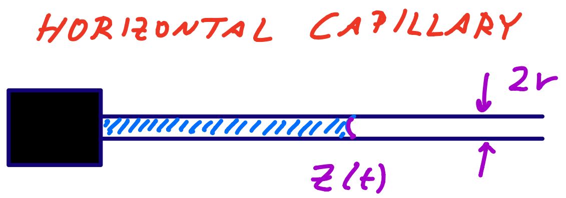
Lecture 8. Marangoni flows

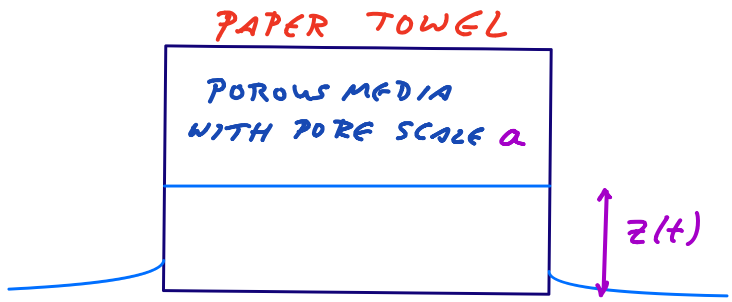
But first, a final word on capillary rise....

Wicking: in the viscous regime, we have

$$\underbrace{\frac{2\sigma\cos\theta}{\rho a}}_{\text{CAPILLARY}} = \underbrace{\frac{8\mu z\dot{z}}{\rho a^2}}_{\text{VISCOUS}} + \underbrace{\rho g}_{\text{GRAVITY}}$$



$\downarrow g$



What if VISCOUS STRESSES $\gg$ GRAVITY?

Balance: $\dfrac{2\sigma a\cos\theta}{8\mu} = z\dot{z} = \dfrac{1}{2}\dfrac{d}{dt}z^2$

$$\Rightarrow \boxed{z = \left(\frac{\sigma a\cos\theta}{2\mu}t\right)^{\frac{1}{2}} \sim \sqrt{t}}$$

WASHBURN'S LAW

Note: front slows down, not due to gravity, but due to increased dissipation as fluid intrudes into tube / porous matrix

Marangoni Flows : those driven or influenced by surface tension gradients

- since $\sigma(T, c, \Gamma)$, Marangoni flows

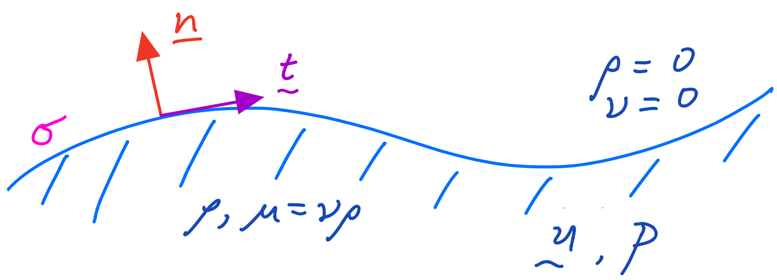
temp chemistry surfactant

may be driven by $\vec{\nabla} T$, $\vec{\nabla} c$, $\vec{\nabla} \Gamma$ at an interface.

Recall N-S eqns:

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \underline{\nabla} \underline{u}\right) = -\underline{\nabla} p + \rho \underline{g} + \mu \nabla^2 \underline{u}$$

$$\underline{\nabla} \cdot \underline{u} = 0$$



Free surface BCs

1. Normal stress : $\underline{n} \cdot \underline{\underline{T}} \cdot \underline{n} = \sigma \, \underline{\nabla} \cdot \underline{n}$

2. Tangential stress : $\underline{n} \cdot \underline{\underline{T}} \cdot \underline{t} = \underline{t} \cdot \underline{\nabla} \sigma$

where $\underline{\underline{T}} = -p \underline{\underline{I}} + 2\mu \underline{\underline{E}}$, and $\underline{\underline{E}} = \frac{1}{2}\left(\underline{\nabla} \underline{u} + \underline{\nabla} \underline{u}^T\right)$

BC 2 indicates that, for a static system, since

$$\underline{n} \cdot \underline{\underline{T}} \cdot \underline{\underline{t}} = 0 \implies \vec{\nabla} \sigma = 0$$

$\therefore$ If $\vec{\nabla} \sigma \neq 0 \Rightarrow$ There must be flow

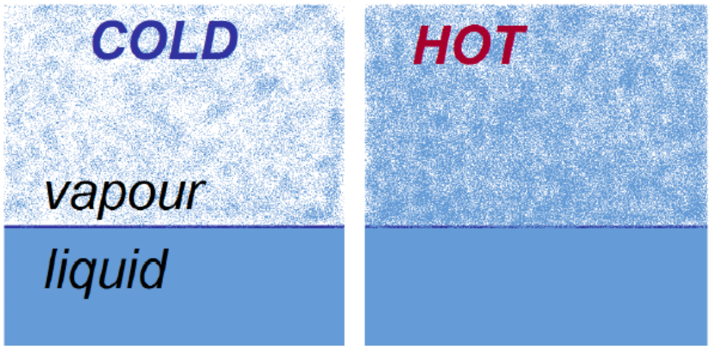
i.e. Marangoni stresses can only be balanced by tangential viscous stresses associated with flow

Thermocapillary Flows

$\sigma(T)$

- Marangoni flows induced by temperature gradients

Note: $\dfrac{d\sigma}{dT} < 0$. Why?



IN HOT ZONE:

$\Rightarrow$ gas phase has more fluid molecules

$\Rightarrow$ surface less energetically unfavourable

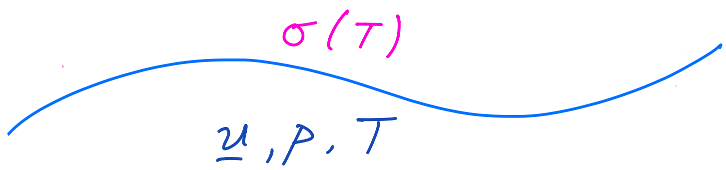
$\Rightarrow$ σ lower

Approach: through the BCs (the $\vec{\nabla}\sigma$ term), NS eqns are coupled to the heat eqn

$$\frac{\partial T}{\partial t} + \underline{u} \cdot \underline{\nabla}\, T = \underline{\kappa}\, \nabla^2 T$$

where $\underline{\kappa}$ = thermal diffusivity

Note: 1. heat eqn must itself be subject to appropriate BCs at the free surface



$\Rightarrow$ can be complicated, e.g. if fluid is evaporating

2. analysis may be simplified in $Pe \ll 1$ limit

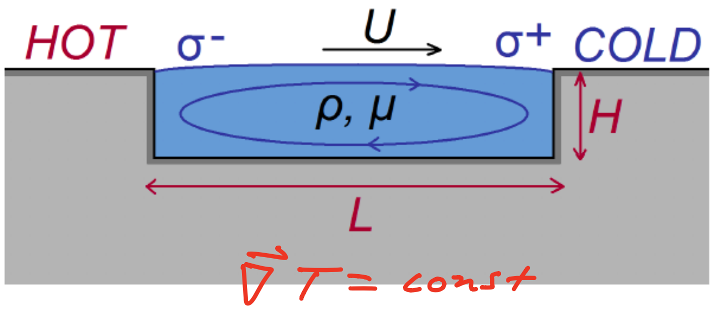
Nondim: $\underline{x} \sim a\,\underline{x}'$, $t \sim \dfrac{a}{U}\, t'$, $\underline{u} \sim U\, \underline{u}'$

$$Pe \left(\frac{\partial T'}{\partial t'} + \underline{u}' \cdot \underline{\nabla}' T' \right) = \nabla'^2 T'$$

where $Pe = \dfrac{Ua}{\kappa} = \underbrace{Re}_{\text{Reynolds}}\, \overbrace{Pr}^{\text{Prandtl}} = \dfrac{Ua}{\nu} \dfrac{\nu}{\kappa}$

In limit of $Pe = \frac{Ua}{K} \ll 1$, one has $\nabla^2 T = 0$.
(often valid at low Re)

Eg. 1. Thermocapillary flow in a slot



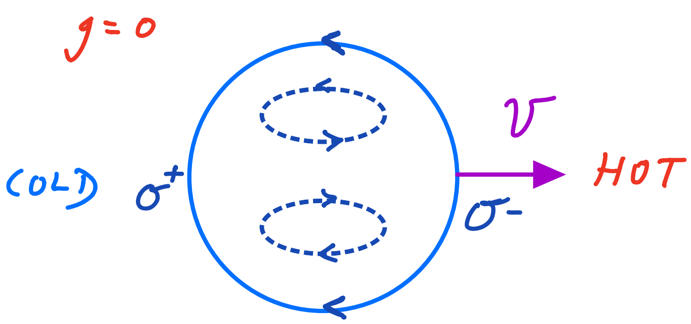
Surface Tangential BC:

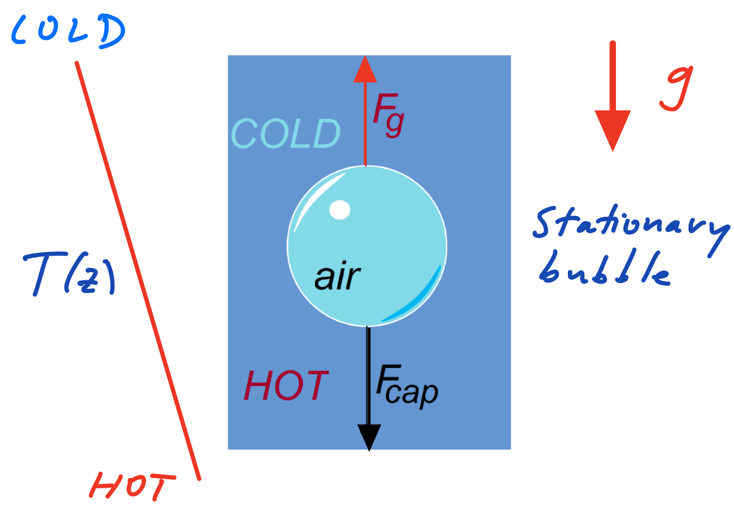
$$\tau = \frac{\Delta\sigma}{L} = \underbrace{\frac{d\sigma}{dT}\frac{\Delta T}{L}}_{\text{Marangoni stress}} = \underbrace{\frac{\mu U}{H}}_{\text{viscous stress}}$$

$$\Rightarrow \boxed{U \sim \frac{1}{\mu}\frac{H}{L}\Delta\sigma}$$

Eg. 2 Thermocapillary Drop Motion

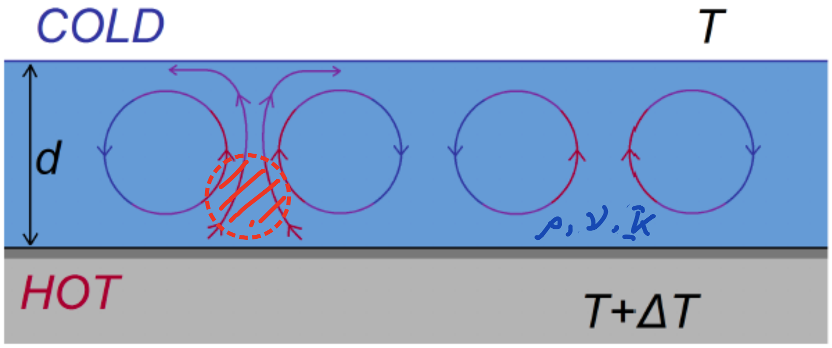
(Young, Goldstein + Block, 1952)





Eg. 3 Thermal Marangoni Convection in a Plane Layer

Recall : Thermal buoyancy-driven convection "Rayleigh–Bénard"



$$\rho(T) = \rho_0\left[1 - \alpha\,(T - T_0)\right]$$

THERMAL EXPANSIVITY

Consider a buoyant blob with characteristic scale d. Near the onset of convection, one expects it to rise with the Stokes velocity:

$$U \sim \frac{g' d^2}{\nu} = \frac{g\,\Delta\rho}{\rho}\frac{d^2}{\nu} = \frac{g\alpha\Delta T d^2}{\nu}$$

The blob will rise, and so convection occur, provided its rise time is less than the time required for it to lose its heat (and so buoyancy) by diffusion.

$$\tau_{RISE} = \frac{d}{U} \sim \frac{d\,\nu}{g\alpha\Delta T d^2}$$

$$\tau_{diffusion} = d^2/\kappa$$

$\therefore$ Criterion for instability:

$$\frac{\tau_{diff}}{\tau_{RISE}} \sim \frac{g\alpha\Delta T d^3}{\kappa\,\nu} \equiv Ra > Ra_{crit} \sim 10^3$$

RAYLEIGH #

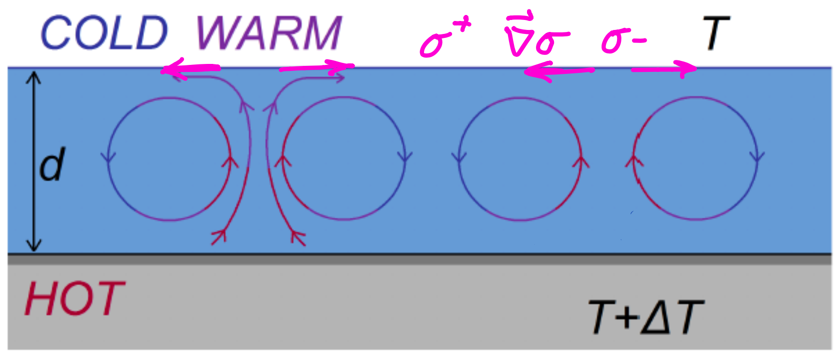
Evolution determined by Ra and $Pr = \nu/\kappa$

Note: $Ra < Ra_c$: pure diffusion; static fluid
$Ra = Ra_c$: onset of convection, steady rolls

As $Ra \uparrow$: increasingly complex flows emerge, onset of time-dependence, periodic flow → period doubling → chaos → turbulen

Thermal Marangoni Convection

- arises through influence of T on σ, rather than density ρ : $\sigma(T) = \sigma_0 - \Gamma(T - T_0)$



Mechanism : if a warm blob surfaces ⇒ prompts surface divergence ⇒ upwelling
- upwelling blob will be warm ⇒ reinforces the thermal perturbation provided it rises before losing its heat via diffusion

Balance Marangoni + viscous stress : $\frac{\Delta\sigma}{d} \sim \frac{\mu v}{d}$

Rise time : $\frac{d}{v} \sim \frac{\mu d}{\Delta\sigma} = \frac{\mu d}{\Gamma \Delta T} \equiv \tau_{RISE}$

Diffusion time : $\tau_{diff} = d^2/\kappa$

Criterion for Instability :

$$\frac{\tau_{diff}}{\tau_{RISE}} \sim \frac{\Gamma \Delta T\, d}{\mu\, \underline{\kappa}} \equiv Ma > Ma_c \sim 100$$

Marangoni #

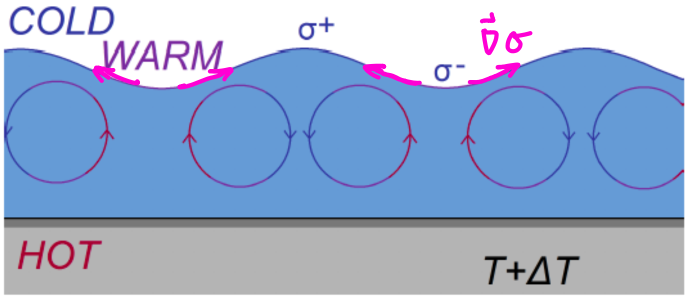
Note : 1. since $Ma \sim d$ and $Ra \sim d^3$, thin layers are most unstable to Marangoni convection

2. Bénard's (1900) original expts performed in millimetric layers of spermaceti were visualizing Marangoni convection, but were misinterpreted by Rayleigh (1916) as being due to buoyancy
   $\Rightarrow$ not recognized until Block (1956)

3. Pearson (1958) performed stability analysis with flat upper surface, deduced $Ma_c = 80$.

4. Scriven + Sternling (1964) considered a deformable $\Rightarrow$ renders system unstable at all Ma.

Note : downwelling/upwelling beneath peaks indicates Marangoni/Ra-B convection



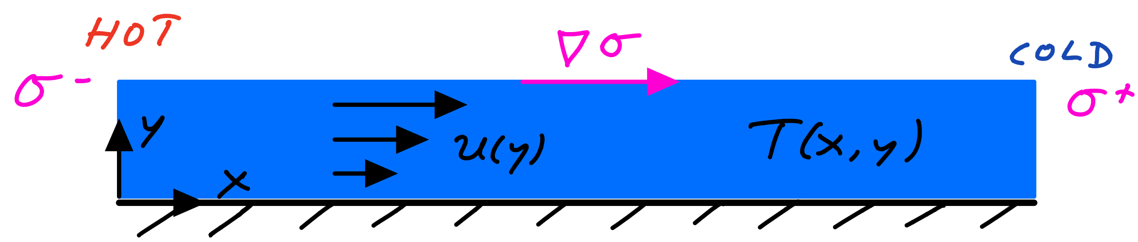
5. Smith (1966) showed that the destabilizing influence of the free surface may be mitigated by gravity

Stability Criterion: $\frac{d\sigma}{dT}\frac{dT}{dz} < \frac{2}{3}\rho g d$

$\Rightarrow$ thin layers prone to instability

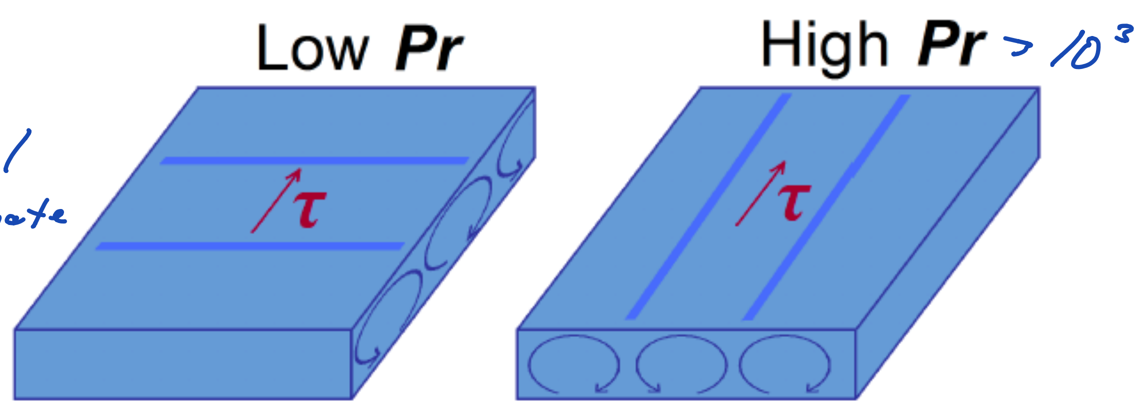
Eg. 4 Marangoni Shear Layers

$g = 0$



- lateral temp gradient $\Rightarrow$ Marangoni stress $\Rightarrow$ shear flow $\Rightarrow$ resulting $T(x,y)$ may destabilize layer to Ma convection
- Smith + Davis (1983 a b) considered case of a flat free surface $\Rightarrow$ system behaviour depends on $Pr = \nu/\kappa$

hydrothermal waves propagate in direction of τ



streamwise vortices

- Bush + Hosoi (2001) considered a deformable free surface