} \eta_N^{5/3} \int_0^1 (1 - y^2)^{\frac{1}{3}} dy$$

$$\Rightarrow \eta_N = \left[\left(\frac{9}{10}\right)^{\frac{1}{3}} \int_0^1 (1-y^2)^{\frac{1}{3}} dy\right]^{-3/5}$$

$$= \left[\left(\frac{9}{10}\right)^{\frac{1}{3}} \frac{\sqrt{\pi}}{5} \frac{\Gamma(\frac{1}{3})}{\Gamma(5/6)}\right]^{-3/5} \simeq 1.133$$

where $\Gamma(x)$ is the Gamma Function

We thus have the shape of the gravity current valid for all time:

$$h(x,t) = \left(\frac{\nu V^2}{g}\right)^{\frac{1}{5}} t^{-\frac{1}{5}} \cdot \left(\frac{9}{10}\right)^{\frac{1}{3}} \eta_N^{\frac{2}{3}} \left[1 - \left(\frac{\eta}{\eta_N}\right)^2\right]^{\frac{1}{3}}$$

where $\eta = \left(\frac{\nu}{gV^3}\right)^{\frac{1}{5}} \frac{x}{t^{\frac{1}{5}}}$ and $\eta_N \simeq 1.133$

Note: analysis only valid provided inertial terms are negligible. According to lubrication theory, we require

$$\frac{UL}{\nu}\frac{h}{L} << 1$$

$$\Rightarrow t >> \left(\frac{V^4}{g^2 \nu^3}\right)^{\frac{1}{7}}$$

eg. for syrup: take $V \sim 10\ cm^2$

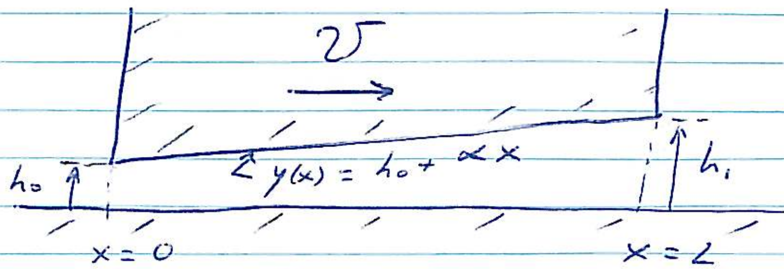
$g \sim 10^3\ cm/s^2$

$\nu \sim 10\ cm^2/s$

$$\Rightarrow t >> 10^{-5/7}s \sim 0.2s$$

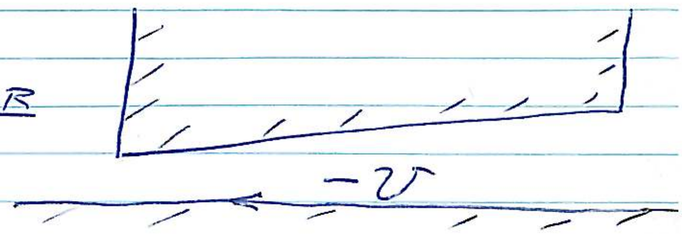
The Slider Bearing   e.g. thrust bearing

- illustrates how a lubrication flow can generate large stresses normal to the boundaries which are capable of supporting loads



AXES FIXED ON LOWER SURFACE

OR



AXES FIXED ON MOVING SLIDER

Take axes fixed on moving slider, and so consider the system

$$\mu \frac{d^2 u}{dy^2} = \frac{dp}{dx} \qquad \star$$

B.C.s 1. $u = -U$ on $y = 0$

2. $u = 0$ on $y = h(x) \equiv h_0 + \alpha x = h_0 + \frac{h_1 - h_0}{L} x$

G.S. of $\star$: $u(x) = \frac{1}{2\mu}\left(\frac{dp}{dx}\right) y^2 + C_1(x)\, y + C_2(x)$

B.C. 1 $\Rightarrow$ $-U = C_2(x)$

2. $\Rightarrow$ $0 = \frac{1}{2\mu}\left(\frac{dp}{dx}\right) h^2 + C_1(x)\, h - U$

$\Rightarrow$ $C_1(x) = \frac{U}{h} - \frac{1}{2\mu}\frac{dp}{dx} h$

$$\Rightarrow u(x,y) = \underbrace{-\frac{1}{2\mu}\frac{dp}{dx}\, y\,(h(x) - y)}_{\text{POISEUILLE}} - \underbrace{U\left(\frac{h(x) - y}{h(x)}\right)}_{\text{COUETTE}}$$

Volume Flux : $Q(x) = \int_0^{h(x)} u(x,y)\, dy$

$$Q = -\frac{1}{2\mu}\frac{dp}{dx}\left[h \cdot \frac{h^2}{2} - \frac{h^3}{3}\right] - Uh + U\frac{1}{\cancel{h}}\frac{h^{\cancel{2}}}{2}$$

$$Q(x) = -\frac{h^3}{12\mu}\frac{dp}{dx} - \frac{1}{2} U h$$

Mass Conservation (in lubrication limit) : $\underbrace{\frac{\partial h}{\partial t}}_{0} + \frac{\partial Q}{\partial x} = 0$ DERIVED IN LEC. 25

$\Rightarrow$ Q indep of x

$$Q_0 = -\frac{h^3}{12\mu}\frac{dp}{dx} - \frac{1}{2}Uh \qquad \text{indep of } x$$

$$\Rightarrow \quad \frac{dp}{dx} = \frac{-6\mu U}{h^2} - \frac{12\mu Q_0}{h^3}$$

$$\frac{dp}{dh} = \frac{dx}{dh}\frac{dp}{dx} = \frac{1}{\alpha}\left(-\frac{6\mu U}{h^2} - \frac{12\mu Q_0}{h^3}\right)$$

$$\Rightarrow \quad p(h) - p(h_0) = \frac{6\mu U}{\alpha}\left(\frac{1}{h} - \frac{1}{h_0}\right) + \frac{6\mu Q_0}{\alpha}\left(\frac{1}{h^2} - \frac{1}{h_0^2}\right) \; ✱$$

Now $p(0) = p(L) = p_0$ and $h(L) = h_1$, so

$$0 = \frac{6\mu U}{\alpha}\left(\frac{1}{h_1} - \frac{1}{h_0}\right) + \frac{6\mu}{\alpha}Q_0\left(\frac{1}{h_1^2} - \frac{1}{h_0^2}\right)$$

$$\Rightarrow \quad \boxed{Q_0 = -U\frac{h_1 h_0}{h_1 + h_0}}$$

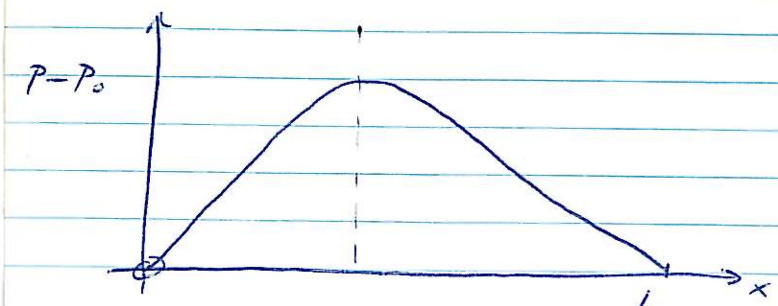
Thus the pressure is given by (from ✱)

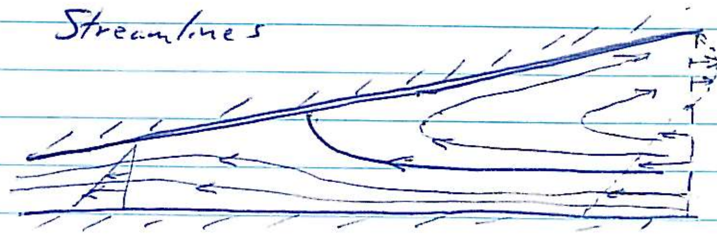
$$p(x) - p_0 = \frac{6\mu U}{\alpha}\left(\frac{h_0 - h}{h h_0}\right) + \frac{-6\mu U}{\alpha}\frac{h_1 \cancel{h_0}}{h_1 + h_0}\left(\frac{h_0^2 - h^2}{h^2 h_0^{\cancel{2}}}\right)$$

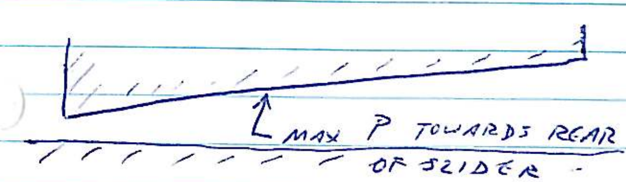
$$= \frac{6\mu U}{\alpha}\left[\frac{h(h_0 - h)(h_1 + h_0) - h_1(h_0^2 - h^2)}{h^2 h_0 (h_1 + h_0)}\right]$$

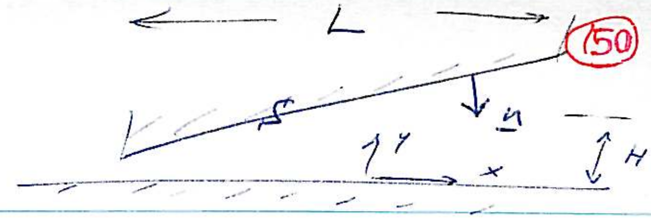
$$= \frac{6\mu U}{\alpha}\left[\frac{h h_0^{\cancel{2}} + h \cancel{h_0} h_1 - \cancel{h^2 h_1} - h^2 \cancel{h_0} - h_1 h_0^{\cancel{2}} + \cancel{h_1 h^2}}{h^2 \cancel{h_0} (h_1 + h_0)}\right]$$

$$= \frac{6\mu U}{\alpha}\frac{(h_1 - h)(h - h_0)}{h^2 (h_1 + h_0)} \qquad \text{vanishes at } h = h_0, h_1$$









Normal (vertical) Force on the slider :

Note : in general hydrodynamic force : $\underset{\sim}{F}_H = -\int_S \underline{n} \cdot \underline{\underline{T}} \, dS$
has both pressure + viscous contributions

Here, $\dfrac{\text{VISCOUS}}{\text{PRESSURE}} = \dfrac{\hat{e}_y \cdot \int_S \mu \, \underline{n} \cdot \left(\nabla \underline{u} + (\nabla \underline{u})^T \right) dS}{\hat{e}_y \cdot \int_S -p \, \underline{n} \, dS}$

$$\simeq \frac{\mu \frac{V}{H} \cdot L}{\mu \frac{VL}{H^2} \cdot L} = \frac{H^2}{L^2} = \varepsilon^2 << 1$$

The dominant hydrodynamic force on the slider is thus

$$F_N = \int_0^L (p - p_0) \, dx = \frac{6\mu V}{\alpha} \int_0^L \frac{h(h_0+h_1) - h^2 - h_0 h_1}{h^2 (h_1 + h_0)} \, dx$$

$$= \frac{6\mu V}{\alpha^2} \int_{h_0}^{h_1} \left[\frac{1}{h} - \frac{1}{(h_0+h_1)} - \frac{1}{h^2} \frac{h_0 h_1}{(h_0 + h_1)} \right] dh$$

$$= \frac{6\mu V}{\alpha^2} \left[\ln \frac{h_1}{h_0} - \frac{h_1 - h_0}{h_0 + h_1} + \left(\frac{1}{h_1} - \frac{1}{h_0} \right) \frac{h_0 h_1}{(h_0 + h_1)} \right]$$

$$= \frac{6\mu V}{\alpha^2} \left[\ln \frac{h_1}{h_0} + \frac{\cancel{h_0 h_1} (h_0 - h_1) + (h_0 - h_1) \cancel{h_0 h_1}}{\cancel{h_0 h_1} (h_0 + h_1)} \right]$$

$$= \frac{6\mu V}{\alpha^2} \left[\ln \left(\frac{h_1}{h_0} \right) - 2 \frac{h_1 - h_0}{h_1 + h_0} \right]$$

Tangential Stress on the slider

$$F_T = -\int_0^L \mu \left. \frac{\partial u}{\partial y} \right|_{y=h} dx$$

Recall $\quad u(x,y) = -\dfrac{1}{2\mu} \dfrac{dp}{dx} y(h-y) + \dfrac{Vy}{h} - V$

$$\left. \frac{\partial u}{\partial y} \right|_{y=h} = -\frac{1}{2\mu} \frac{dp}{dx} (h - 2h) + \frac{V}{h} = \frac{1}{2\mu} \frac{dP}{dx} h + \frac{V}{h}$$

$\Rightarrow$ Note : $\dfrac{dp}{dx} = -\dfrac{6\mu V}{h^2} - \dfrac{12 \mu Q_0}{h^3} = 6\mu V \left(-\dfrac{1}{h^2} + \dfrac{2 h_1 h_0}{h_1 + h_0} \dfrac{1}{h^3} \right)$

$$\Rightarrow \left. \frac{\partial u}{\partial y} \right|_{y=h} = \frac{6 \cancel{\mu} V}{2 \cancel{\mu}} \left(-\frac{1}{h} + \frac{2 h_1 h_0}{h_1 + h_0} \frac{1}{h^2} \right) + \frac{V}{h}$$

$$= -2 \frac{V}{h} + \frac{6 V h_1 h_0}{h_1 + h_0} \frac{1}{h^2}$$