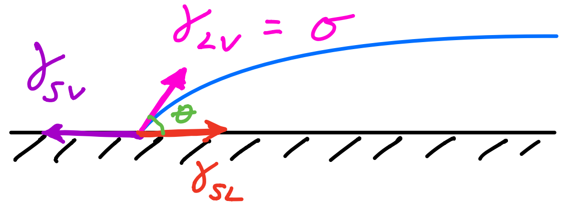
Lecture 7. Capillary rise

Recall: Spreading Parameter: $S = \gamma_{SV} - (\gamma_{SL} + \gamma_{LV})$

Spreading condition:

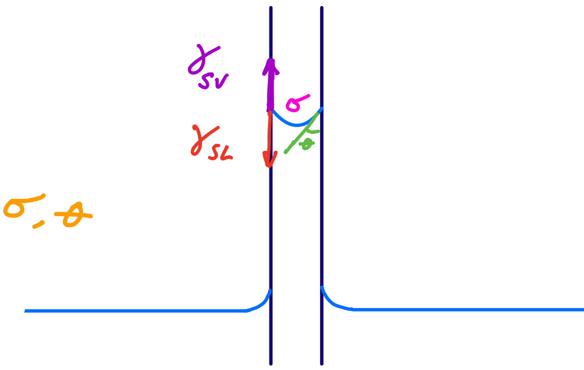
$$S > 0$$



Define: Imbibition Parameter

$$I = \gamma_{SV} - \gamma_{SL}$$

$$= \gamma_{LV}\cos\theta = \underbrace{\sigma\cos\theta}_{\text{depends only on } \sigma,\,\theta}$$



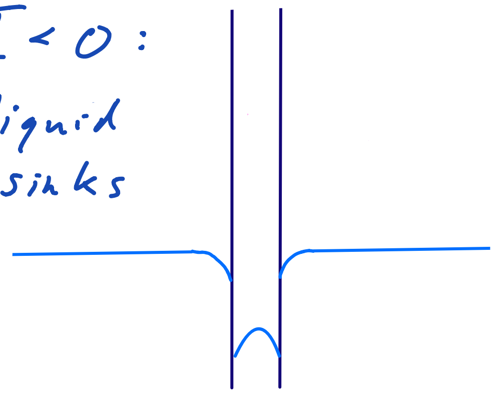
Note: in capillary rise, motion does not change the energy of the liquid-vapor interface

- since $I = S + \gamma_{LV}$, the imbibition condition, $I > 0$ is always more easily met than the spreading condition, $S > 0$

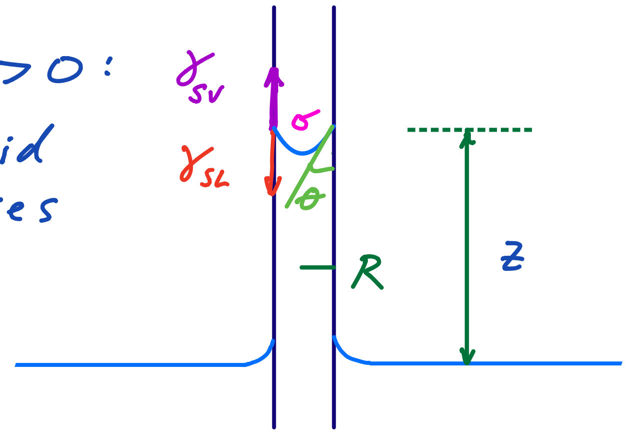
$\Rightarrow$ most liquids soak sponges and other porous media, but complete wetting is far less common.

Energetics

$I < 0$: liquid sinks



$I > 0$: liquid rises



Energy of column:

MENISCUS AREA

$$E = \underbrace{(\gamma_{SL} - \gamma_{SV})\, 2\pi R z}_{\text{DIFF. SURFACE ENERGY}} + \underbrace{\tfrac{1}{2}\rho g R^2 \pi z^2}_{\text{GPE}} + \underbrace{\sigma A_m}_{\text{MENISCUS ENERGY}}$$

$$= -2\pi R z I + \tfrac{1}{2}\rho g R^2 \pi z^2 + \sigma A_m$$

$$= \text{MIN when } \frac{dE}{dz} = 0, \text{ i.e. at } z = H,$$

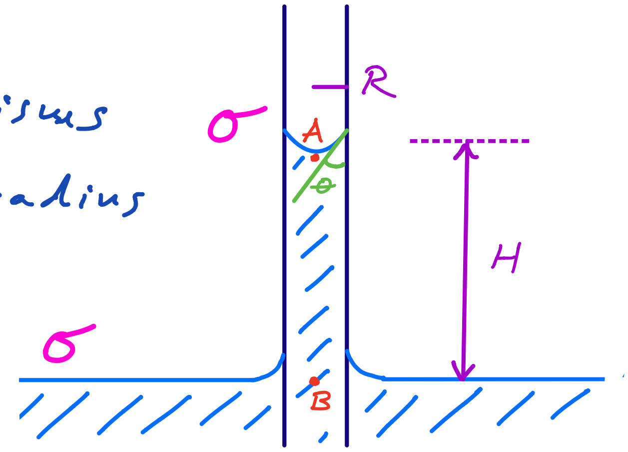
where $H = 2\,\dfrac{\gamma_{SV} - \gamma_{SL}}{\rho g R} = \dfrac{2I}{\rho g R}$

$$\boxed{H = 2\,\frac{\sigma \cos\theta}{\rho g R}}$$

Note: 1. describes both capillary rise + descent: sign of H depends on whether $\theta > \frac{\pi}{2}$ or $\theta < \frac{\pi}{2}$

2. $H \uparrow$ as $\theta \downarrow$: H_{MAX} for $\theta = 0$.

Pressure Argument



Provided $R \ll l_c$, the menisus will be a spherical cap with radius

$$R' = \frac{R}{\cos\theta}$$

Thus $P_A = P_0 - \dfrac{2\sigma\cos\theta}{R}$

We also know $P_B = P_0$, and so

$$P_A = P_0 - \frac{2\sigma\cos\theta}{R} = P_0 - \rho g H$$

$$\Longrightarrow \quad H = \frac{2\sigma\cos\theta}{\rho g R} \quad \text{as previously.}$$

Capillary Rise : Dynamics

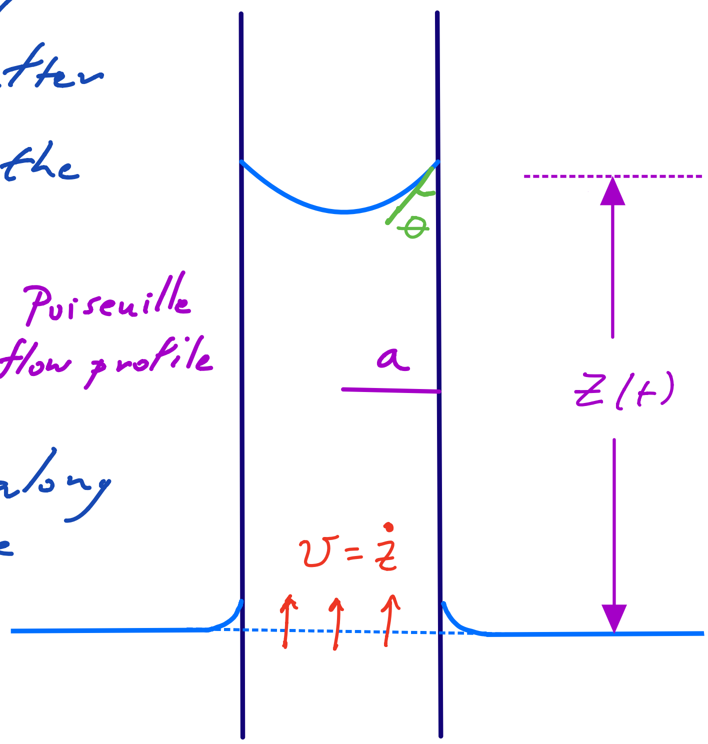
The column rises due to σ (if $I > 0$), its rise being resisted by some combination of fluid inertia, viscosity, gravity, dynamic pressure.

$$(\underbrace{m}_{\text{INERTIA}} + \underbrace{m_a}_{\text{ADDED MASS}})\,\ddot{z} = \underbrace{2\pi a\,\sigma\cos\theta}_{\text{SURFACE TENSION}} - \underbrace{mgz}_{\text{GRAVITY}} \qquad \otimes$$

$$- \underbrace{\pi a^2 \tfrac{1}{2}\rho\dot{z}^2}_{\text{DYNAMIC PRESSURE}} - \underbrace{2\pi a\,\tau_v \cdot z}_{\text{VISCOUS STRESS}}$$

where $m = \pi a^2 z \rho$ is column mass.

When flow in tube is fully developed Poiseuille flow (after $\tau \sim a^2/\nu$), we can estimate the viscous stress.



$$u(r) = 4\dot{z}\left(1 - \frac{r^2}{a^2}\right)$$

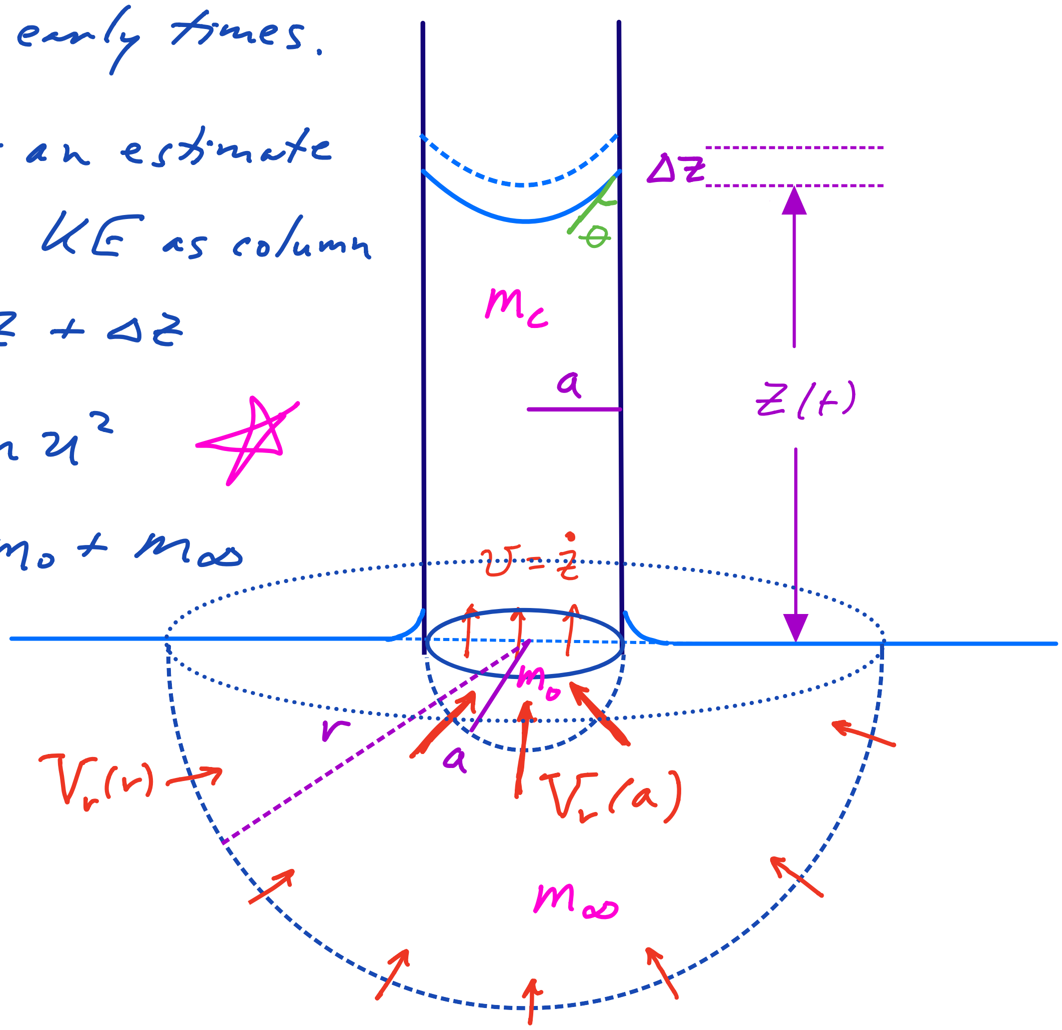
$$\Rightarrow \quad F = \dot{z}\,\pi a^2 \text{ is flux along tube}$$

Stress on outer wall:

$$\tau_v = \mu \left.\frac{du}{dr}\right|_{r=a} = -\frac{8\mu}{a}\dot{z}$$

Next, we need to estimate the added mass, which will dominate at early times.

Doing so requires an estimate of the change in KE as column rises from z to $z + \Delta z$



$$\Delta E_k = \Delta \tfrac{1}{2} m u^2$$

where $m = m_c + m_0 + m_\infty$

In column:

$m_c = \pi a^2 z \rho$

and $u = v$

In spherical cap, radial flow matches onto vertical

$$m_0 = \frac{2\pi}{3} a^3 \rho , \quad u \simeq U$$

In outer region, radial inflow extends to ∞ but decays

Volume Conservation : $F = \pi a^2 U = 2\pi a^2 V_r$

$\Rightarrow \quad V_r(a) = U/2$ is the magnitude of the radial inflow at $r = a$

Continuity: $2\pi a^2 V_r(a) = 2\pi r^2 V_r(r)$

$$\Rightarrow \quad V_r(r) = \frac{a^2}{r^2} V_r(a) = \frac{a^2}{2r^2} U$$

By defn, $\frac{1}{2} m_\infty^{eff} U^2 = \frac{1}{2} \int_a^\infty V_r^2(r) \underbrace{dm}_{\rho 2\pi r^2 dr}$

$$\Rightarrow m_\infty^{eff} = \frac{1}{U^2} \int_a^\infty \rho \left(\frac{a^2}{2r^2} U \right)^2 \cdot 2\pi r^2 dr$$

$$= \frac{1}{2} \rho \pi a^3$$

Now, back to $\star$:

$$\Delta E_k = \frac{1}{2} \Delta (m_c + \cancel{m_0} + \cancel{m_\infty}) U^2 + \frac{1}{2} m \, 2 U \Delta U$$

$$= \frac{1}{2} \Delta m_c \cdot U^2 + \frac{1}{2} (m_c + m_0 + m_\infty^{eff}) \cdot 2 U \Delta U$$

$$= \frac{1}{2} (\pi a^2 \rho \Delta z) U^2 + \Big(\underbrace{\pi a^2 \rho z}_{m_c} + \underbrace{\frac{2}{3} \pi a^3 \rho + \frac{1}{2} \pi a^3 \rho}_{\text{ADDED MASS} = \frac{7}{6} \pi a^3 \rho} \Big) U \Delta U$$

Subing for $m = \pi a^2 z \rho$, $m_a = \frac{7}{6}\pi a^3 \rho$ and $\tau_v = -\frac{8\mu}{a}\dot{z}$, we obtain from ⊗
(after dividing by $\pi a^2 \rho$)

$$\boxed{\left(z + \frac{7}{6}a\right)\ddot{z} = \frac{2\sigma\cos\theta}{\rho a} - \frac{1}{2}\dot{z}^2 - \frac{8\mu z\dot{z}}{\rho a^2} - gz}$$

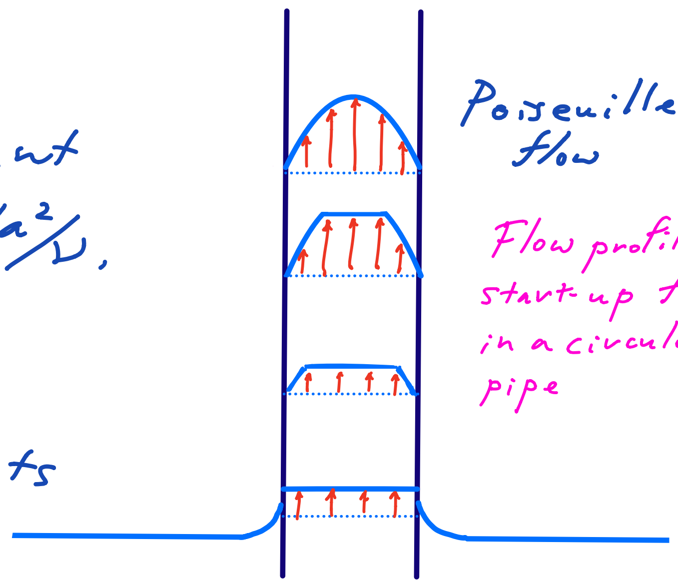
INERTIA, ADDED MASS, CAPILLARY, DYNAMIC PRESSURE, VISCOUS DRAG, GRAVITY

Recall: balancing capillary force + gravity $\Rightarrow$ RISE HEIGHT

But how do we get there?

Inertial Regime

- the timescale of establishment of Poiseuille flow is $\tau^* = 4a^2/\nu$, is that of vorticity to diffuse across tube

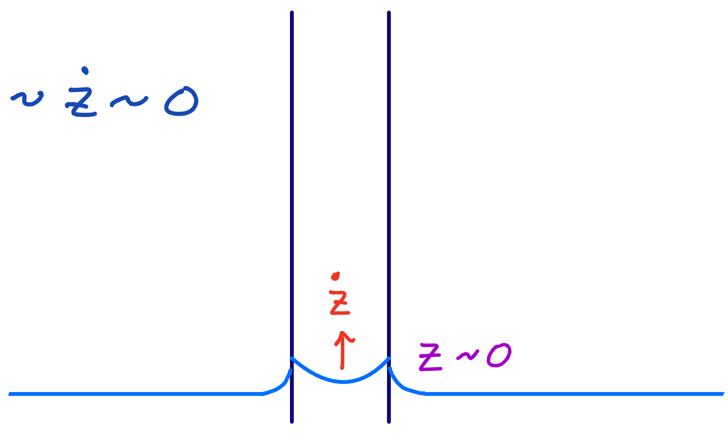


Until this time, viscous effects are negligible, and rise is resisted primarily by fluid inertia.

Early Inertial Regime : $z \sim \dot{z} \sim 0$

$$\Rightarrow \frac{7}{6}a\ddot{z} = \frac{2\sigma\cos\theta}{\rho a}$$

Integrate, using $z(0) = 0$
$\dot{z}(0) = 0$



$$\Rightarrow \quad \boxed{z(t) = \frac{6}{7}\frac{\sigma\cos\theta}{\rho a^2}\, t^2}$$

Once $z \geq \frac{7}{6}a$, one must also consider the column mass, and so solve

$$\left(z + \frac{7}{a}\right)\ddot{z} = \frac{2\sigma\cos\theta}{\rho a}$$

As the column accelerates from $\dot{z} = 0$, $\dot{z}^2$ becomes important, so one enters the ...

Late Inertial Regime:

$$\frac{1}{2}\dot{z}^2 = \frac{2\sigma\cos\theta}{\rho a}$$

$$\Rightarrow \quad \boxed{z = \left(\frac{4\sigma\cos\theta}{\rho a}\right)^{\frac{1}{2}} t}$$

Note: $v = \dot{z} = \left(\frac{4\sigma\cos\theta}{\rho a}\right)^{\frac{1}{2}}$ indep of g, μ.

Viscous Regime: $t \gg \tau^*$

For $t \gg \tau^*$, inertial effects become negligible and we have

$$\frac{2\sigma\cos\theta}{\rho a} - \frac{8\mu z\dot{z}}{\rho a^2} - gz = 0$$

$$\Rightarrow H - z = \frac{8\mu z\dot{z}}{\rho g a^2} \quad \text{where } H = \frac{2\sigma\cos\theta}{\rho g a}$$

Now nondimensionalize: $z^* = z/H$

$t^* = t/\tau$ where $\tau = \dfrac{8\mu H}{\rho g a^2}$

$$\Rightarrow \quad \dot{z}^* = \frac{1}{z^*} - 1$$

$$\Rightarrow \quad dt^* = \frac{z^*}{1-z^*}\, dz^* = \left(-1 - \frac{1}{1-z^*}\right) dz^*$$

$$t^* = \left[-z^* - \ln(1-z^*)\right] \quad \boxtimes$$

Note: as $t^* \to \infty$, $z^* \to 1$

$\Rightarrow$ 2 distinct viscous regimes emerge

Early Viscous Regime: $z^* \ll 1$

$$\ln(1-z^*) \simeq -z^* - \frac{1}{2} z^{*2}$$

$$\boxtimes \Rightarrow \quad t^* = \frac{1}{2} z^{*2} \Rightarrow z^* = \sqrt{2t^*}$$

Redimensionalize: $$\boxed{z = \left(\frac{\sigma a \cos\theta}{2\mu} t\right)^{\frac{1}{2}}}$$

Late Viscous Regime: $z^* \simeq 1$

Here $t^* = \left[\underbrace{-z^*}_{\text{small}} - \ln(1-z^*)\right] \simeq -\ln(1-z^*)$

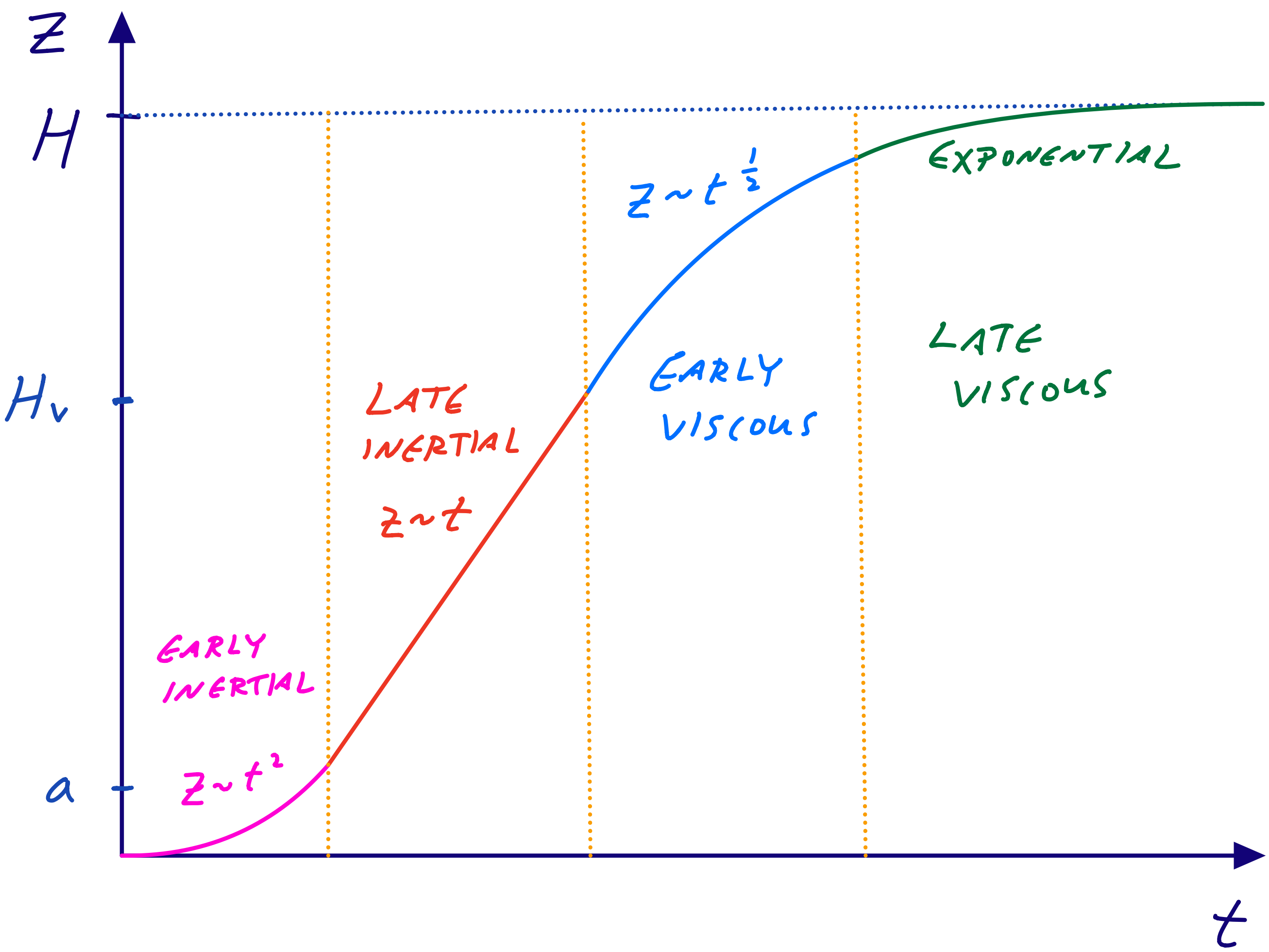
$$\Rightarrow \quad z^* = 1 - \exp(-t^*)$$

Redimensionalize:

$$\boxed{z/H = 1 - \exp^{-t/\tau}}$$

where $\tau = \dfrac{8\mu H}{\rho g a^2}$

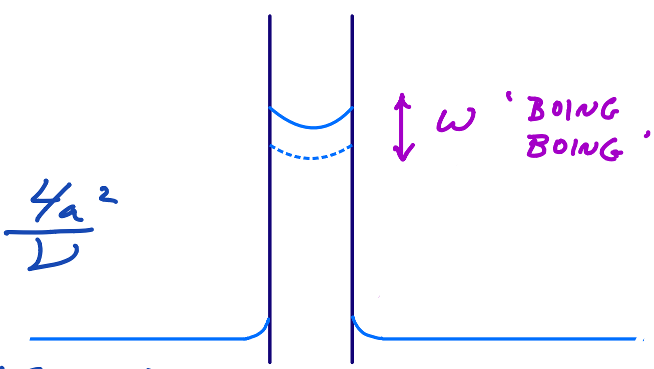
SUMMARY : 4 distinct regimes



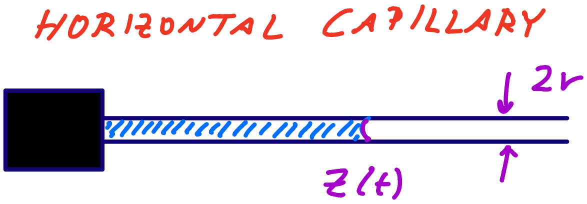
Note: if timescale of rise $<< \tau^* = \dfrac{4a^2}{\nu}$, inertia dominates

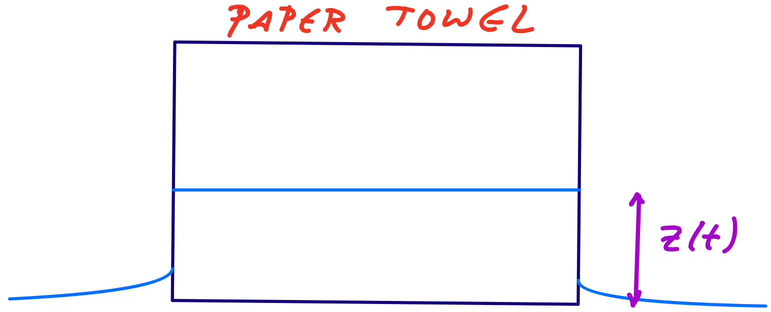
i.e. $H << U_{inertia}\, \tau^*$

$$= \left(\frac{4\sigma\cos\theta}{\rho a}\right)^{\frac{1}{2}} \frac{4a^2}{\nu}$$



$\Rightarrow$ inertial overshoot, oscillations arise.


HORIZONTAL CAPILLARY
2r
z(t)


PAPER TOWEL
z(t)