18.357: Lecture 13

Fluid sheets, bells and fishbones

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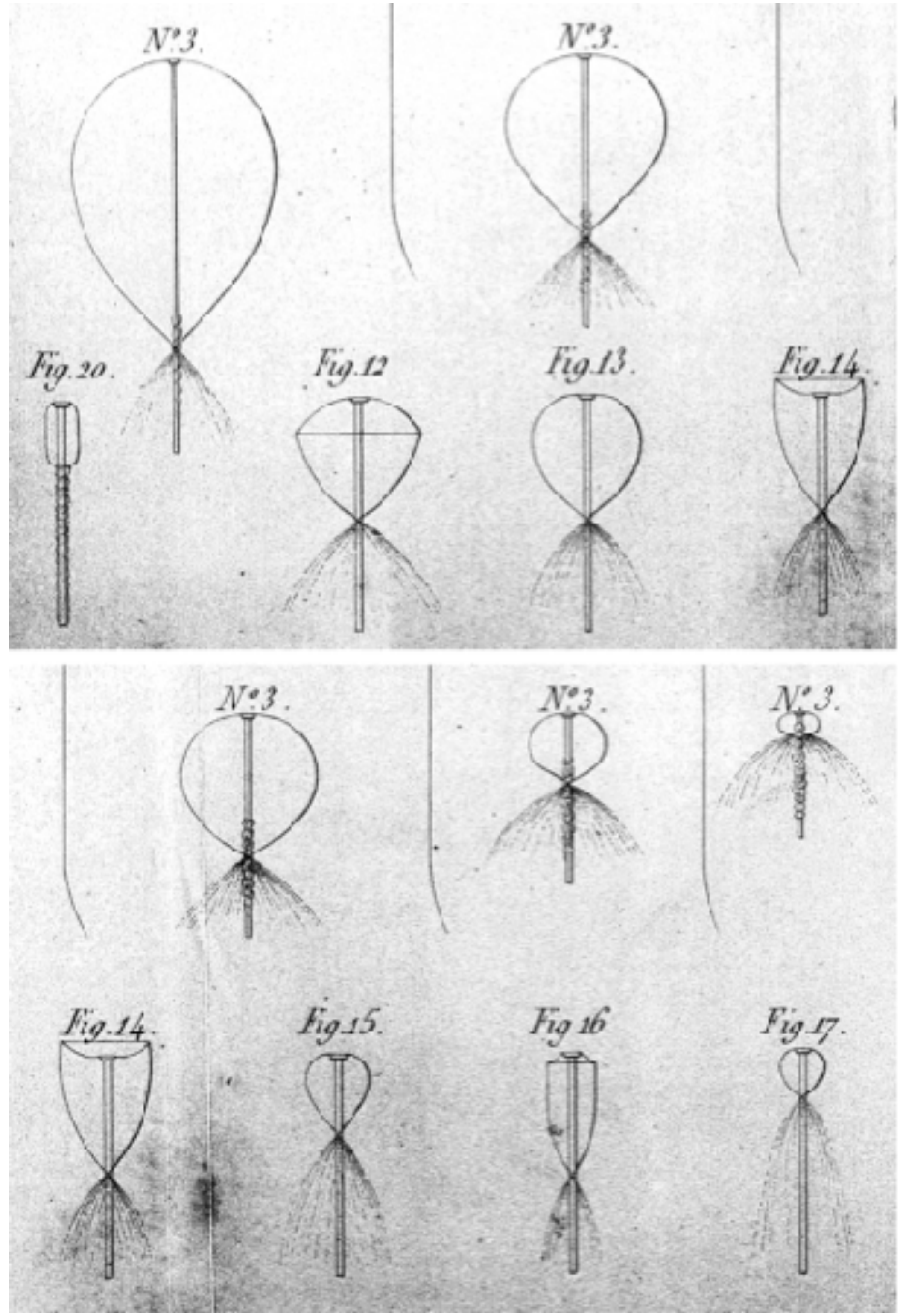
Savart (1833)

Rayleigh (1879)

Taylor (1959abc)



Industrial application: spray atomization

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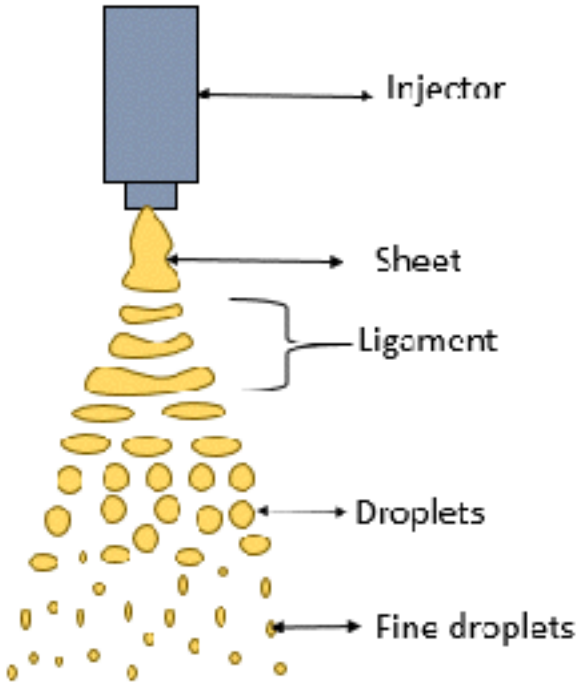
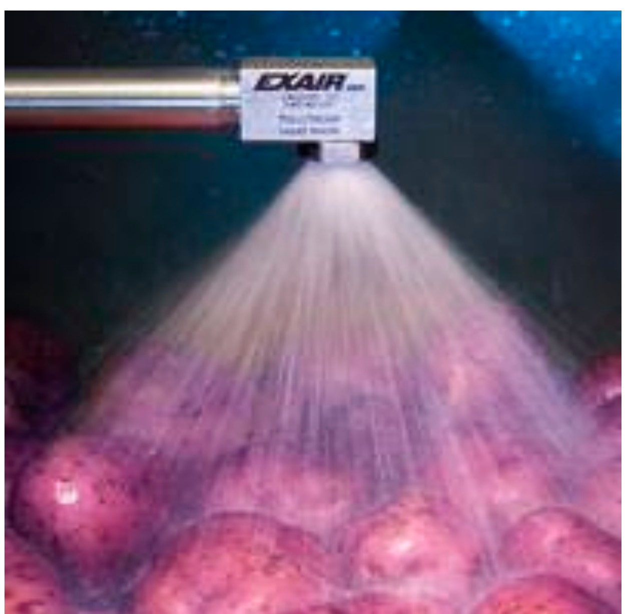
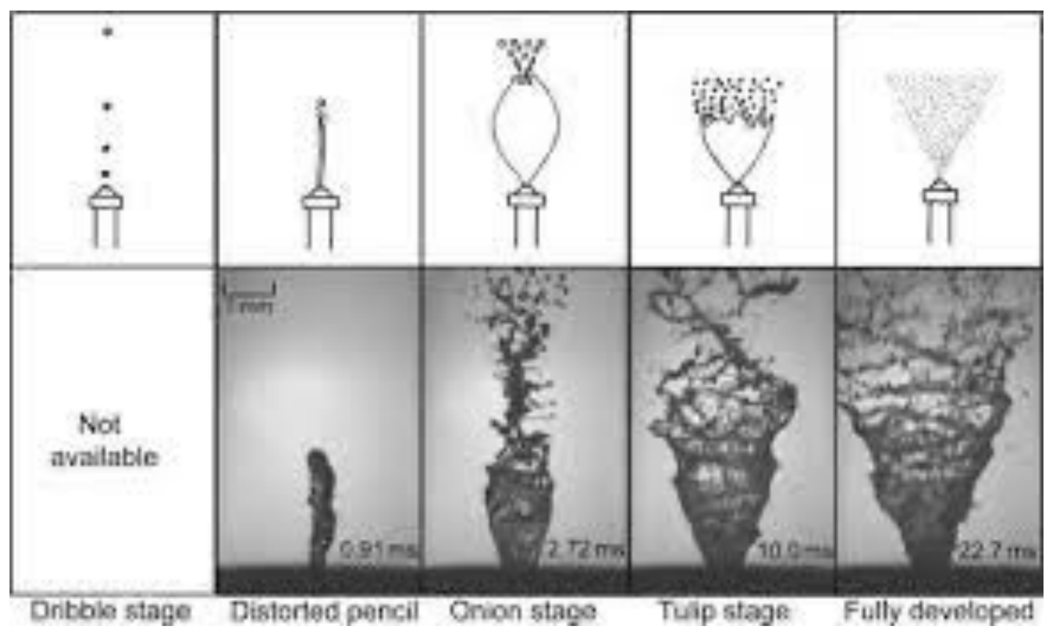
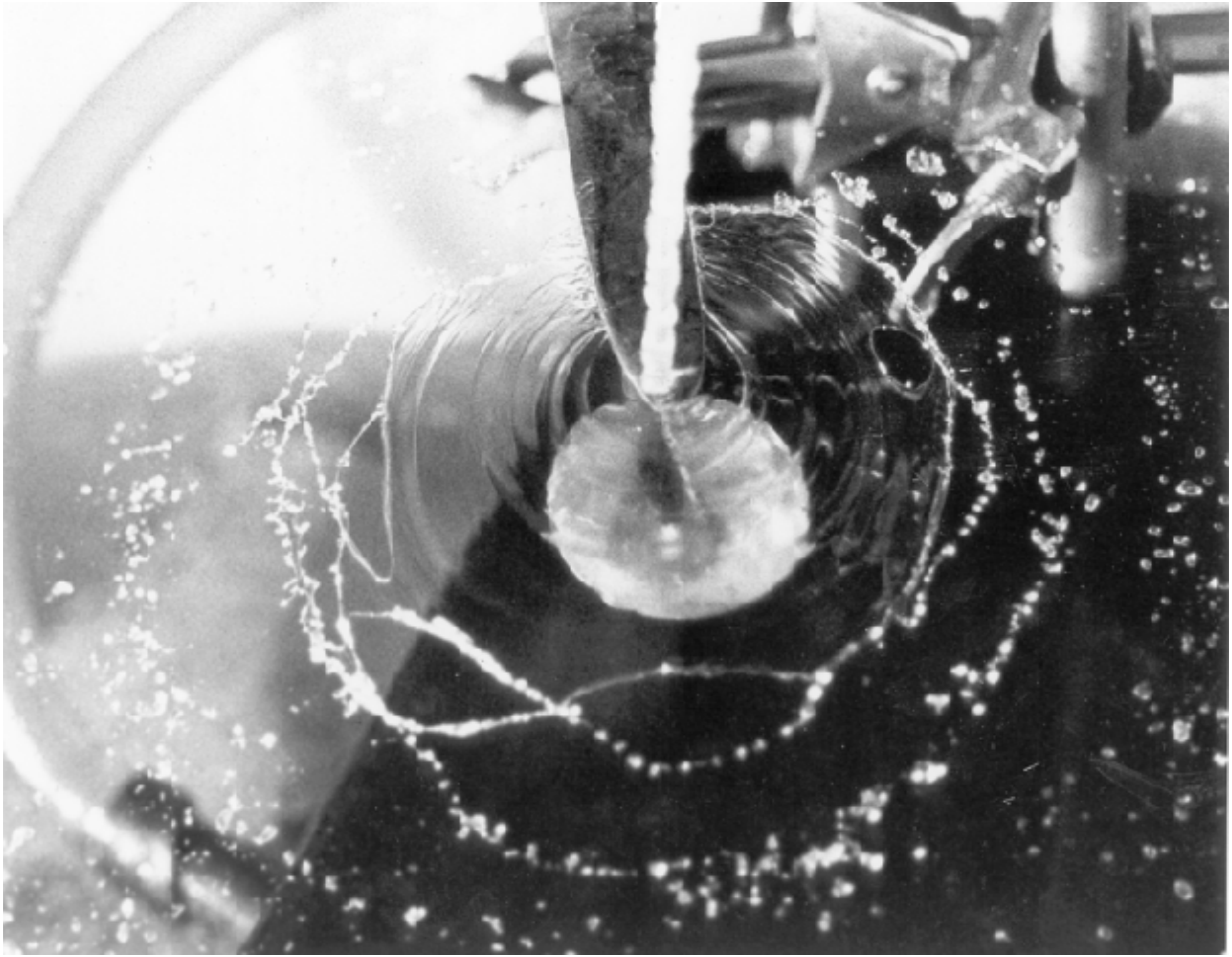
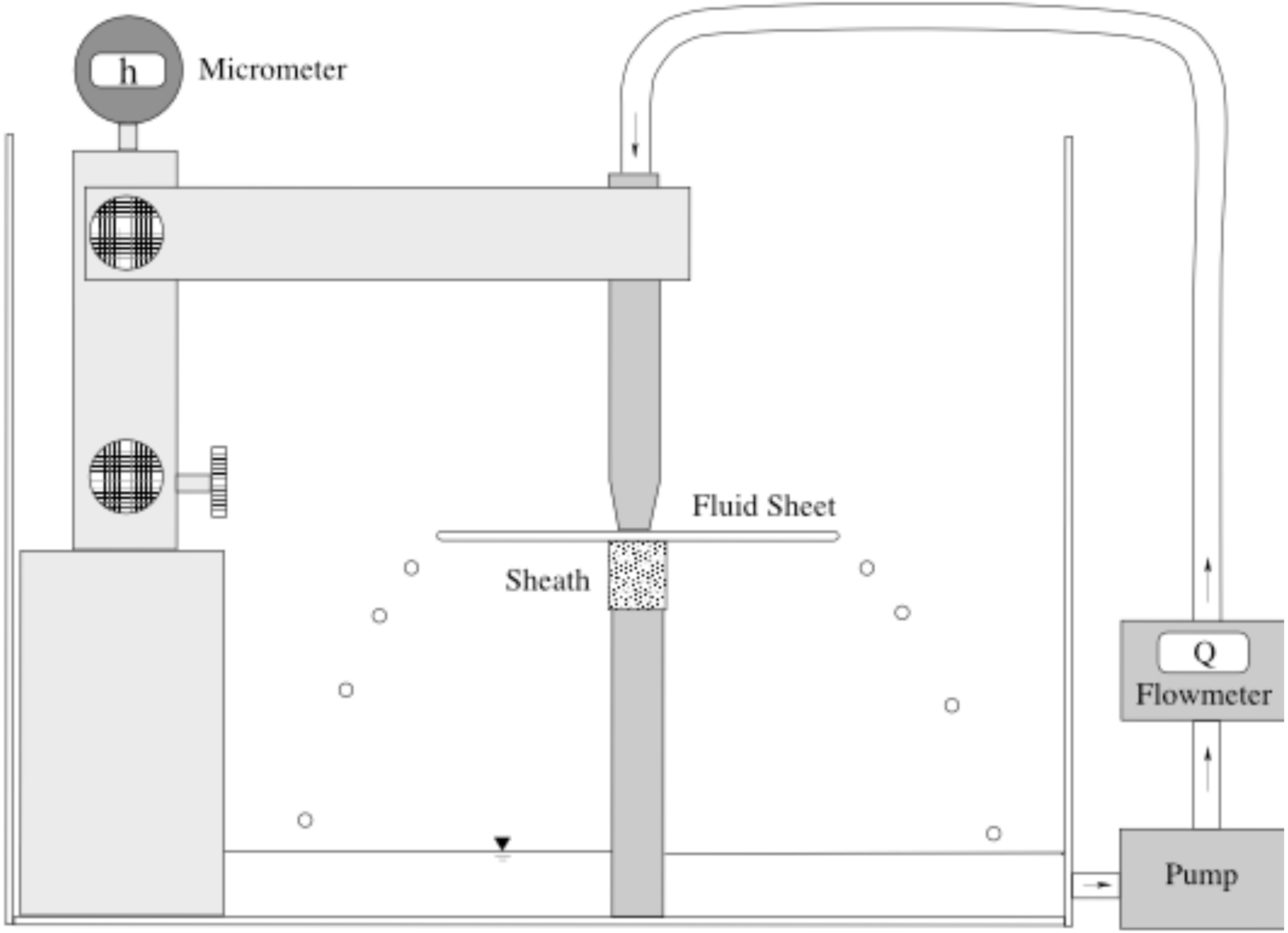










Glycerol-water solutions: $\nu \sim 1 - 100$ cS

Recall …

Curvature force on rim:

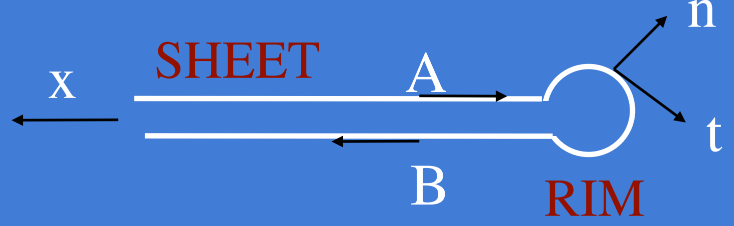
$$F_c = \int_S \sigma\,(\nabla\cdot\mathbf{n})\,\mathbf{n}\,dS$$

2D Surfaces

Frenet-Serret equation: $$(\nabla\cdot\mathbf{n})\,\mathbf{n} = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}\ell}$$

$$F_c = \int_C \sigma\,(\nabla\cdot\mathbf{n})\,\mathbf{n}\,d\ell = \int_A^B \sigma\,\frac{d\mathbf{t}}{d\ell}\,d\ell = \sigma(\mathbf{t}_\mathbf{B} - \mathbf{t}_\mathbf{A})$$

E.g. Force/length on edge of planar sheet



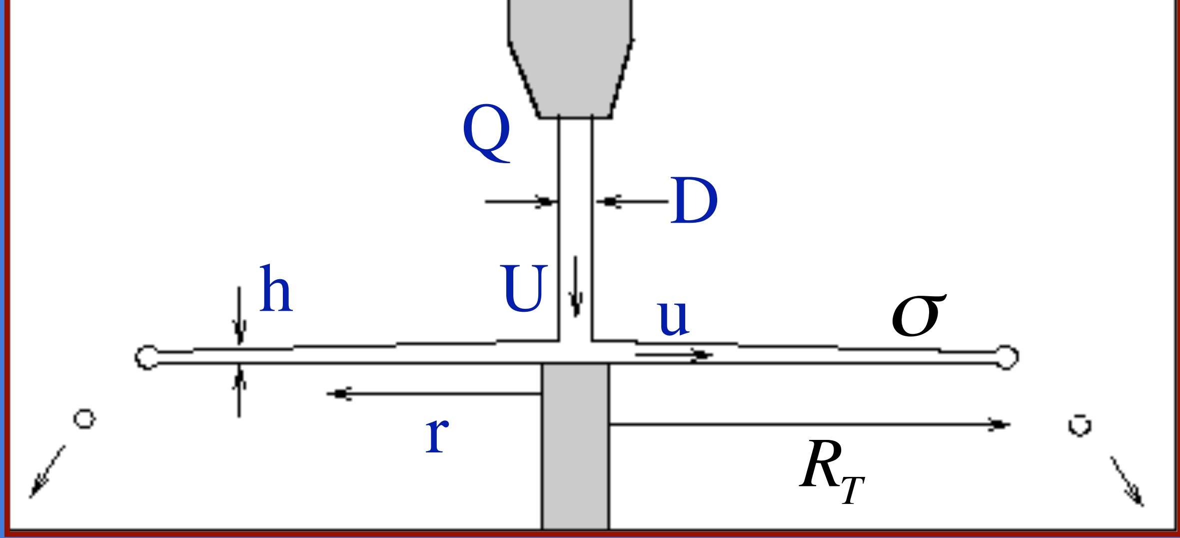
$$F_c = \sigma(\mathbf{t}_\mathbf{B} - \mathbf{t}_\mathbf{A}) = 2\sigma\,\mathbf{x}$$

independent of detailed rim shape

Stable sheet

(Savart 1833, Taylor 1959)

$$We = \frac{\rho U^2 D}{\sigma} < 1000$$



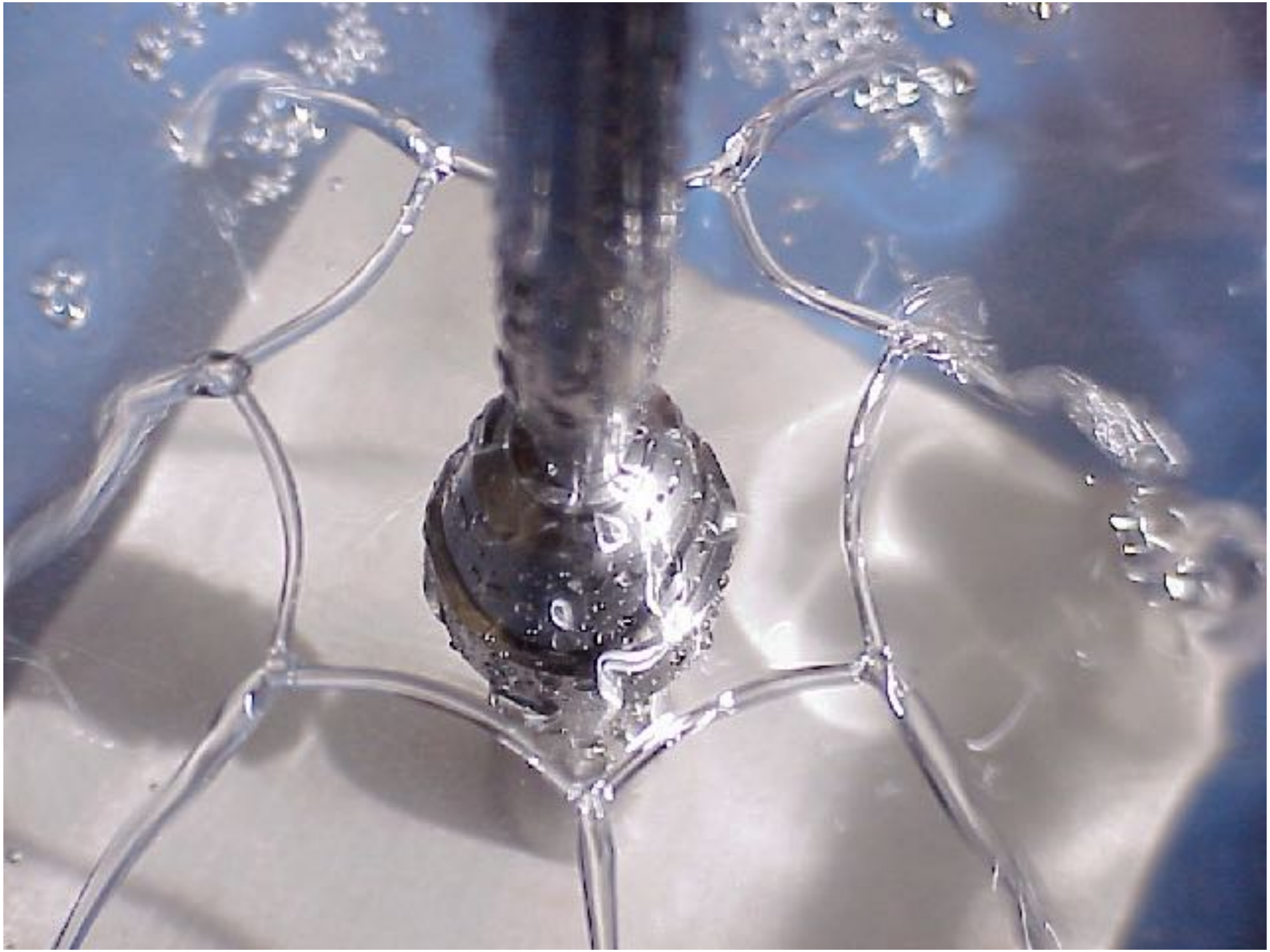
- sheet radius: balance of surface tension and inertia

$$\rho u^2 h \sim 2\sigma$$

- sheet thickness : $h = \frac{Q}{2\pi r u} \sim \frac{1}{r}$

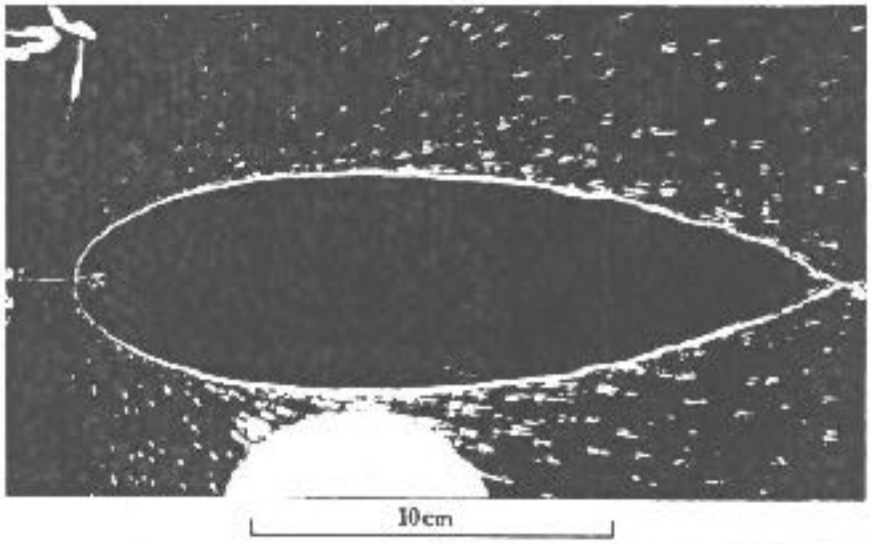
- Taylor radius: $$R_T = \frac{\rho Q u}{4\pi\sigma}$$

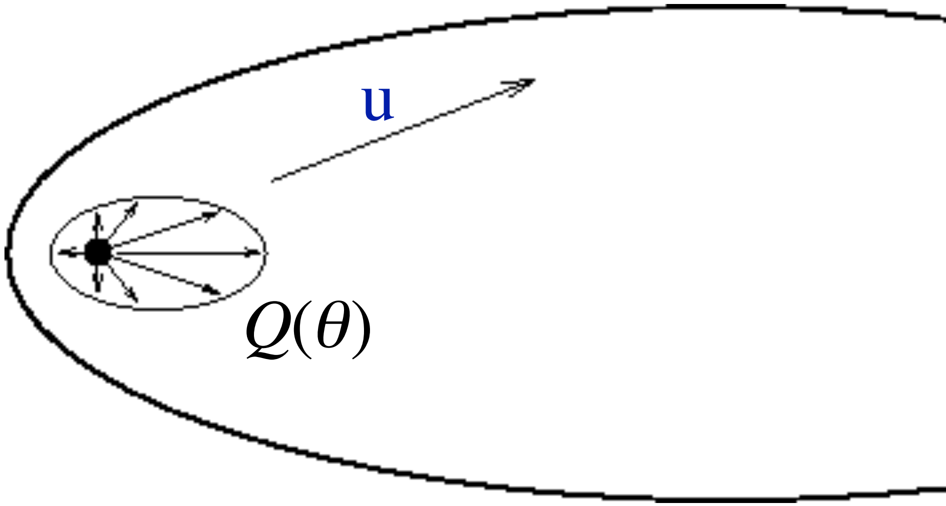
- toroidal sheet rim releases drops through Rayleigh instability



Formation of thin flat sheets of water

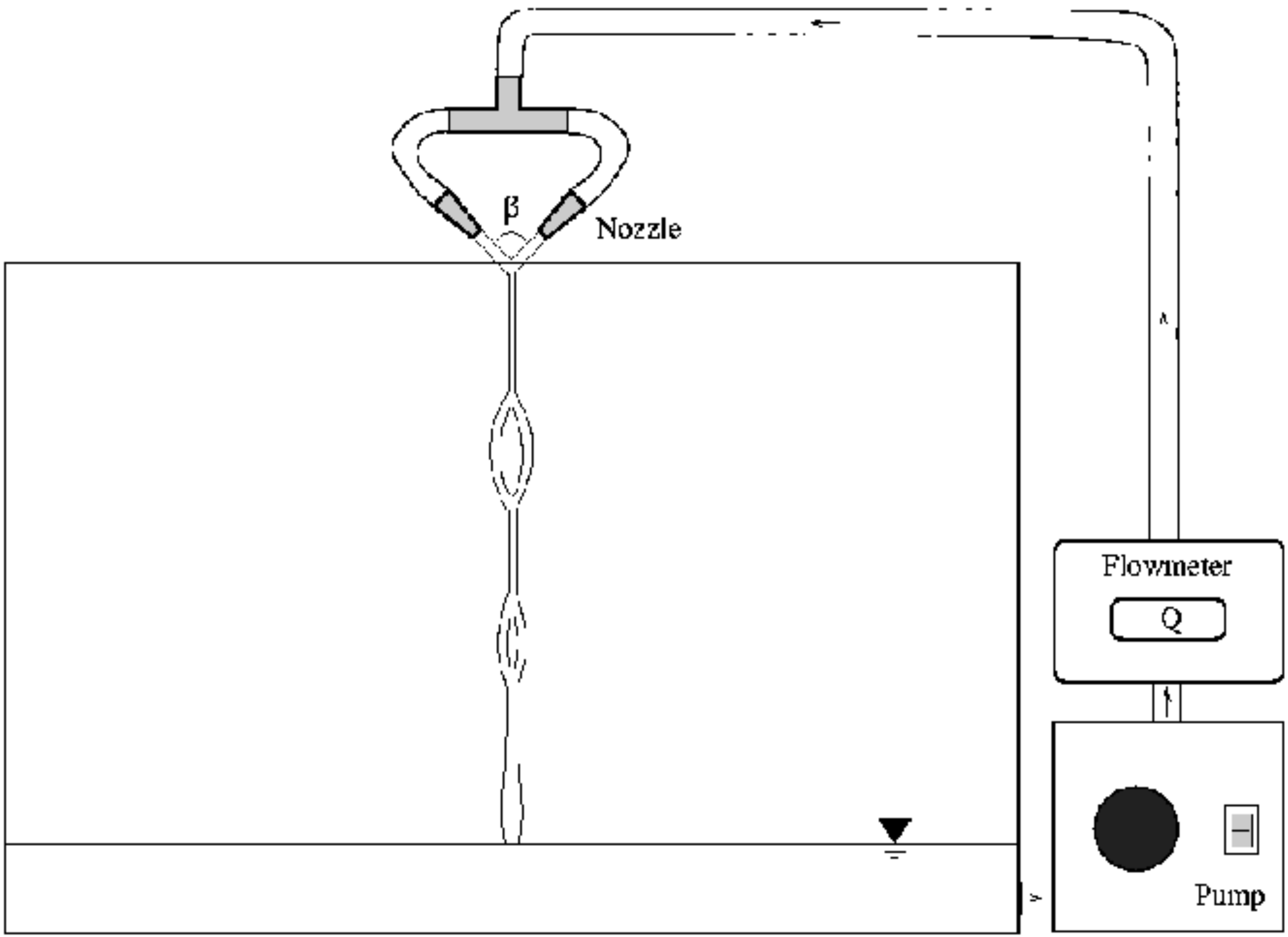
(Taylor 1960)

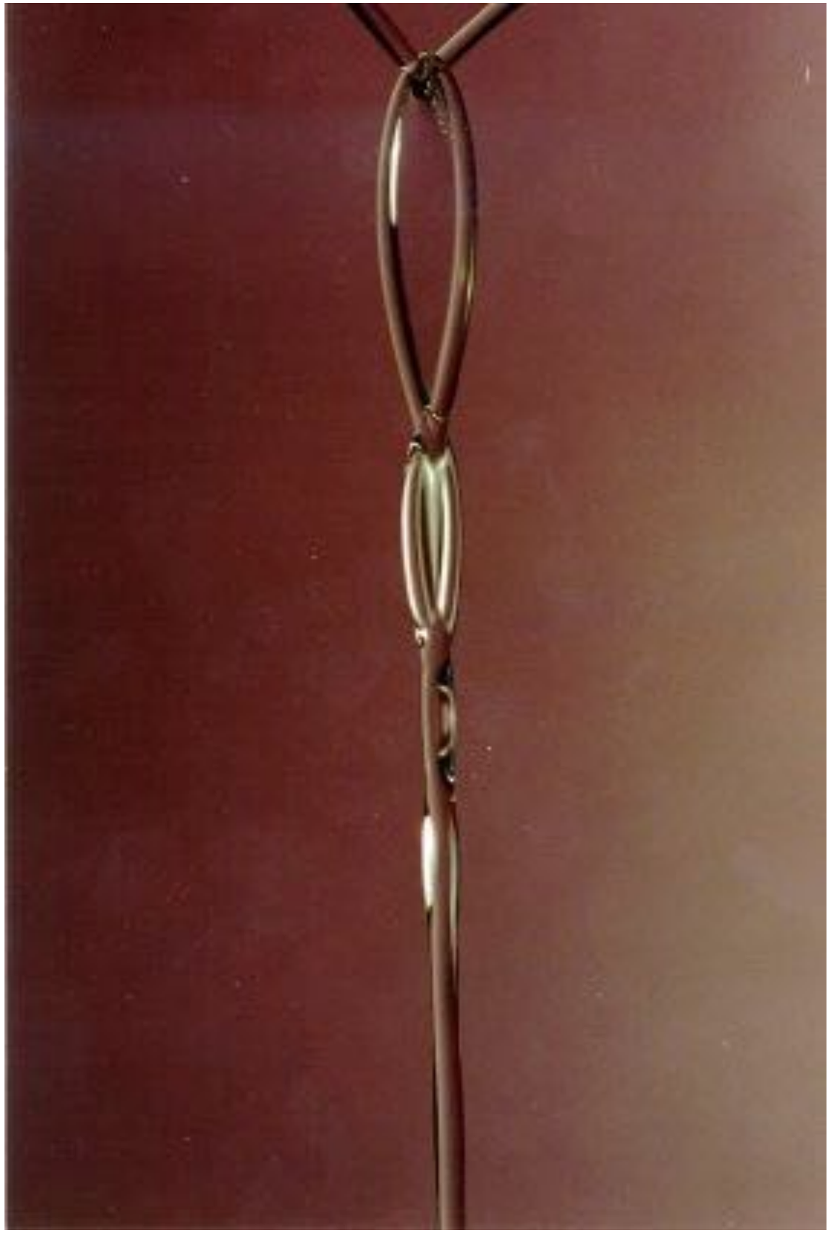


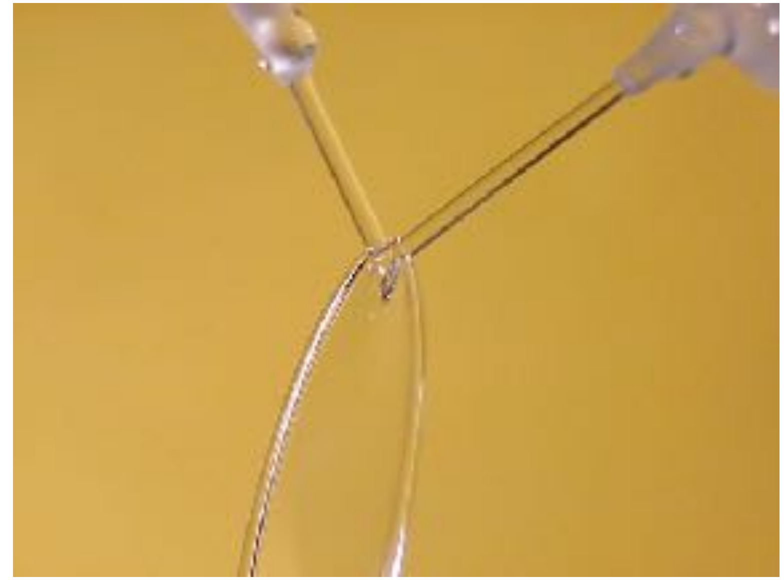


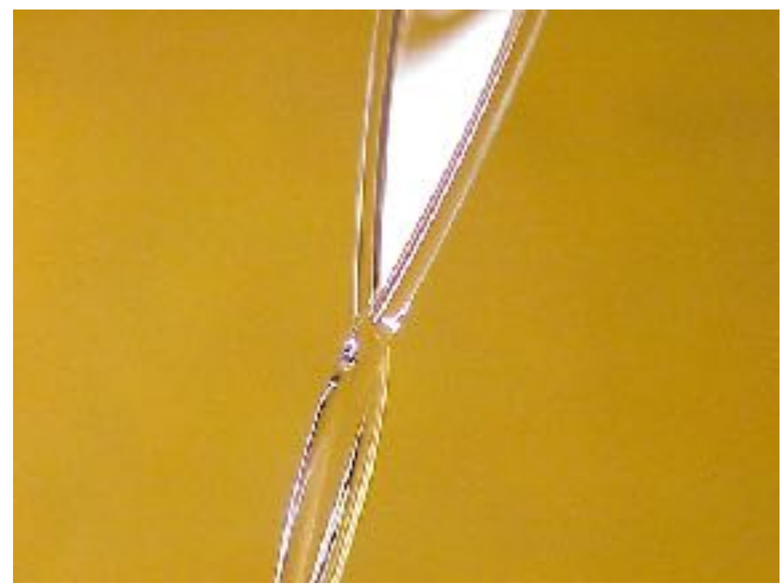
- unstable rims eject droplets
- sheet shape prescribed by balance: $\rho u_n^2 h \sim 2\sigma$
- sheet thickness: $h = \dfrac{Q(\theta)}{2\pi r u}$
- Taylor radius: $$R_T = \frac{\rho u_n^2 Q(\theta)}{4\pi \sigma u}$$
- flux distribution $Q(\theta)$ deduced experimentally, or calculated

Sheets with stable rims


β
Nozzle
Flowmeter
Q
Pump



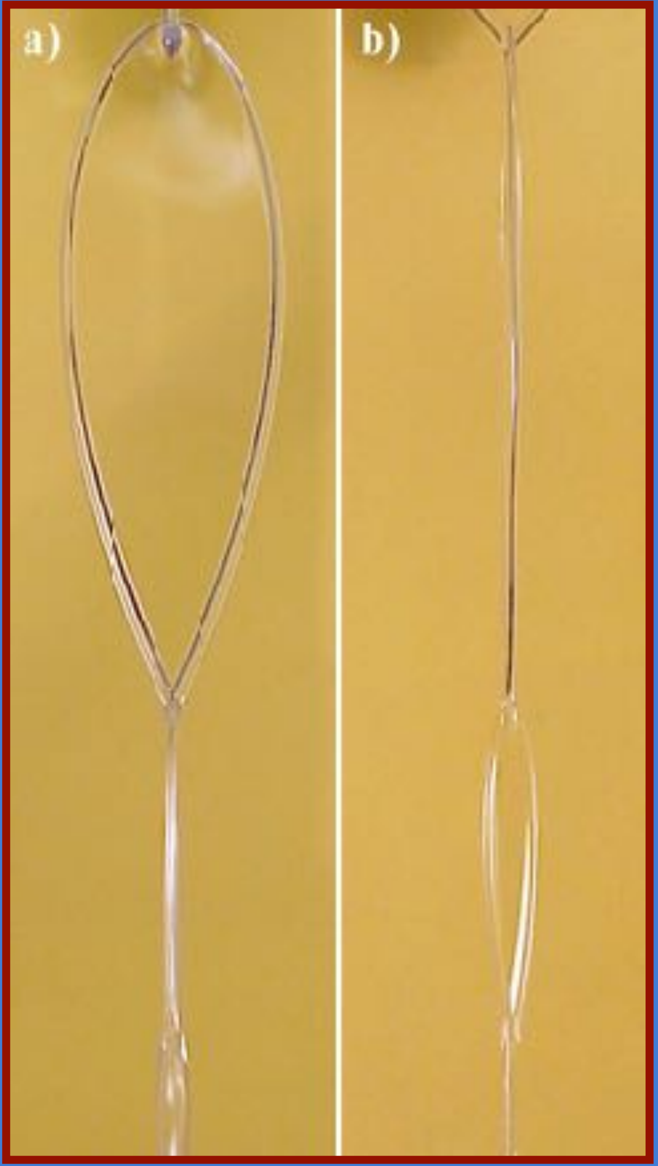




Fluid Chains

- ubiquitous in high Re sheet motion

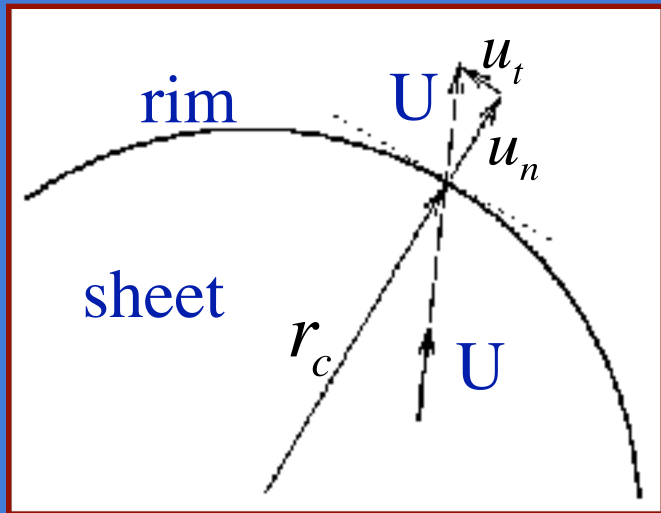
  e.g. pour wine from a lipped jug

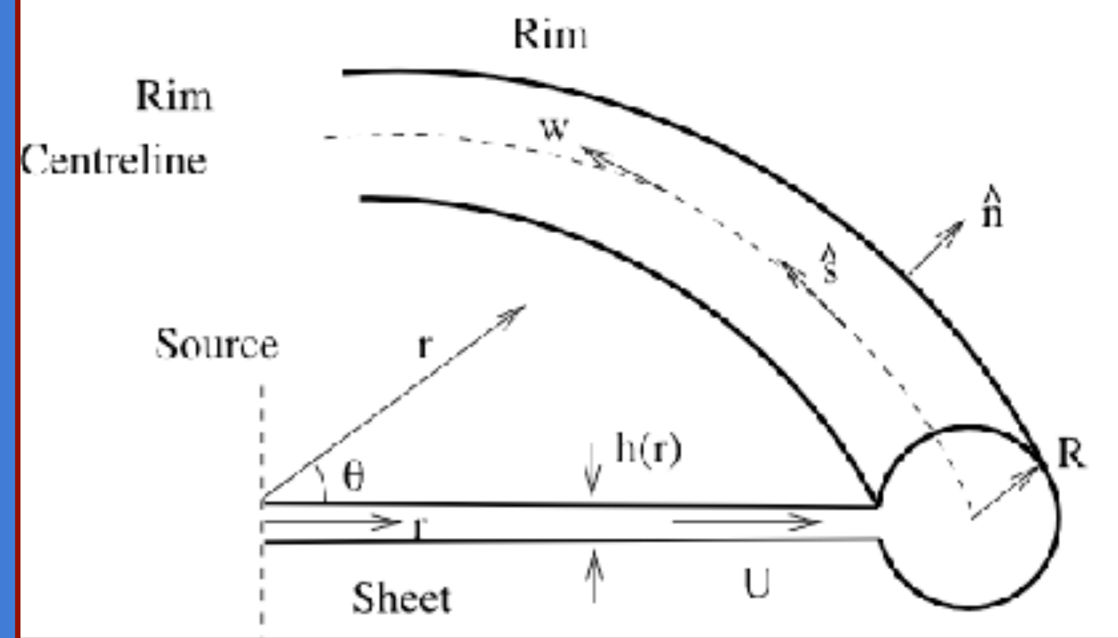


Physical Picture

- colliding jets generate fluid sheets in orthogonal plane
- sheet develops rims and closes through influence of σ
- rim jets again collide … ad infinitum
- successive links decrease in size through viscous damping
- chain eventually coalesces into a cylindrical stream

Rim Dynamics





Mass conservation in rim:

$$\frac{\partial}{\partial s}(\pi R^2 w) = u_n h$$

flux from sheet

Normal force balance:

$$\rho u_n^2 h + \pi \rho R^2 w^2 \frac{1}{r_c} = 2\sigma + \sigma \pi \frac{R}{r_c}$$

inertia centripetal curvature

Tangential force balance:

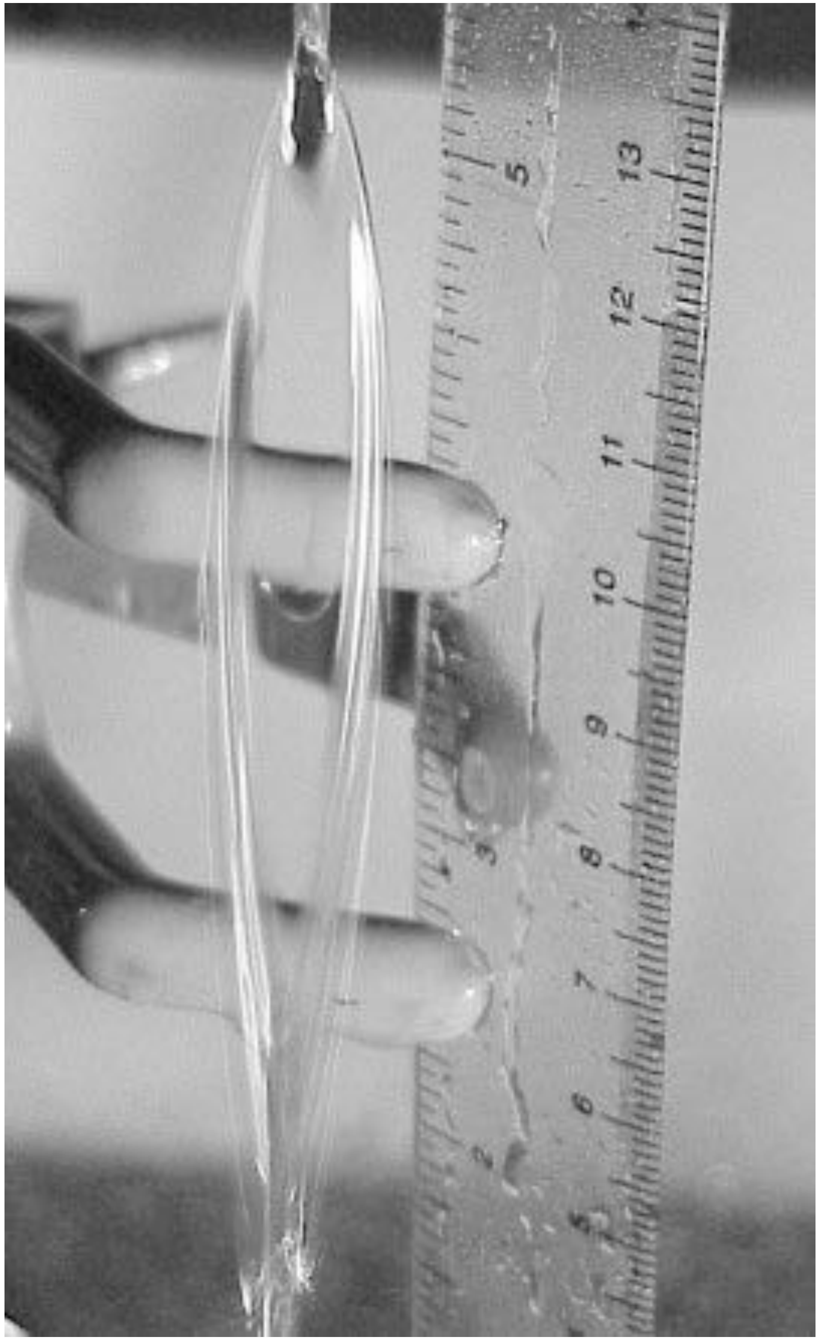
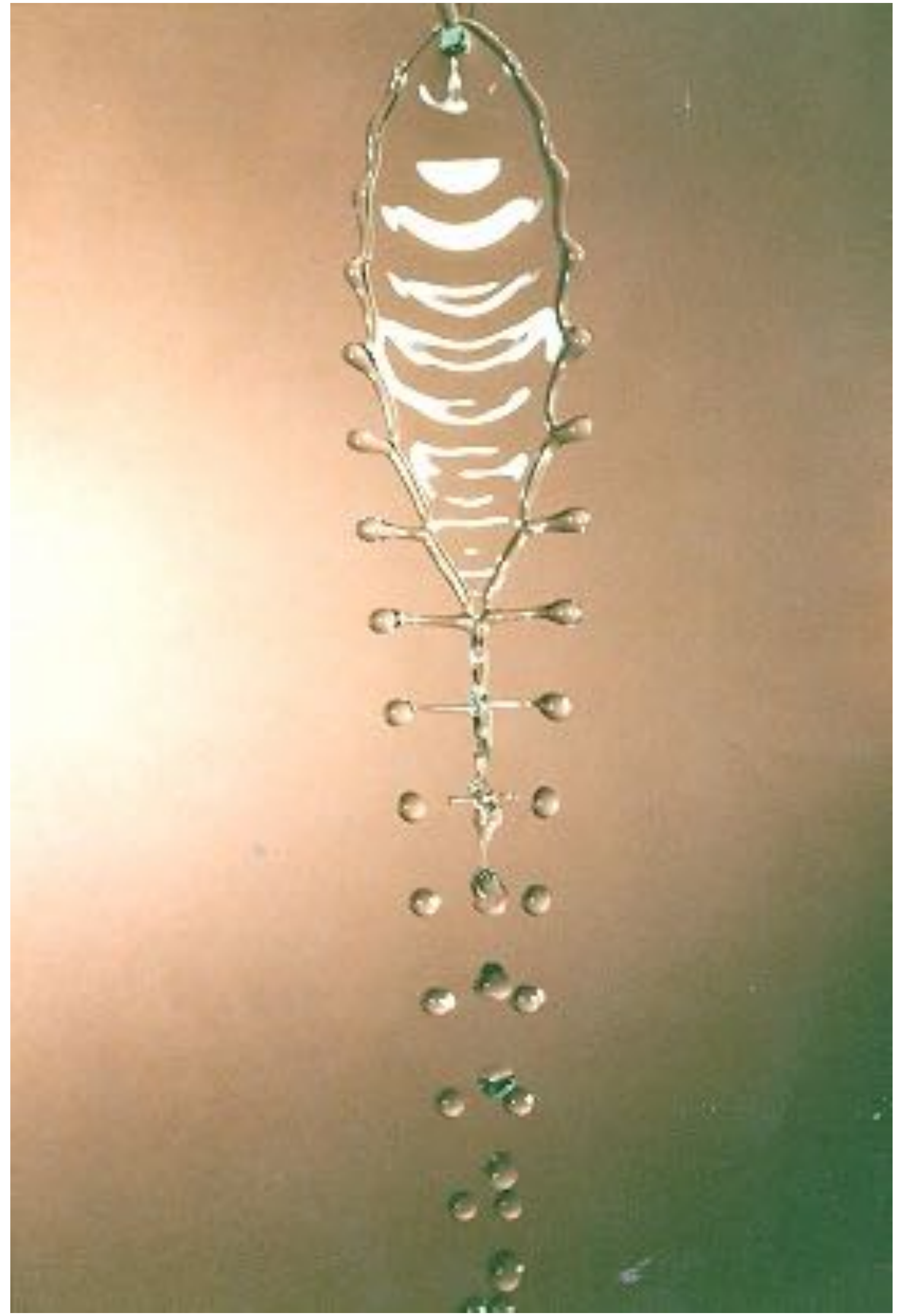
$$\rho \frac{\partial}{\partial s}(\pi R^2 w^2) = -\pi R^2 \sigma \frac{\partial}{\partial s}(\nabla \cdot \mathbf{n}) + 3\mu \frac{\partial}{\partial s}\left(R^2 \frac{\partial w}{\partial s}\right) + \rho h\, u_t u_n$$

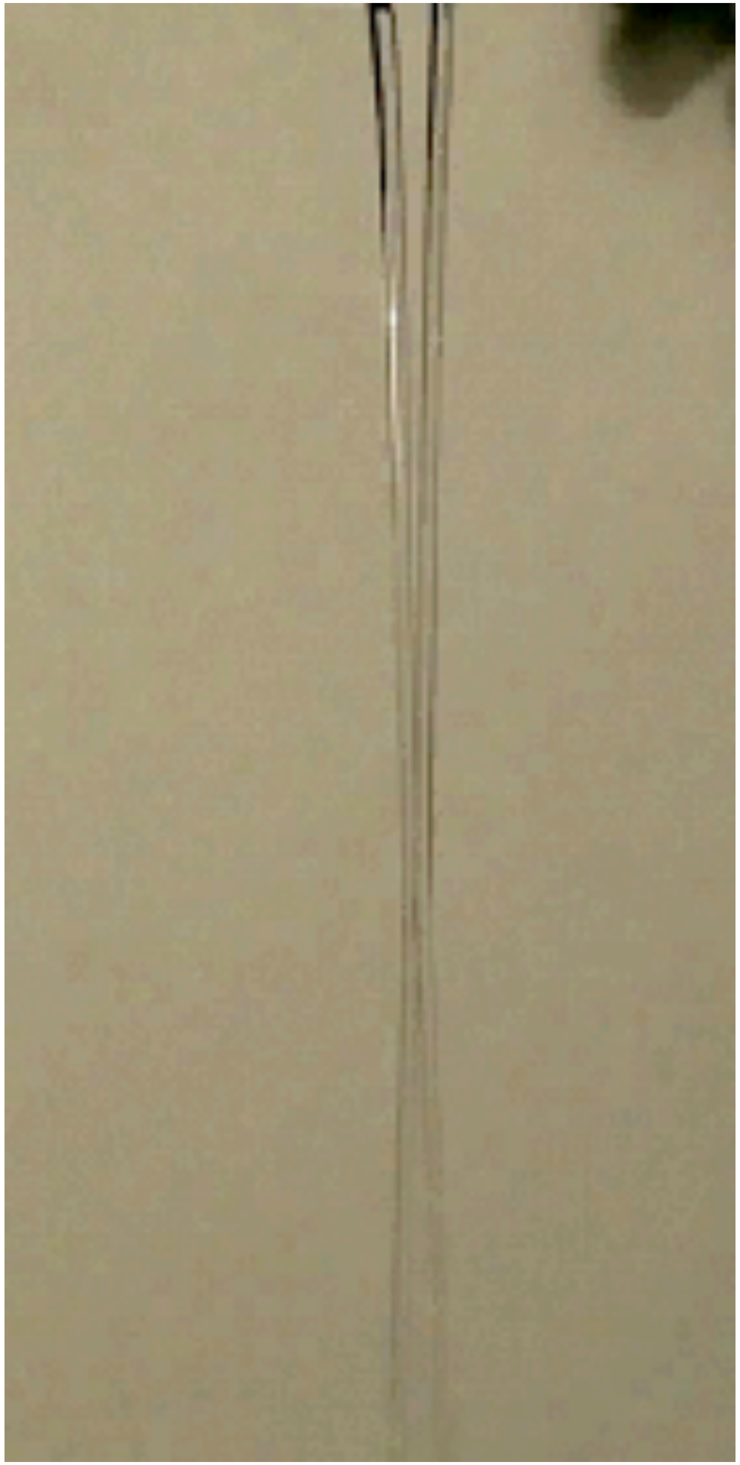
gradient in curvature pressure

viscous resistance

tang. mom. flux from sheet

Rim instability

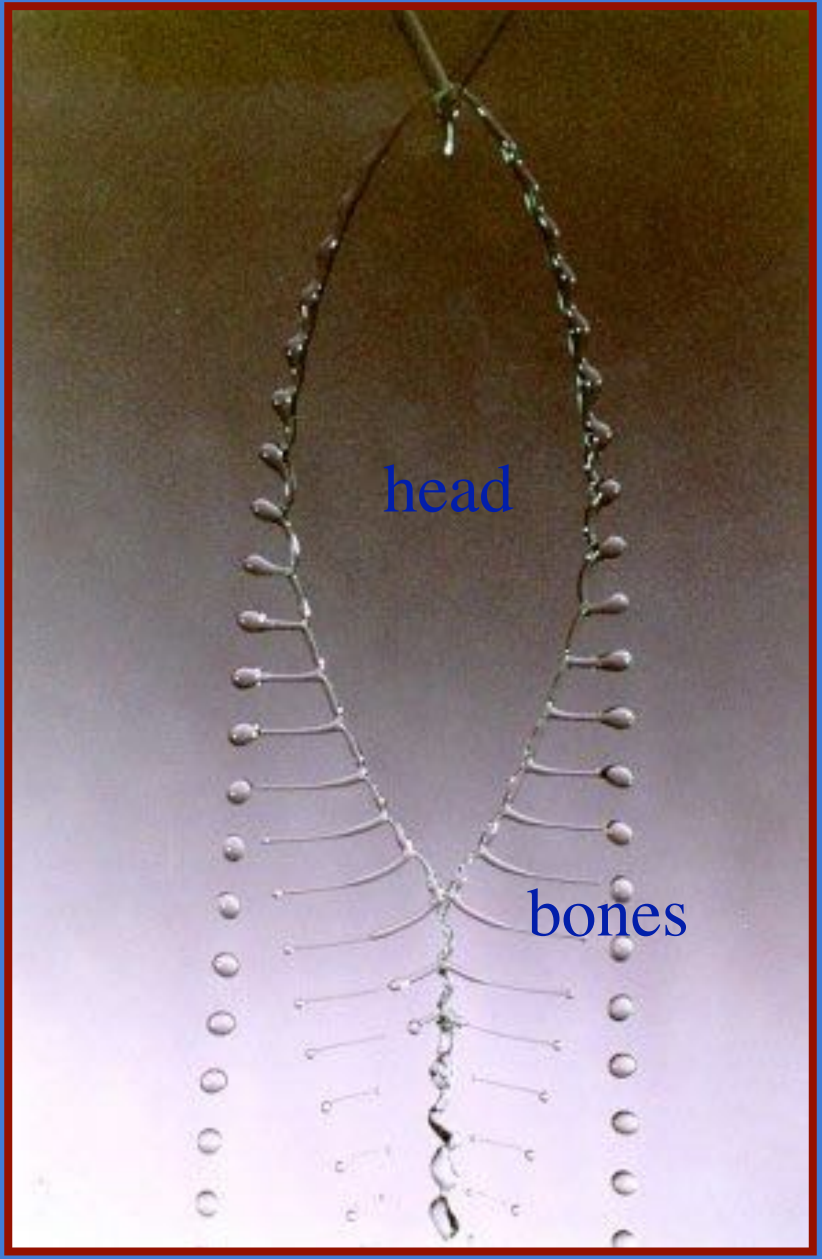





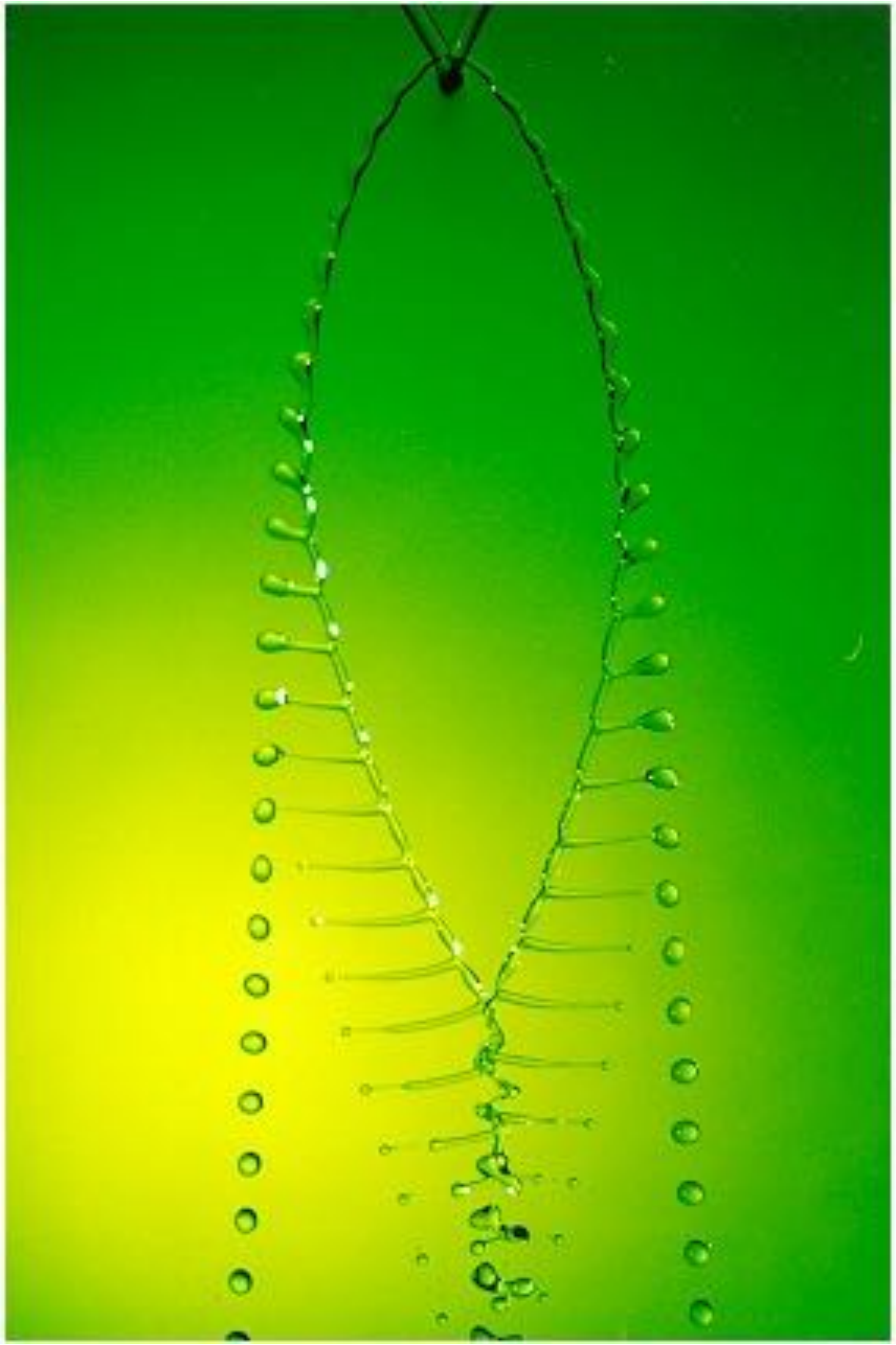
Fluid Fishbones

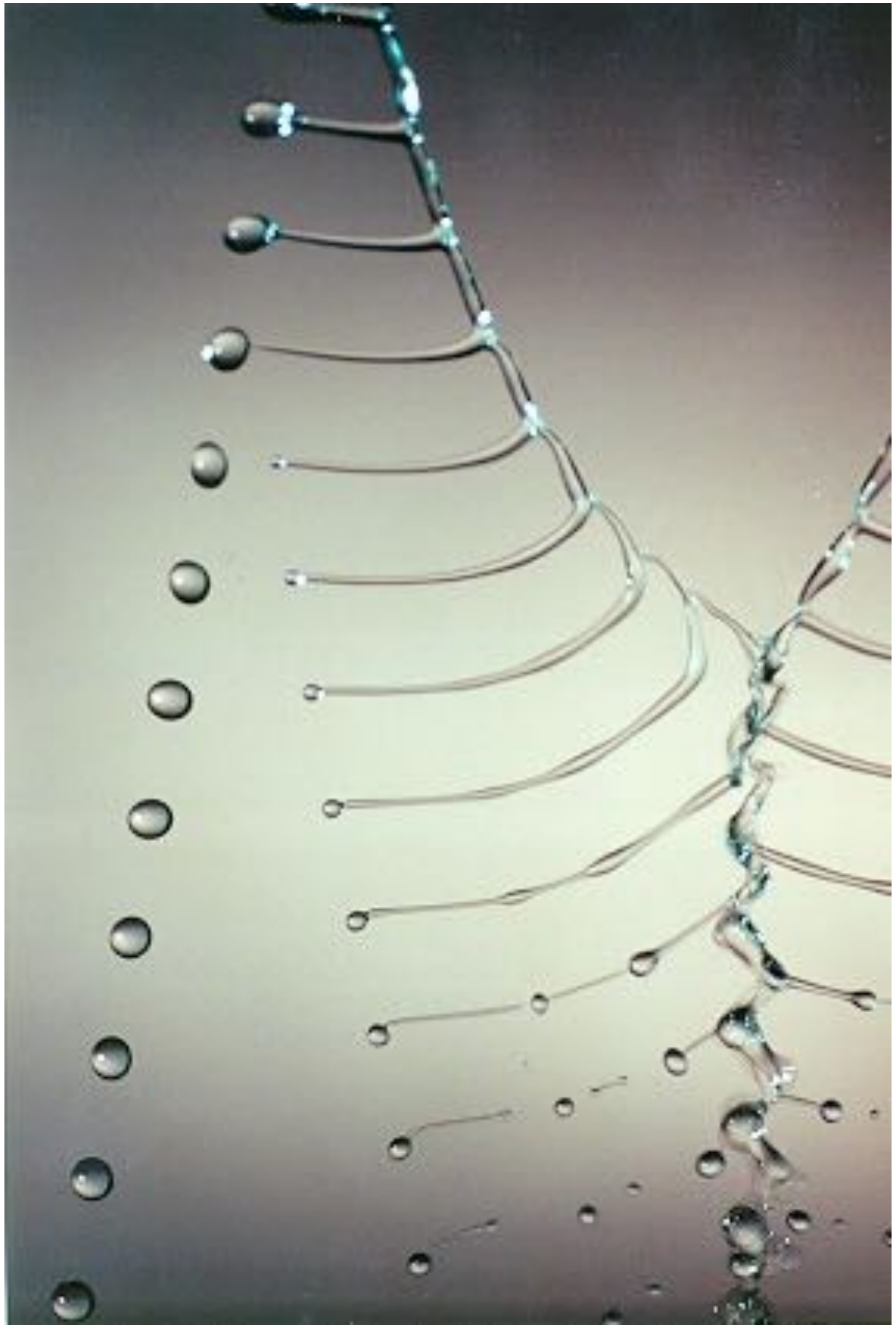
Physical Picture

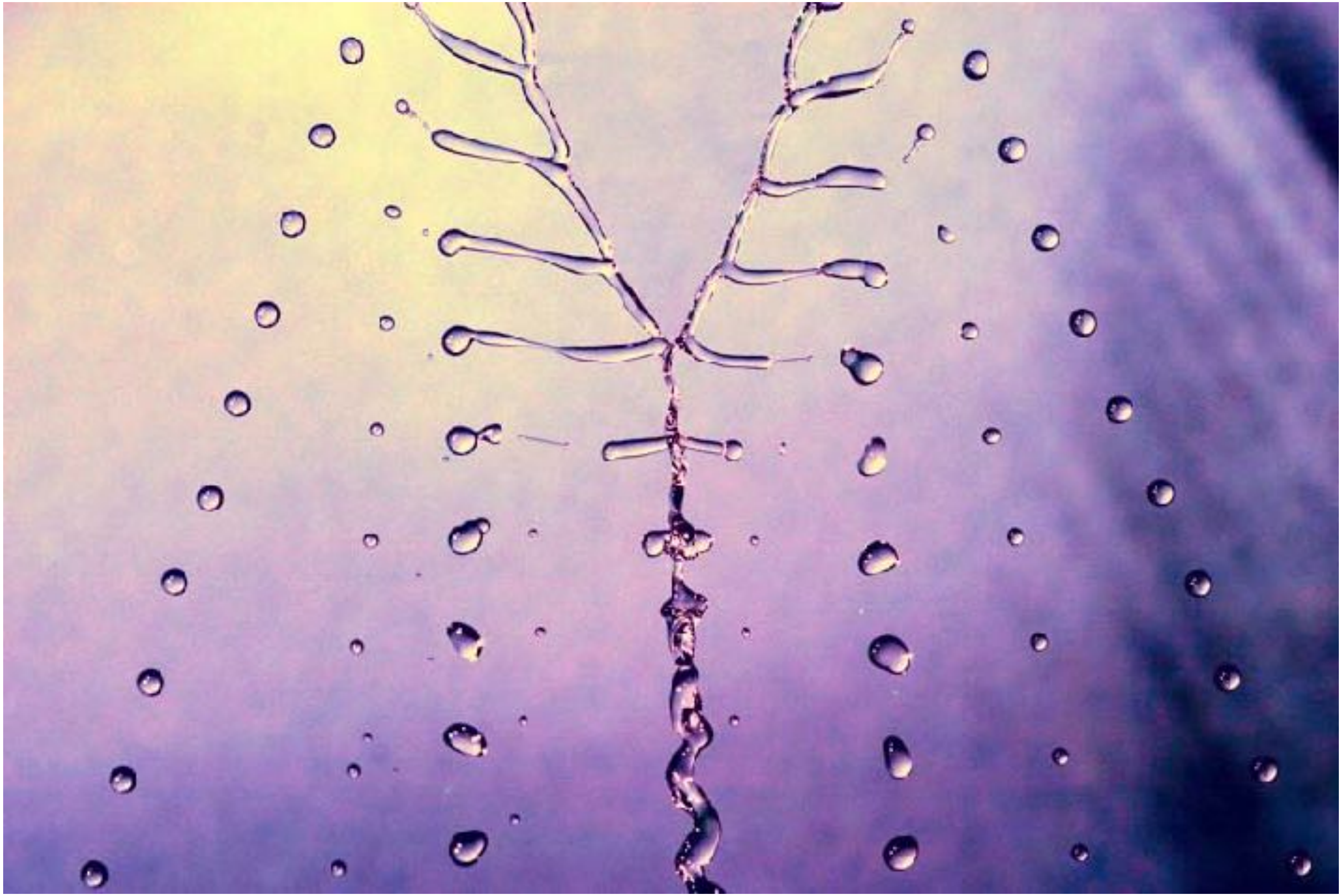
- capillarity instability develops on bounding rims
- bulbous regions flung outwards by centripetal force
- fluid tendrils, fishbones drawn out
- capillary instability of fishbones leads to elaborate wake structure

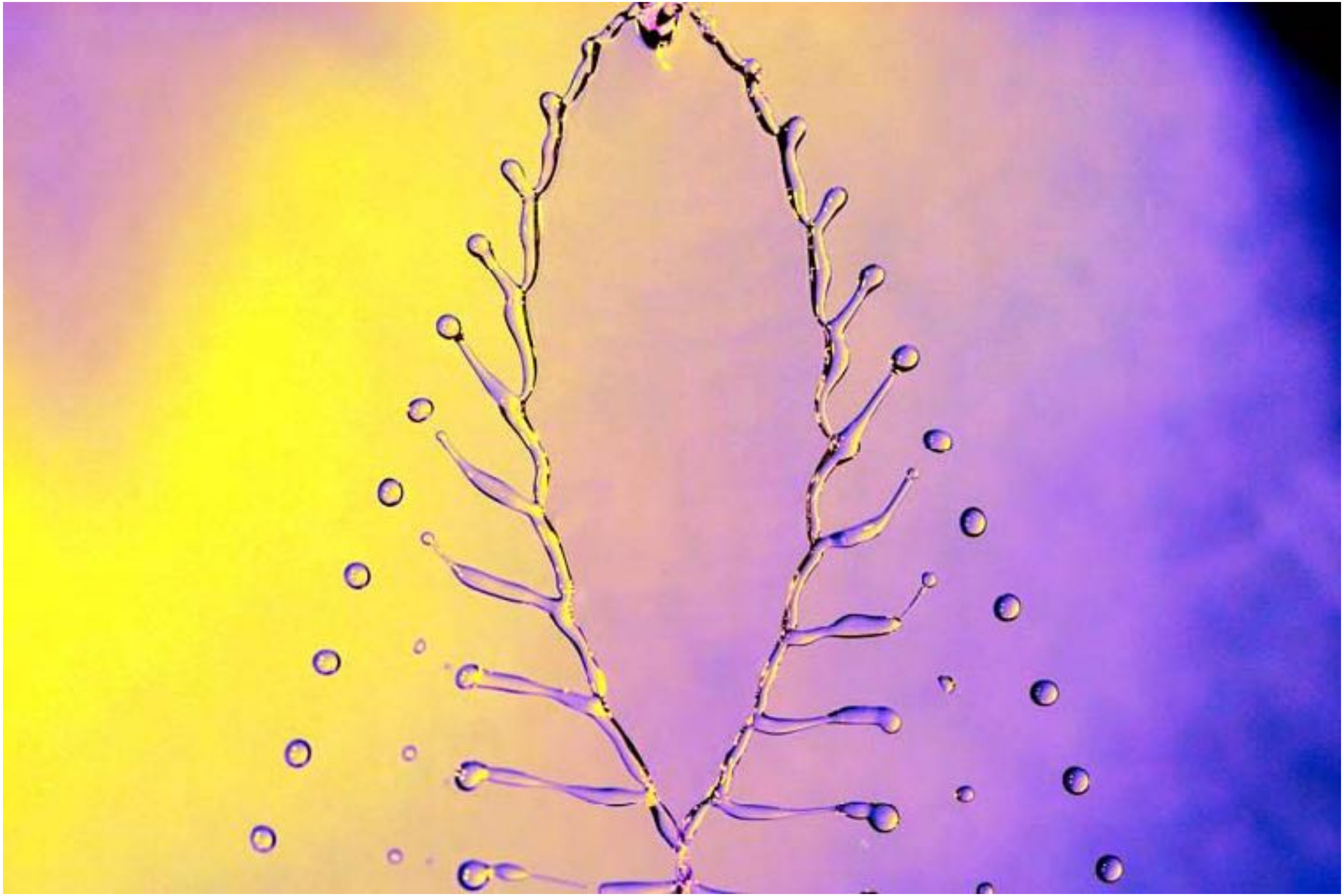


Bush & Hasha (2004)





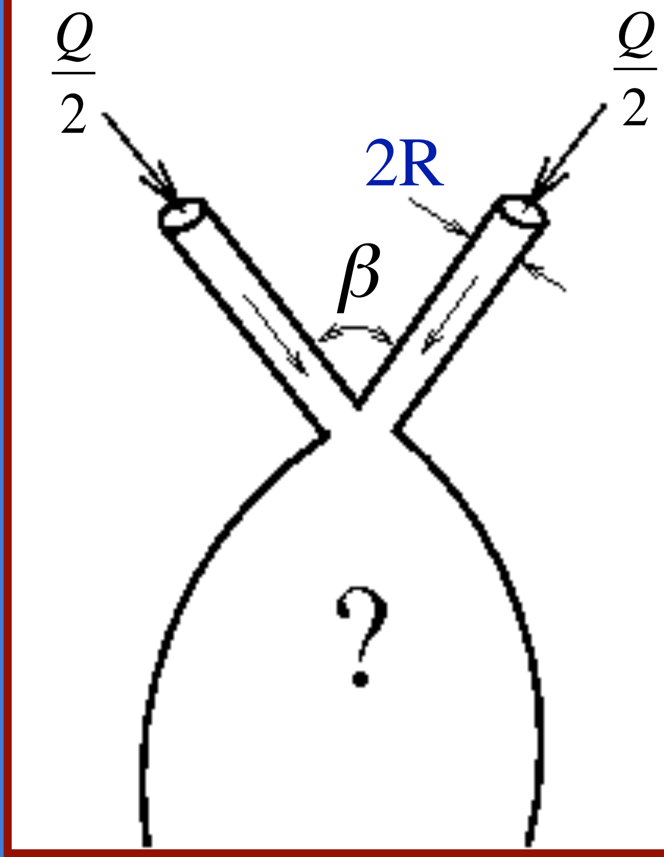






FISH OUT OF WATER
A kinetic sculpture by
Jeff Lieberman & Daniel Paluska

Colliding jets

DIMENSIONAL ANALYSIS



Physical variables: $R, Q, \rho, \sigma, \nu, \beta, g$

Fundamental units: M , L , T

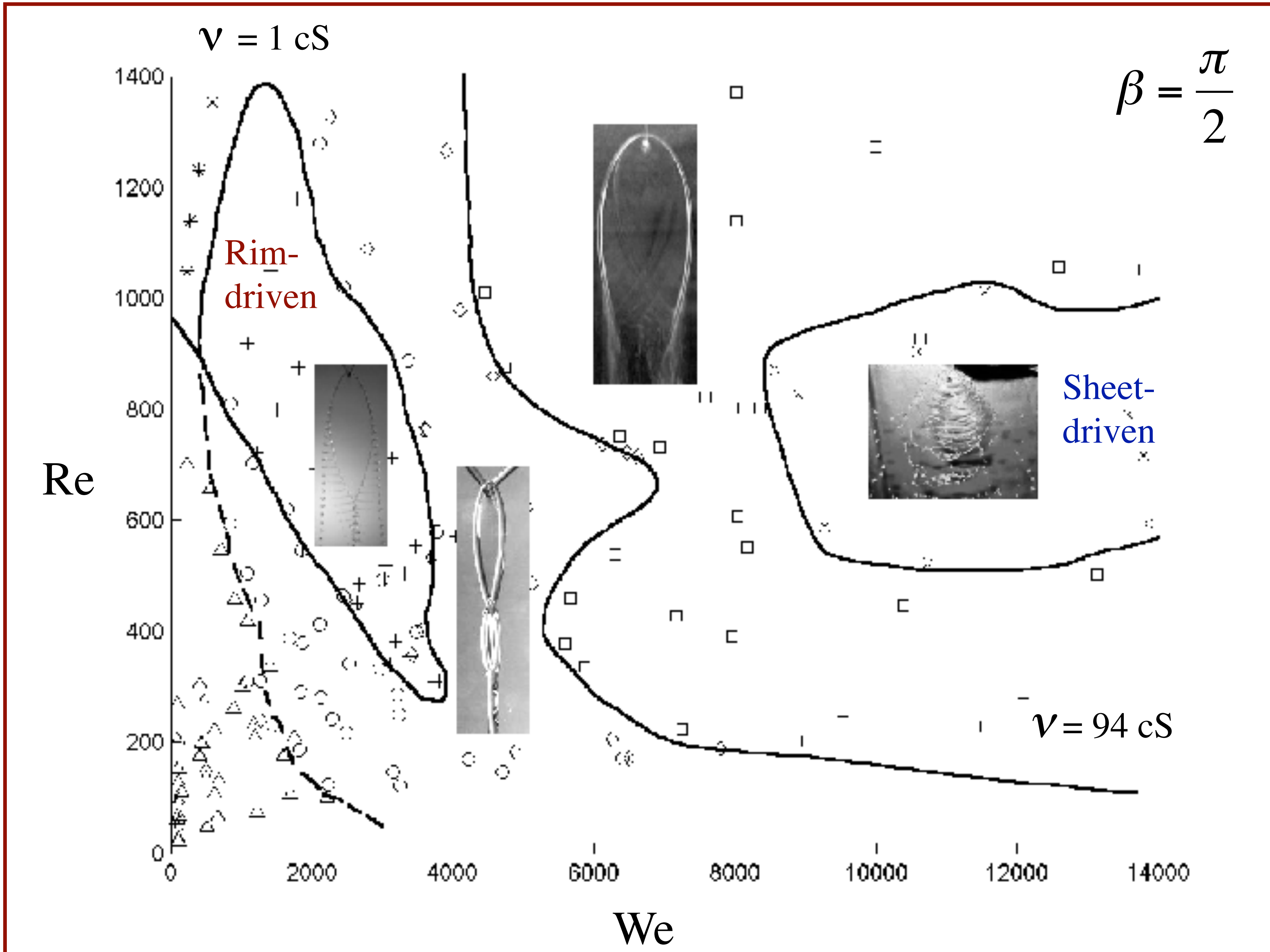
Buckingham's Thm: 4 dimensionless groups

Impact angle: β

Reynolds number: $\mathrm{Re} = \frac{Q}{\nu R} = \frac{\text{INERTIA}}{\text{VISCOSITY}}$

Weber number: $\mathrm{We} = \frac{\rho Q^2}{\sigma R^3} = \frac{\text{INERTIA}}{\text{CURVATURE}}$

Froude number: $\mathrm{Fr} = \frac{\mathrm{Q}^2}{\mathrm{gR}^5} = \frac{\text{INERTIA}}{\text{GRAVITY}} >> 1$



Bush & Hasha (JFM, 2004)

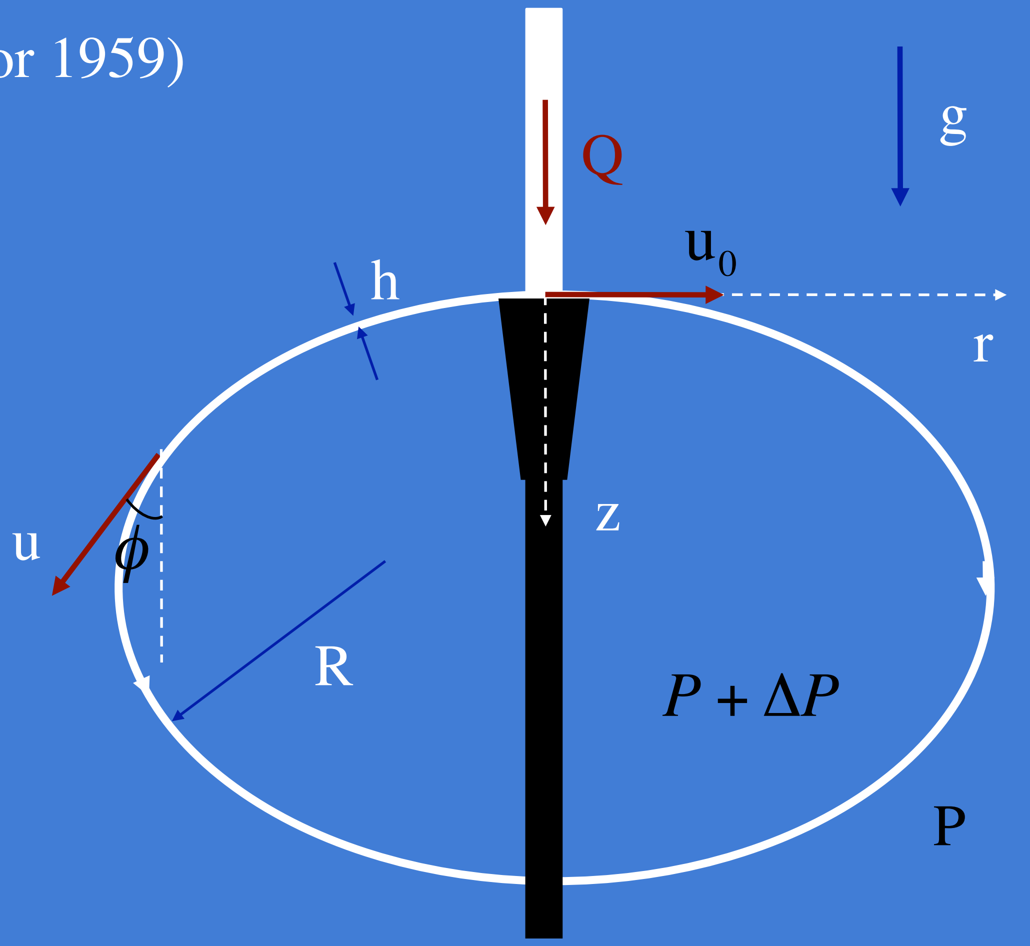
Sagging sheets

The water bell

Water bells in the garden

Water bells (Savart 1833, Taylor 1959)

- form prescribed by balance of inertia, gravity and capillarity



Continuity: $Q = 2\pi h r u$

Energy conservation:

$$u^2 = 2gz + u_0^2$$

Normal force balance:

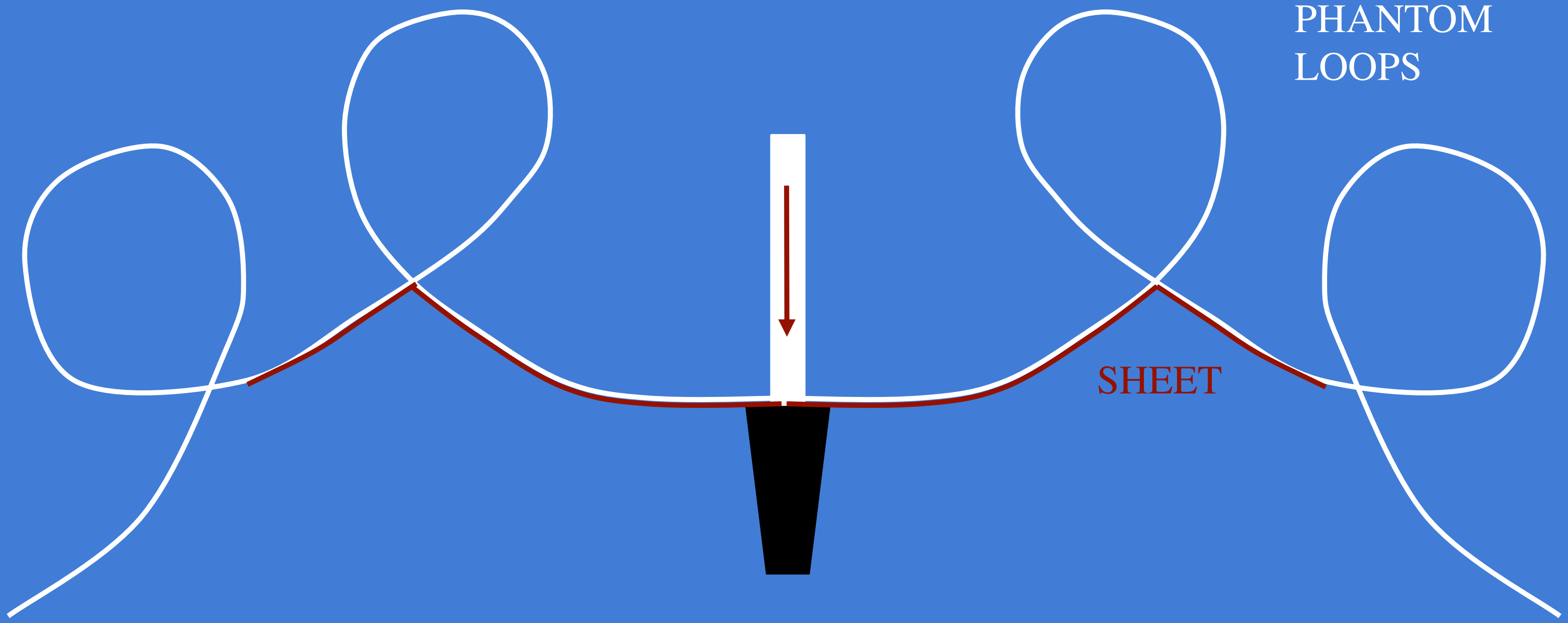
$$\underbrace{\frac{2\sigma}{R} + \frac{2\sigma\cos\phi}{r}}_{\text{curvature}} + \underbrace{\rho g h \sin\phi}_{\text{gravity}} = \Delta P + \underbrace{\frac{\rho h u^2}{R}}_{\text{inertia}}$$

- bell closes owing to influence of out-of-plane curvature

Cusps on sheets

Cusps on sheets

- may arise for sheets that are initially concave upwards



- in this case, numerical integration of governing equations suggests that sheet will be self-intersecting
- Bark et al. (1979) suggest that the cusps arise at the lines of intersection of these phantom loops

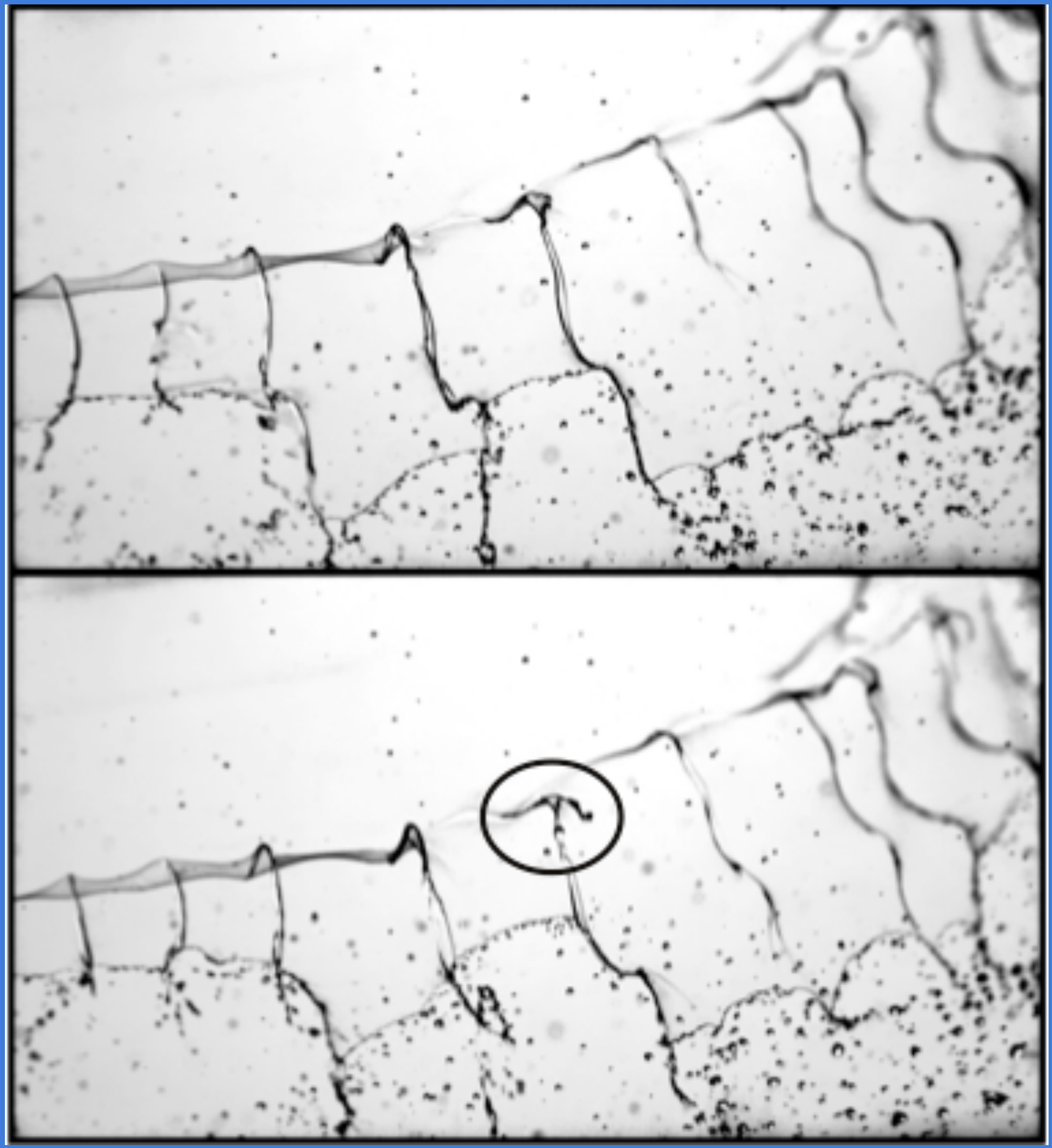
→ **SURELY NOT!**

The latest word on fluid sheets...

Lhuissier & Villermaux (2011)

Cusps on sheets

Instability of flapping sheets



Jets and sheets in rotation

Instabilities of rotating jets

with Nikos Savva

- thread destabilized by rotation owing to influence of centripetal force

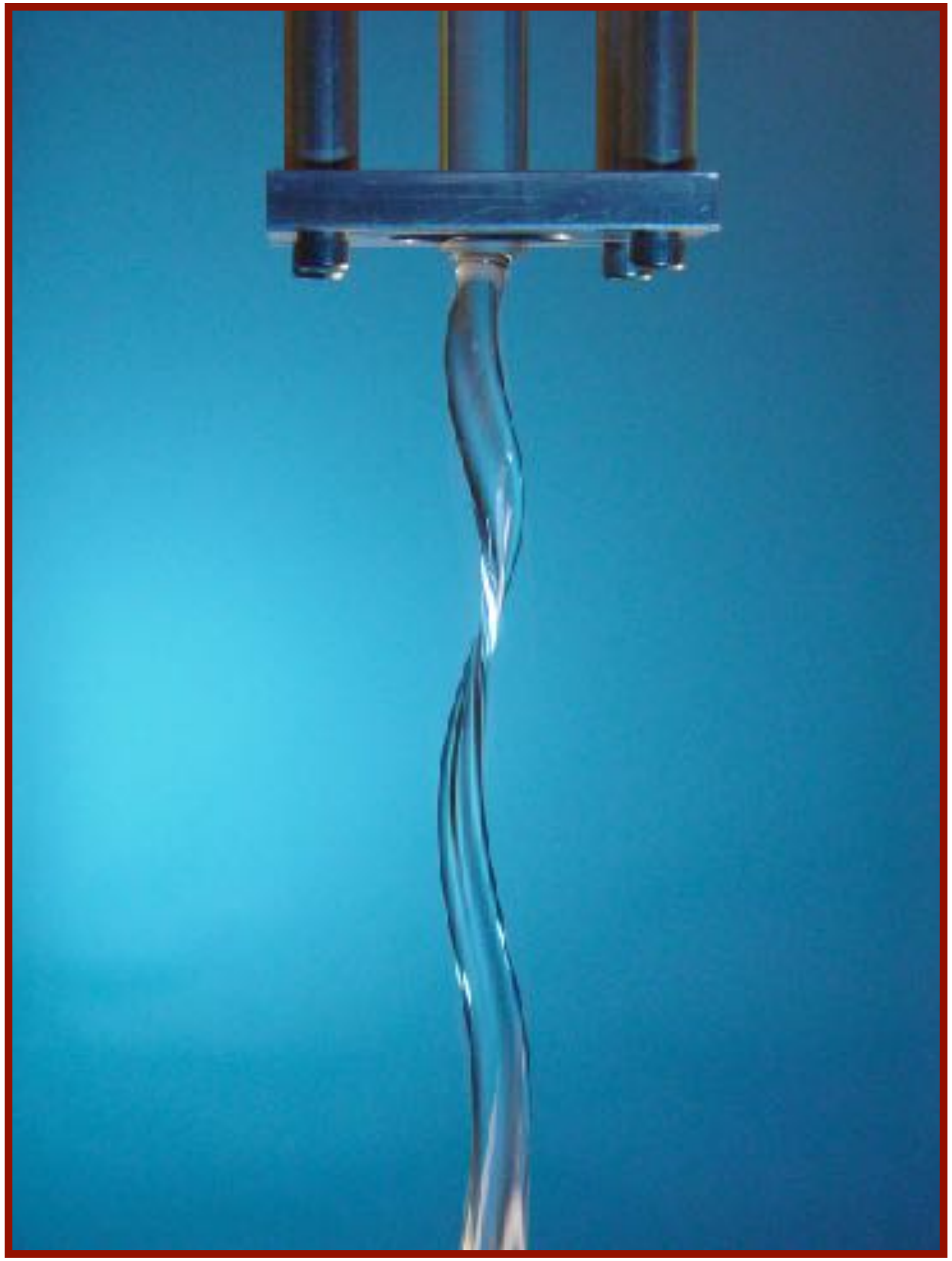
- above a critical $\Sigma = \dfrac{\rho \Omega^2 a^3}{\sigma}$,

  most unstable mode is nonaxisymmetric

- reminiscent of symmetry-breaking in rotating drop

Weidman (1987)

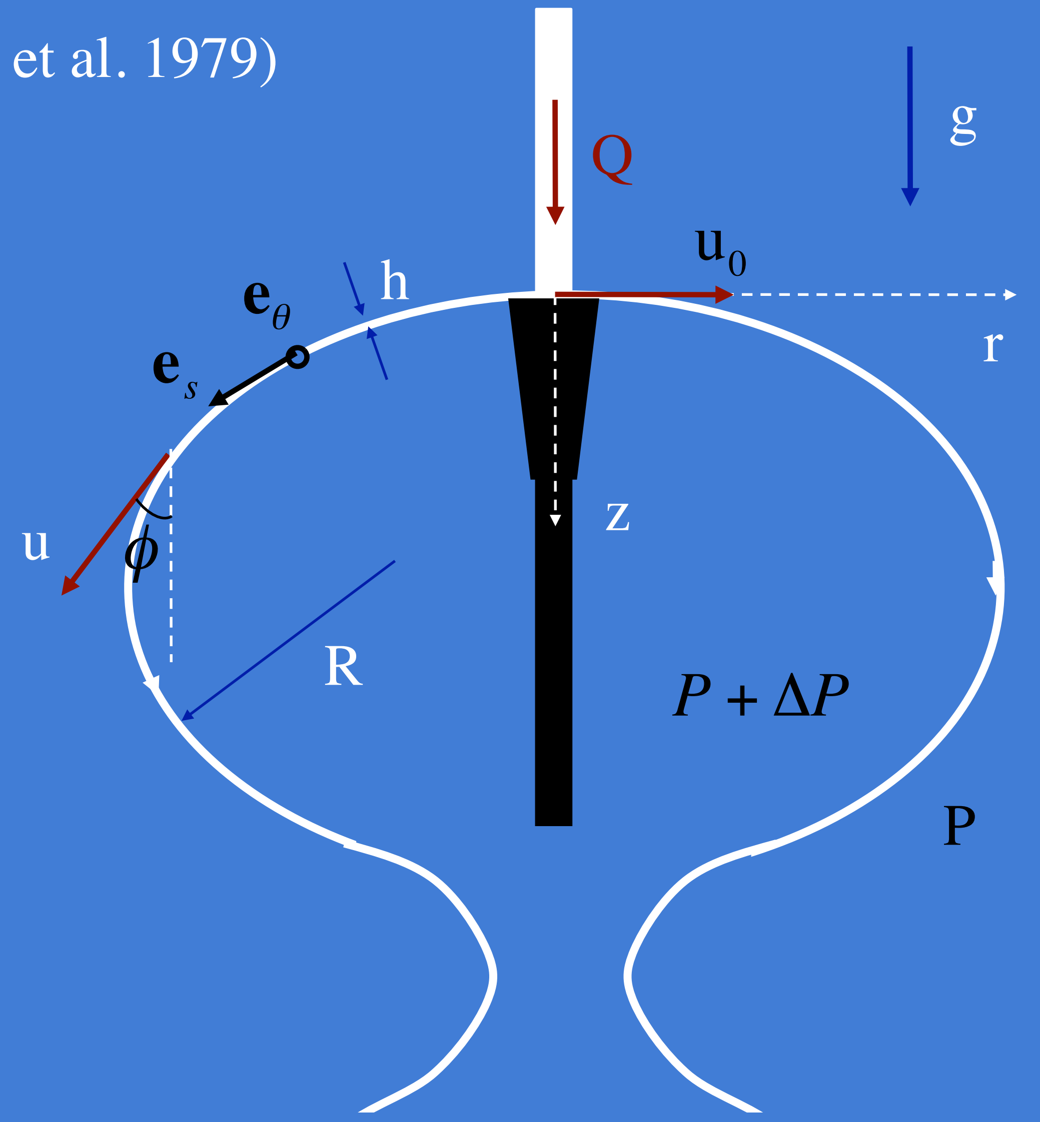
Peter Rhines





Swirling water bells

(Bark et al. 1979)



Sheet velocity: $\mathbf{v} = u\,\mathbf{e}_s + v\,\mathbf{e}_\theta$

Continuity: $Q = 2\pi h r u$

Energy conservation:

$$u^2 + v^2 = 2gz + u_0^2 + v_0^2$$

Conservation of angular momentum:

$$v\,r = v_0\,r_0$$

Normal force balance:

$$\frac{2\sigma}{R} + \frac{2\sigma\cos\phi}{r} + \rho g h \sin\phi = \Delta P + \frac{\rho h u^2}{R} + \underbrace{\frac{\rho h v^2 \cos\phi}{r}}$$

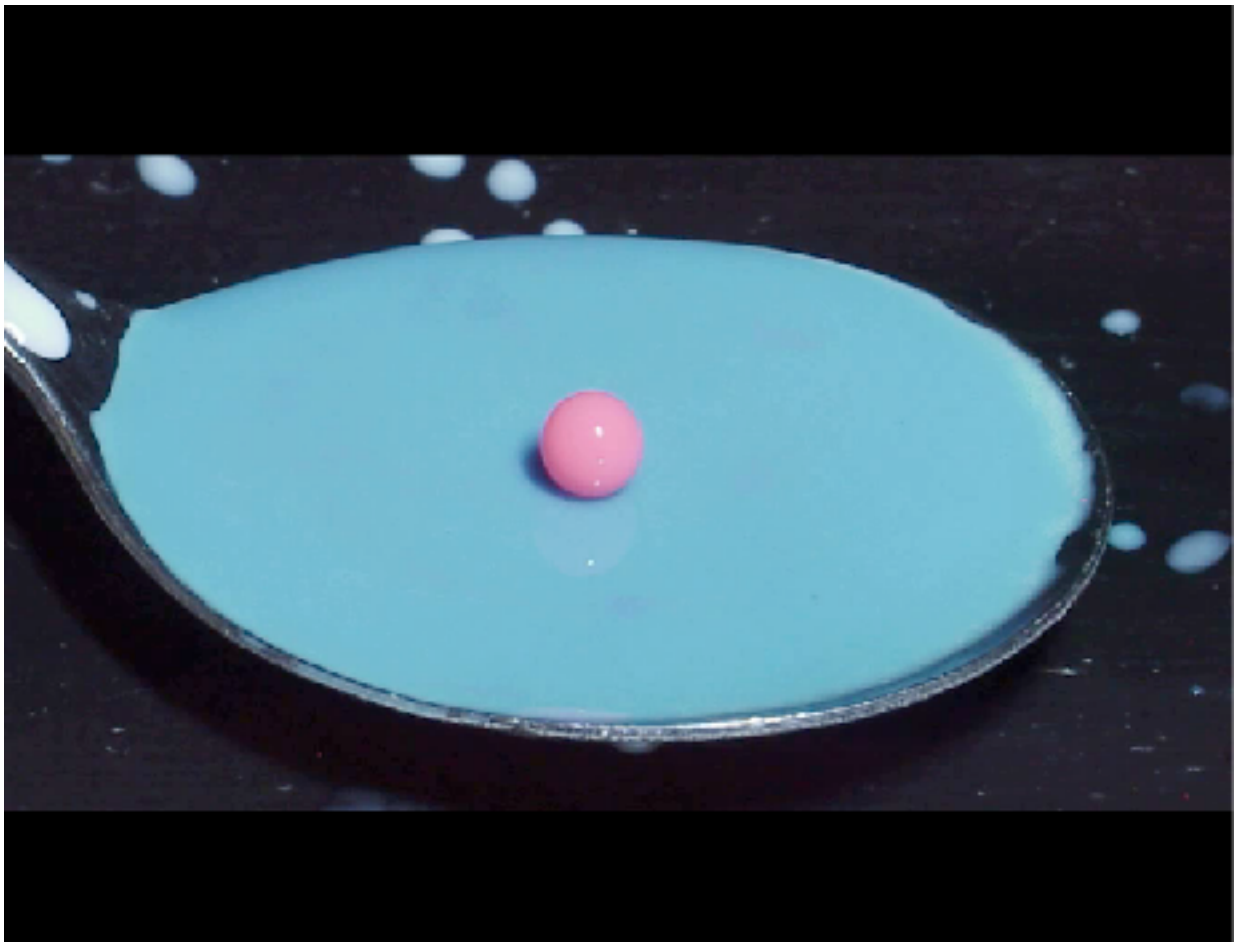
- bell fails to close owing to influence of …..

CENTRIPETAL FORCE

Rain drop hits a puddle

What forms do we expect?

Fluid-fluid impact

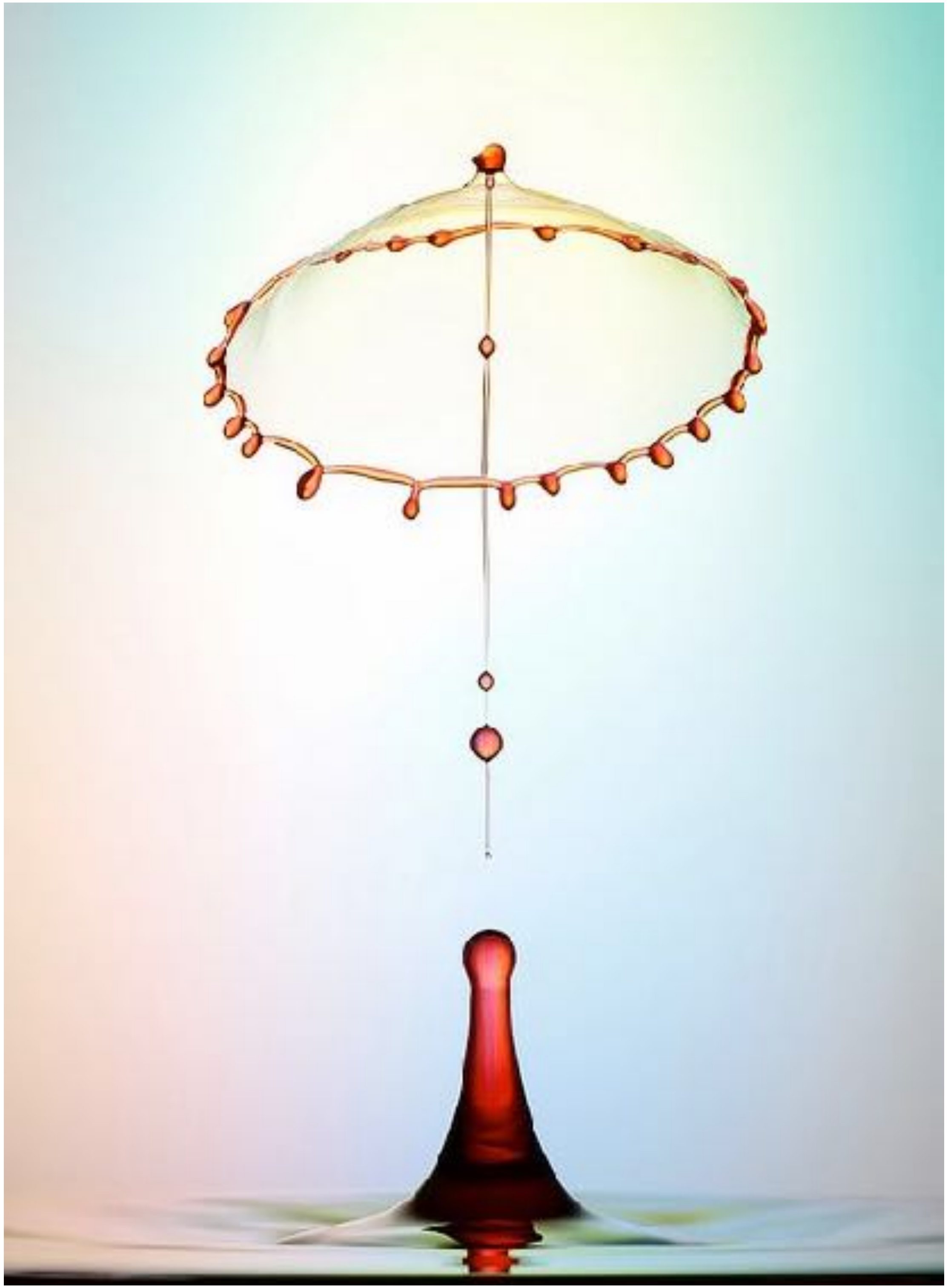


The Edgerton crown

What forms do we expect?

Fluid-fluid impact

How do we rationalize the resulting forms?





"I never before realized so strongly the splendour and beauty of the mere physical forms of Nature.

A wonderful thing is the curious repetition of the same forms, of the same design almost, in the shape of the falling water.

It gave me a sense of how completely what seems to us the wildest liberty of Nature is restrained by governing laws."

\- **Oscar Wilde**, on viewing Niagara Falls (the Canadian side)