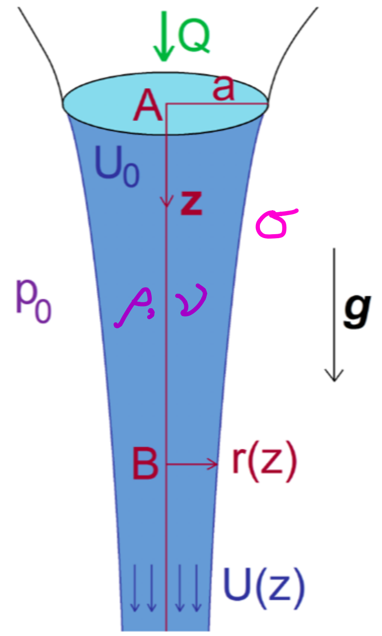
Lecture 11. Fluid jets

The shape of a falling fluid jet



We assume the jet Reynolds number $Re = \frac{Q}{a\nu}$ is suff. large that we can neglect viscous effects. We further assume that the jet speed is indep. of radius, so can be described as $U(z)$.

Deduce $r(z)$ and $U(z)$.

Applying Bernoulli's Thm at points A and B:

$$\frac{1}{2}\rho U_0^2 + \rho g z + P_A = \frac{1}{2}\rho U^2(z) + P_B$$

The local curvature of slender threads:

$$\underline{\nabla} \cdot \underline{n} = \frac{1}{R_1} + \frac{1}{R_2} \simeq \frac{1}{r}$$

$$\therefore \quad P_A \simeq P_0 + \frac{\sigma}{a}, \qquad P_B \simeq P_0 + \frac{\sigma}{r}$$

Bernoulli: $\frac{1}{2}\rho U_0^2 + \rho g z + P_0 + \frac{\sigma}{a} = \frac{1}{2}\rho U^2(z) + P_0 + \frac{\sigma}{r}$

$$\Rightarrow \frac{U(z)}{U_0} = \Bigg[1 + \underbrace{\frac{2}{Fr}\frac{z}{a}}_{\text{JET ACCELERATES DUE TO } g} + \underbrace{\frac{2}{We}\left(1 - \frac{a}{r}\right)}_{\text{JET SLOWS DOWN DUE TO } \sigma}\Bigg]^{\frac{1}{2}} \quad \star$$

where $Fr = \dfrac{U_0^2}{ga} = \dfrac{\text{INERTIA}}{\text{GRAVITY}}$ = Froude number

$$We = \frac{\rho U_0^2 a}{\sigma} = \frac{\text{INERTIA}}{\text{CURVATURE}} = \text{Weber number}$$

Now flux conservation ensures

$$Q = 2\pi \int_0^r U(z)\, r(z)\, dr = \pi r^2 U(z) = \pi a^2 U_0$$

from which we find

$$\frac{r(z)}{a} = \left(\frac{U_0}{U(z)}\right)^{\frac{1}{2}} = \left[1 + \frac{2}{Fr}\frac{z}{a} + \frac{2}{We}\left(1 - \frac{a}{r}\right)\right]^{-\frac{1}{4}}$$

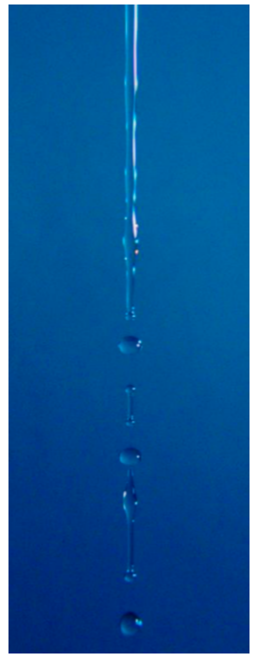
This may be solved algebraically to yield $r(z)/a$ by subbing into ☆ to deduce $U(z)$.

In the $We \to \infty$ limit, one finds

$$\frac{r}{a} = \left(1 + \frac{2gz}{U_0^2}\right)^{-\frac{1}{2}}, \qquad \frac{U(z)}{U_0} = \left(1 + \frac{2gz}{U_0^2}\right)^{\frac{1}{2}}$$

The Rayleigh – Plateau Instability

- the instability of a $\underbrace{\text{quiescent}}_{\text{static}}$ fluid cylinder bound by σ
- neglect influence of gravity g and viscosity ν
- consider a cylinder of radius R_0



Internal pressure:

$$P_0 = \sigma \underline{\nabla} \cdot n = \frac{\sigma}{R_0}$$

assuming $P_{atm} = 0$.

We consider the evolution of infinitesimal varicose perturbations, which enables us to linearize the governing eqns.

The perturbed columnar surface takes the form:

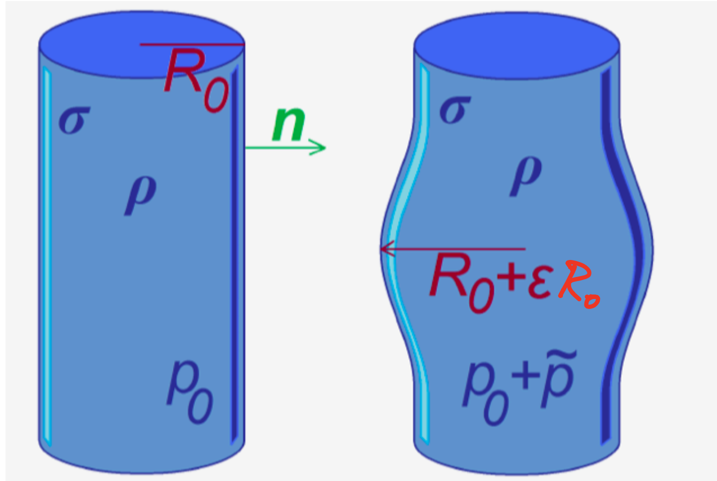
$$\tilde{R} = R_0 \left(1 + \varepsilon e^{\omega t + ikz}\right) \qquad \text{where } k = \frac{2\pi}{\lambda}$$

Find k for which $Re[\omega] > 0$, in which case the perturbation will grow. Also, we'll find the mode k that maximizes $Re[\omega]$, the fastest growing mode.

Note: perturbation amplitude $\varepsilon \ll 1$
ω is the growth rate
$k = 2\pi/\lambda$ is the wavenumber.

Define: $\tilde{u}_r$ = radial perturbation velocity
$\tilde{u}_z$ = axial " "
$\tilde{p}$ = perturbation pressure

Substituting $\tilde{u}_r$, $\tilde{u}_z$, $\tilde{p}$ into Navier–Stokes eqns and retaining terms only to order ε yields:
[NB: $\underline{u} \cdot \underline{\nabla}\, \underline{u} \to 0$]



Momentum eqns:

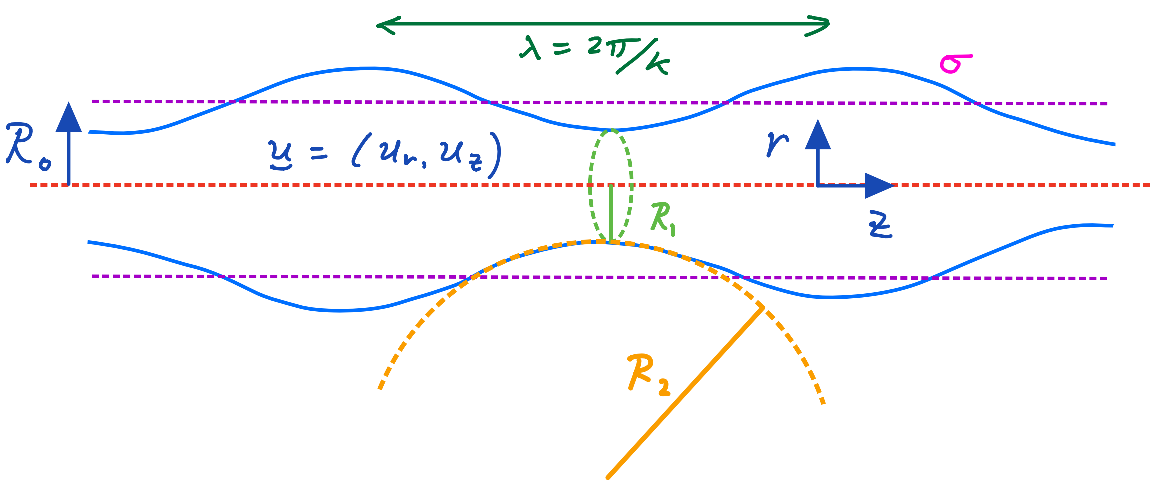
$$\frac{\partial \tilde{u}_r}{\partial t} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial r}, \qquad \frac{\partial \tilde{u}_z}{\partial t} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial z}$$

Continuity: $$\frac{\partial \tilde{u}_r}{\partial r} + \frac{\tilde{u}_r}{r} + \frac{\partial \tilde{u}_z}{\partial z} = 0$$

We anticipate that all perturbations have the same form as the surface disturbance, so

$$\tilde{u}_r = R(r)\, e^{\omega t + ikz}, \qquad \tilde{u}_z = Z(r)\, e^{\omega t + ikz}$$

$$\tilde{p} = P(r)\, e^{\omega t + ikz}$$



Subing $\bar{u}_r$, $\bar{u}_z$, $\bar{p}$ into linearized NS yields

Momentum: $\omega R = -\frac{1}{\rho}\frac{dP}{dr}$ $\otimes$, $\omega Z = \frac{-ik}{\rho} P$ $\star$

Continuity: $\frac{dR}{dr} + \frac{R}{r} + ikZ = 0$

Subing in Z from $\star$, then take $\frac{d}{dr}$ to deduce

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} - \left[1 + (kr)^2\right] R = 0$$

This corresponds to the modified Bessel eqn of order 1, whose solutions are the modified Bessel fns of the first and second kind, $I_1(kr)$ and $K_1(kr)$. We note that $K_1(kr) \to \infty$ as $r \to 0$; thus, the well-behavedness of our sol'n requires

$$R(r) = C' I_1(kr)$$

where C' is a const to be determined by application of appropriate boundary conditions.

Deduce $P(r)$ from $R(r)$ by integrating $\otimes$:

$$P(r) = -\frac{\omega\rho C'}{k} I_0(kr) \quad *$$

and using the Bessel fn identity $I_0'(x) = I_1(x)$

Now we apply the boundary conditions.

1. Kinematic Condition :

$$\frac{\partial \tilde{R}}{\partial t} = \underline{u} \cdot \underline{n} \simeq \bar{u}_r \quad \text{at } r \simeq R_0$$

$$\Rightarrow R_0 \varepsilon \omega = C_1 I_1(kR_0)$$

$$\Rightarrow C_1 = \frac{\varepsilon \omega R_0}{I_1(kR_0)}$$

Thus $*$ $\Rightarrow$ $\boxed{P(r) = -\frac{\varepsilon \omega^2 \rho R_0}{k} \frac{I_0(kR_0)}{I_1(kR_0)}}$ ☆

2. Normal Stress BC :

$$P_0 + \tilde{P} = \sigma \underline{\nabla} \cdot \underline{n} = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \quad \boxtimes$$

where $\frac{1}{R_1} = \frac{1}{R_0(1+\varepsilon e^{\omega t + ikz})} \simeq \frac{1}{R_0}\left(1 - \varepsilon e^{\omega t + ikz}\right)$

$$\frac{1}{R_2} = -\frac{\partial}{\partial z}\left[\frac{\partial R_1/\partial z}{\left[1+\left(\frac{\partial R_1}{\partial z}\right)^2\right]^{\frac{1}{2}}}\right] \simeq -\frac{\partial^2 R_1}{\partial z^2}$$

$$= \varepsilon k^2 R_0 e^{\omega t + ikz}$$

Subing into $\boxtimes$:

$$\cancel{P_0} + \tilde{P} = \cancel{\frac{\sigma}{R_0}} - \frac{\varepsilon \sigma}{R_0}\left(1 - k^2 R_0^2\right) e^{\omega t + ikz}$$

Compare to ☆ to find :

$$\tilde{p} = -\frac{\Sigma\sigma}{R_0}\left(1-k^2R_0^2\right)e^{\omega t + ikz}$$

$$= -\frac{\Sigma\omega^2\rho R_0}{k}\,\frac{I_0(kR_0)}{I_1(kR_0)}\,e^{\omega t+ikx}$$

Dispersion Relation : relates ω to k

$$\boxed{\omega^2(k) = \frac{\sigma}{\rho R_0^2}\,k\,\frac{I_1(kR_0)}{I_0(kR_0)}\left(1-k^2R_0^2\right)}$$

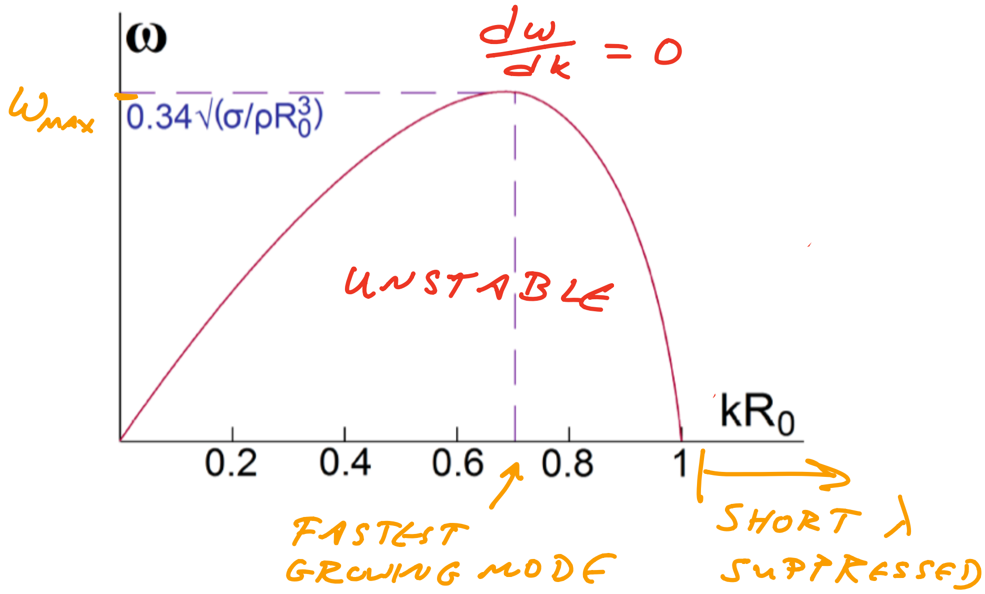
Note : 1. The column is only unstable to wavelengths $\lambda = \frac{2\pi}{k}$ that exceed the circumference of the jet.

2. Fastest growing mode occurs when

$$\frac{d\omega}{dk} = 0,$$

i.e. $kR_0 = 0.697$, when the wavelength

$$\lambda = \frac{2\pi}{k} \simeq 9.02\,R_0.$$

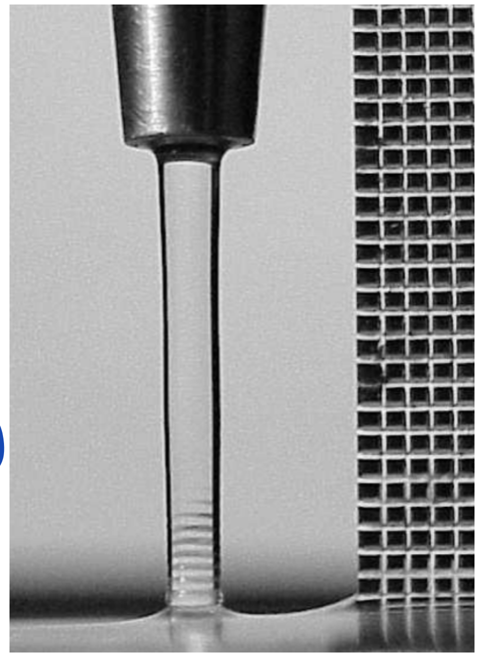


3. By inverting the MAX growth rate, ω_{max}, one finds break-up time, $t_{break} \simeq 2.91\sqrt{\frac{\rho R_0^3}{\sigma}}$

e.g. a water jet of diameter 1 cm,

$t_{break} \sim \frac{1}{8}$ sec, which is roughly consistent with casual observation.

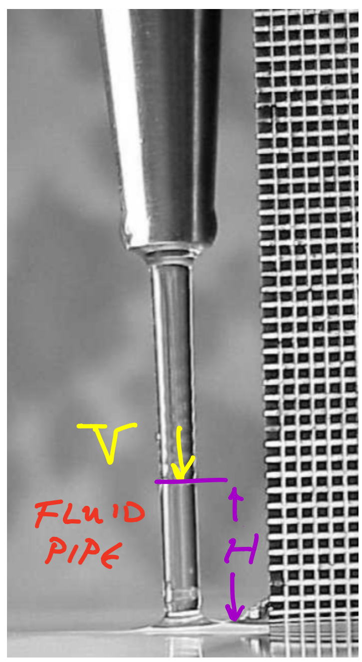
(4.) When a vertical jet impinges on a fluid bath, a standing field of waves may be excited on its base. Requiring the wave phase speed to be equal to the jet speed V:



$$V^2 = \frac{\omega^2}{k^2} = \frac{\sigma}{\rho k R_0^2}\frac{I_1(kR_0)}{I_0(kR_0)}\left(1-k^2R_0^2\right)$$

Provided the jet speed $V(z)$ is known, one may rationalize the observed λ.

Now, add soap to the bath, generating Marangoni stresses that rigidify the surface of the "fluid pipe".



Once fluid enters the pipe, a boundary layer develops on its inner wall, owing to NO-SLIP BC there.

Balance viscous + Marangoni stresses on the pipe surface:

$$\underbrace{\rho\nu}_{\mu}\frac{V}{\delta_H} \sim \frac{\Delta\sigma}{H} \quad *$$

We expect the b.l. to grow with distance to pipe entrance z according to a Blasius b.l.

$$\frac{\delta}{a} \sim \left(\frac{\nu z}{a^2 V}\right)^{\frac{1}{2}}$$

Subbing for $\delta(H)$ into $*$ yields

$$\boxed{H \sim \frac{(\Delta\sigma)^2}{\rho\mu V^3}}$$

increases with $\Delta\sigma$
decreases with V

$\Rightarrow$ Hancock + Bush, "Fluid Pipes", JFM (2003).

5. We have considered here the inviscid case. Viscosity acts to <u>increase</u> the wavelength of the most unstable mode relative to that in the inviscid case, $\sim 9R_0$.