# Lecture 12: Instability of a fiber coating; Film retraction

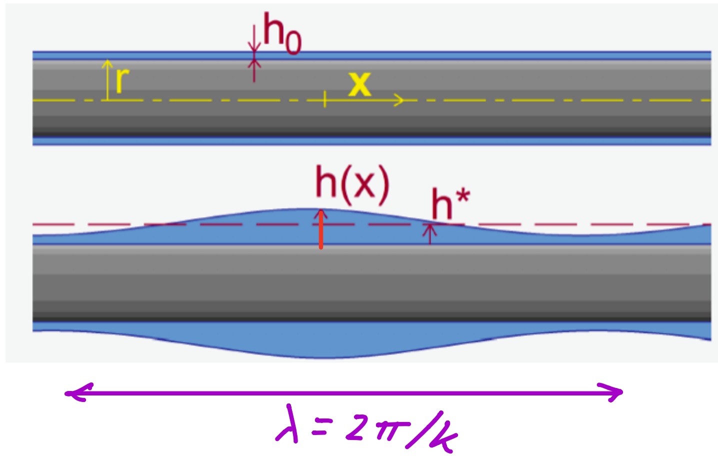
## Instability of a fluid coating on a fiber

Define mean thickness:

$$h^* = \int_0^\lambda h(x)\,dx$$

so interface thickness

$$h = h^* + \epsilon \cos kx$$




Continuity: $\displaystyle\int_0^\lambda \pi (r+h)^2\,dx = \int_0^\lambda \pi (r+h_0)^2\,dx$

$$\Rightarrow \int_0^\lambda \left(r + h^* + \epsilon\cos kx\right)^2 dx = (r+h_0)^2 \lambda$$

$$\int_0^\lambda (r+h^*)^2 + \epsilon^2\cos^2 kx + \cancel{2(r+h^*)\epsilon\cos kx}\,dx$$
$$= (r+h_0)^2\lambda$$

$$\Rightarrow (r+h^*)^2\lambda + \epsilon^2 \lambda/2 + 0 = (r+h_0)^2\lambda$$

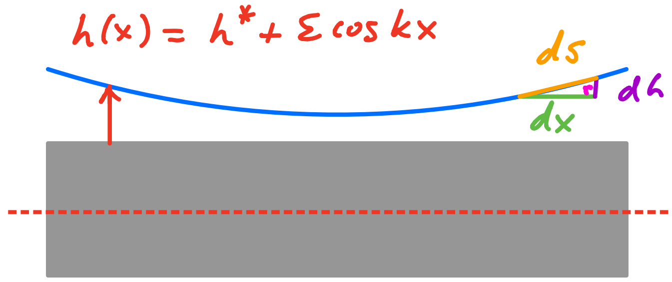
$$\Rightarrow (r+h^*)^2 = (r+h_0)^2\left[1 - \frac{1}{2}\frac{\epsilon^2}{(r+h_0)^2}\right]$$

$$\Rightarrow \boxed{h^* = h_0 - \frac{1}{4}\frac{\epsilon^2}{r+h_0}}$$

So, when does perturbation reduce surface energy?

<u>i.e.</u> when is $\displaystyle\int_0^\lambda 2\pi(r+h)\,ds < 2\pi(r+h_0)\lambda$ ?

Note: $ds^2 = dh^2 + dx^2$




$$\Rightarrow \quad ds = dx\left[1 + \left(\frac{dh}{dx}\right)^2\right]^{\frac{1}{2}}$$

$$\simeq dx\left[1 + \frac{1}{2}\varepsilon^2 k^2 \sin^2 kx\right]$$

$$\text{since } \frac{dh}{dx} = -\varepsilon k \sin kx$$

$$\Rightarrow \int_0^\lambda (r+h)\,ds = \int_0^\lambda (r + h^* + \varepsilon\cos kx)\left(1 + \tfrac{1}{2}\varepsilon^2 k^2 \sin^2 kx\right)dx$$

$$= (r+h^*)\lambda + \frac{1}{4}(r+h^*)\varepsilon^2 k^2 \lambda + O(\varepsilon^3)$$

So, $\star$ holds provided:

$$(\cancel{r} + h^*)\lambda + \frac{1}{4}(r+h^*)\varepsilon^2 k^2 \lambda < (\cancel{r} + h_0)\lambda$$

Sub in $(h^* - h_0)$ from $\boxed{\times}$ :

$$-\frac{1}{4}\frac{\cancel{\varepsilon^2}}{r+h_0} + \frac{1}{4}(r+h^*)\cancel{\varepsilon^2} k^2 < 0$$

Result indep of $\varepsilon$: $k^2 < (r+h_0)^{-1}(r+h^*)^{-1}$

$$\simeq \frac{1}{(r+h_0)^2}$$

i.e. wavelength $\boxed{\lambda = \frac{2\pi}{k} > 2\pi(r+h_0)}$

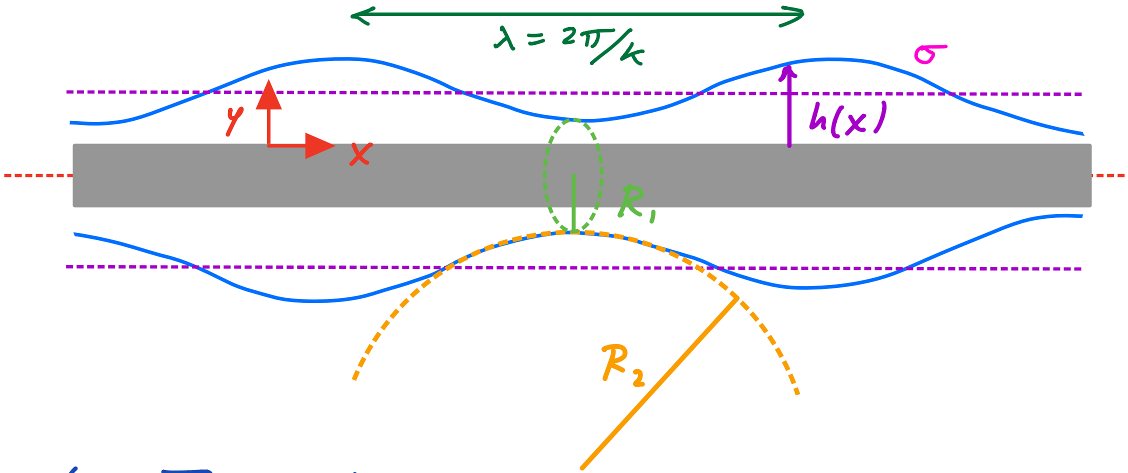
$\Rightarrow$ comparable to standard Ra–P.

All long $\lambda$ will grow, but which grows fastest?

$\Rightarrow$ determined by dynamics, not just geometry.

Consider dynamics in the thin-film limit,

$h_0 \ll r \implies$ LUBRICATION THEORY.




## Dynamics of Instability (Rayleigh 1879)

Poiseuille flow (Couette):

$$\eta \frac{d^2 v}{dy^2} - \frac{dp}{dx} = 0$$

← GRADIENT IN CURVATURE PRESSURE, INDEP of $y$.

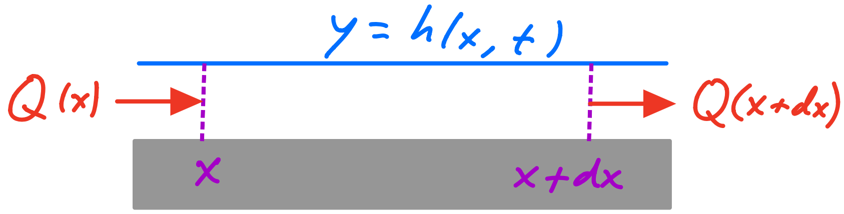
$$\implies \frac{dv}{dy} = \frac{1}{\eta} \frac{dp}{dx} (y - h)$$

$$v(y) = \frac{1}{\eta} \frac{dp}{dx} \left( \frac{y^2}{2} - hy \right)$$

Flux: $$Q_{/cm} = \int_0^h v(y)\, dy = -\frac{1}{3\eta} \frac{dp}{dx} h^3$$

Lubrication: $$Q(x+dx) - Q(x) = -\frac{dh}{dt} dx$$

$$\implies \frac{dQ}{dx} = -\frac{h_0^3}{3\eta} \frac{d^2 p}{dx^2} = -\frac{dh}{dt}$$



**Curvature pressure** : $p(x) = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$

$$= \sigma\left(\frac{1}{r+h} - \frac{d^2h}{dx^2}\right)$$

$$\Rightarrow \frac{dh}{dt} = \frac{\sigma h_0^3}{3\eta}\frac{d^2}{dx^2}\left[\frac{1}{r+h(x)} - h_{xx}\right]$$

$$= \frac{\sigma h_0^3}{3\eta}\frac{d^2}{dx^2}\left[\frac{1}{r+h^*} - \frac{\varepsilon\cos kx}{(h^*+r)^2} + \varepsilon k^2\cos kx\right]$$

since $h(x,t) = h^* + \varepsilon(t)\cos kx$

$$h_x = -\varepsilon k\sin kx, \quad h_{xx} = -\varepsilon k^2\cos kx = -k^2h$$

$$h_t = \cancel{h^*_t} + \frac{d\varepsilon}{dt}\cos kx$$

$$\Rightarrow \cancel{\cos kx}\,\frac{d\varepsilon}{dt} = \frac{\sigma h_0^3}{3\eta}\varepsilon\,\cancel{\cos kx}\left(\frac{k^2}{(r+h^*)^2} - k^4\right)$$

$$\therefore \frac{d\varepsilon}{dt} = \beta\varepsilon$$

DISPERSION RELATION

where $\boxed{\beta = \frac{\sigma h_0^3}{3\eta}\left[\frac{k^2}{(r+h_0)^2} - k^4\right]}$

**Recall:** $h^* = h_0 - \frac{1}{4}\varepsilon^2\frac{1}{(r+h_0)^2} \simeq h_0$

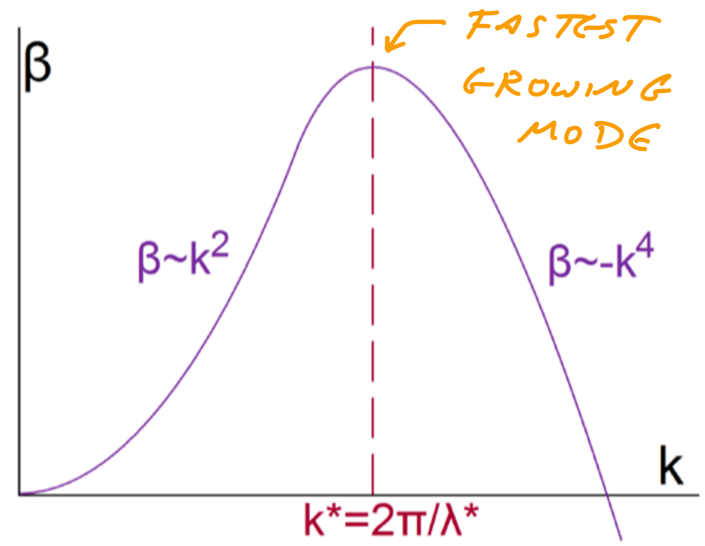
**Fastest Growing Mode:**

Arises when $\frac{d\beta}{dk} = 0 = \frac{2k^*}{(r+h_0)^2} - 4k^{*3}$

$$\Rightarrow \boxed{\lambda^* = 2\pi\sqrt{2}\,(r+h_0)}$$

$$\approx 10\,(r+h_0)$$

similar to inviscid Rayleigh – Plateau




We also see the timescale of instability:

$$\tau^* = \frac{2\pi}{\beta(k^*)} = \frac{12\,\eta\,(r+h_0)^4}{\sigma\, h_0^3}$$

## Scaling Argument for Pinch-off time?

$$\frac{\mu V}{h_0^2} \sim \nabla p \sim \nabla\left[\sigma\, \nabla \cdot n\right]$$

VISCOUS STRESS

CURVATURE PRESSURE

$$\sim \frac{1}{r}\,\sigma\,\frac{h_0}{r^2}$$

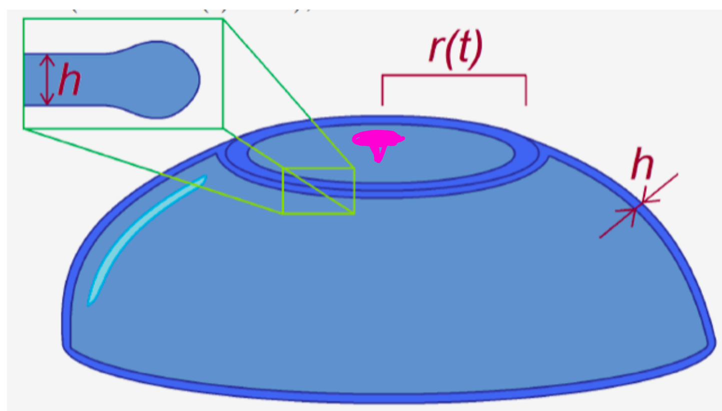
$$\Rightarrow V \sim \frac{r}{\tau_p} \sim \frac{\sigma}{\mu}\,\frac{h_0^3}{r^3}$$

$$\Rightarrow \tau_p \sim \frac{\mu}{\sigma}\,\frac{r^4}{h_0^3}$$

$\Rightarrow$ COMPARE TO COMPLETE RESULT

# Rupture of a Soap Film (Culick 1960, Taylor 1960)

- neglect viscous effects, consider inviscid limit



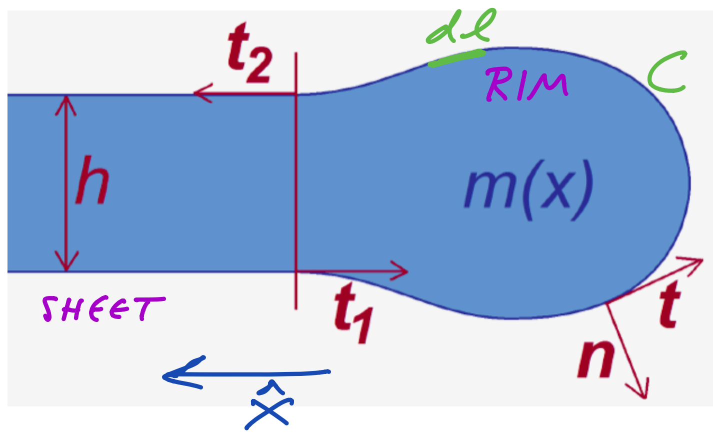

<u>Note</u>: the force/length acting on the rim may be calculated via Frenet-Serret eqns:

$$\underline{F}_c = \int_c \sigma \, \underline{\nabla} \cdot \underline{n} \, \underline{n} \, dl \quad \text{where } (\underline{\nabla} \cdot \underline{n})\underline{n} = \frac{d\underline{t}}{dl}$$

$$= \int_c \sigma \frac{d\underline{t}}{dl} \, dl$$

$$= \sigma (\underline{t}_1 - \underline{t}_2)$$

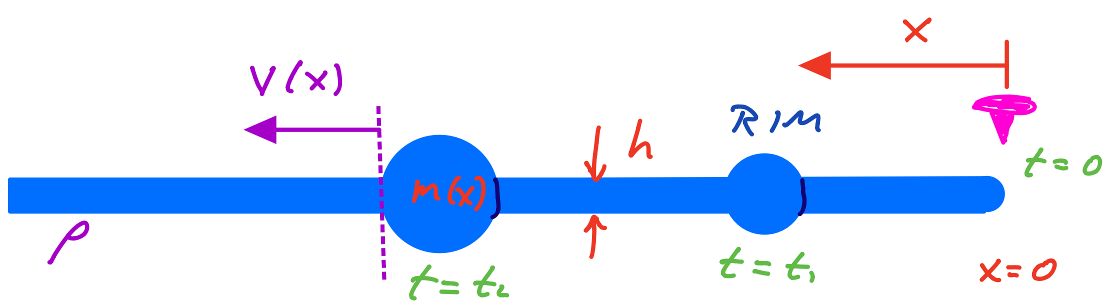
$$= 2\sigma \hat{x}$$




At time $t=0$, a planar sheet is punctured at $x=0$ and retracts in the $\hat{x}$-direction under the influence of $\underline{F}_c = 2\sigma \hat{x}$.

<u>Observation</u>: rim engulfs the film

- no upstream disturbance



**Rim mass** : $m(x) = \rho h x$

**Rim speed** : $V = \frac{dx}{dx}$

Inertial force on rim = rate of change of rim momentum

$$F_I = \frac{d}{dt}(mV) = V\frac{d}{dx}(mV)$$

$$= V^2 \frac{dm}{dx} + mV\frac{dV}{dx}$$

$$= \frac{1}{2}V^2 \rho h + \frac{1}{2}\frac{d}{dx}(mV^2)$$

**Force Balance** : Curvature Force = Inertial Force

$$\Rightarrow \quad 2\sigma = \frac{d}{dx}\left(\frac{1}{2}mV^2\right) + \frac{1}{2}\rho h V^2$$

Integrating from 0 to $x$ :

$$\int_0^x 2\sigma\, dx = \int_0^x d\left(\frac{1}{2}mV^2\right) + \frac{1}{2}\rho h \int_0^x V^2\, dx$$

$$2\sigma x = \frac{1}{2}\rho h x V^2 + \frac{1}{2}\rho h \int_0^x V^2\, dx$$

SURFACE ENERGY RELEASED/LENGTH

K.E. OF RIM

ENERGY REQUIRED TO ACCELERATE RIM

Now assume $V$ is indep of $x$ (observed in lab),

so that $\int_0^x V^2\, dx = xV^2$

$$\Rightarrow 2\sigma x = \rho h x V^2$$

$$\boxed{V = \left(\frac{2\sigma}{\rho h}\right)^{\frac{1}{2}}}$$

Culick Speed of film retraction

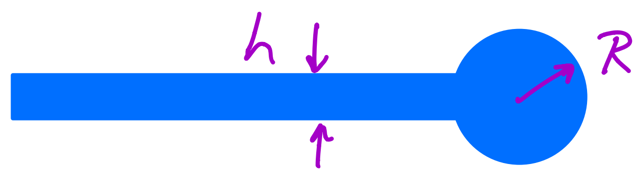
Note: 1. for a water/soap film of $h \sim 150\mu m$

$\Rightarrow$ $V \sim 3 \cdot 10^2$ cm/s very high

2. Surface area of rim/length : $P = 2\pi R$

where $m = \rho h x = \pi \rho R^2$

$\Rightarrow$ $R(x) = \sqrt{\frac{hx}{\pi}}$

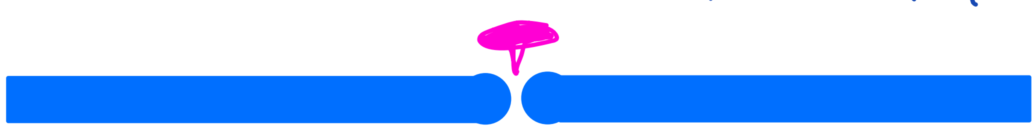



$\therefore$ Rim surface energy : $\sigma P = \sigma 2\pi \sqrt{\frac{hx}{2\pi}}$

Total surface energy : $\sigma\left[2x + 2(\pi h x)^{\frac{1}{2}}\right]$

Scale : $\frac{SA_{rim}}{SA_{sheet}} \sim \frac{2\pi\sqrt{hx}}{2x} \sim \pi\left(\frac{h}{x}\right)^{\frac{1}{2}} << 1$

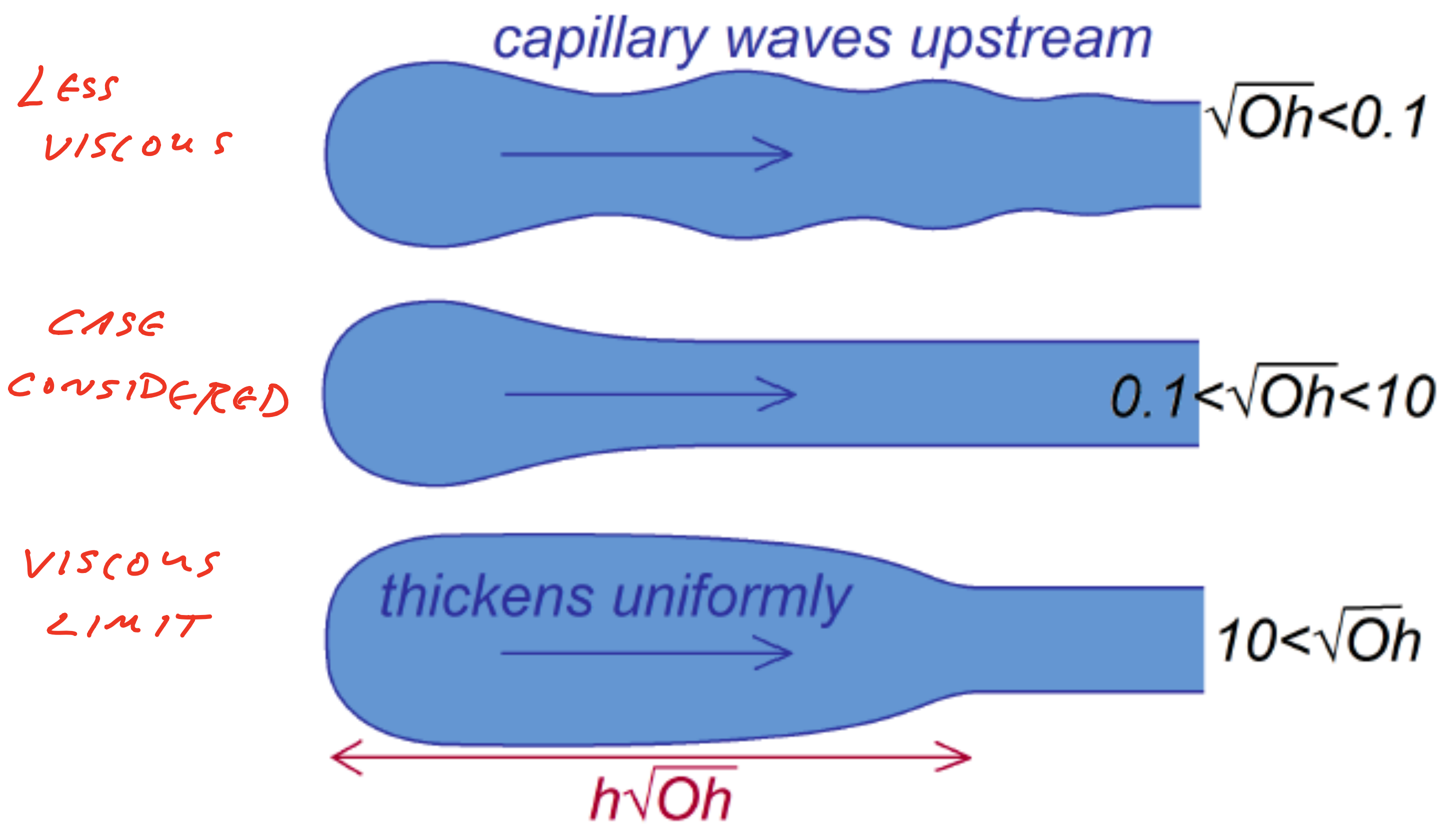
for $x >> h$



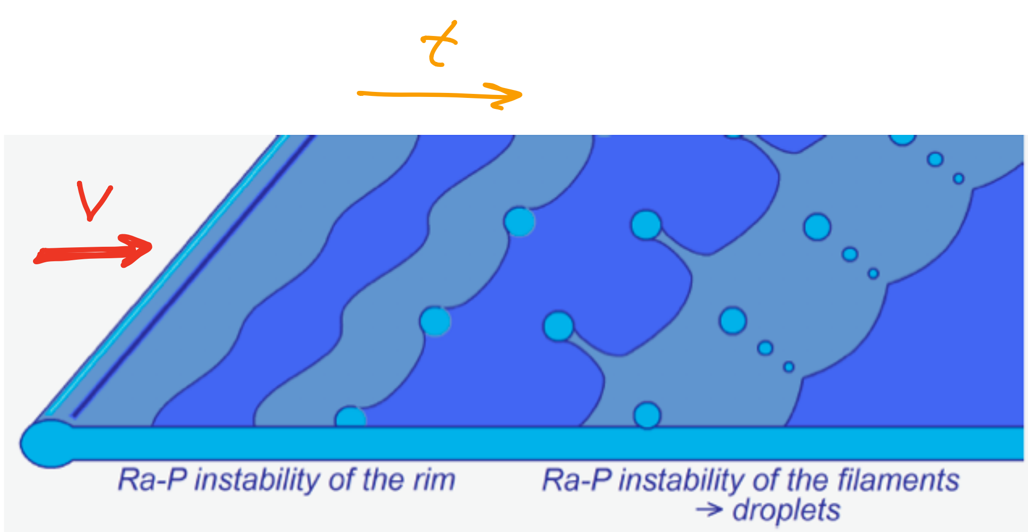
But, a circular film may self-heal if the hole is suff. small.

3. For dependence on geometry (i.e. circular hole) and influence of viscosity, see Savva + Bush (JFM, 2009)

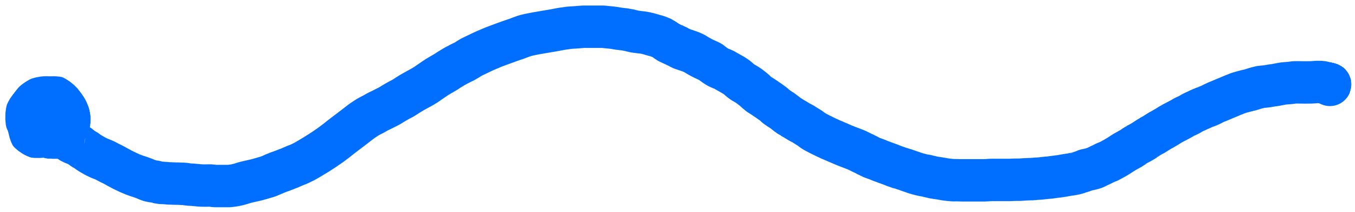
4. form of rim/sheet evolution depend on Ohnesorge # : $\sqrt{Oh} = \frac{\mu}{\sqrt{2h\rho\sigma}}$




5. The growing cylindrical rim is subject to Ra-P → often pinches into drop



6. At very high speed, shear stresses caused by air induce flapping.



⇒ like a flapping flag, with flag elasticity replaced by Marangoni elasticity

Lhuissier + Villermaux (PRL, 2009)