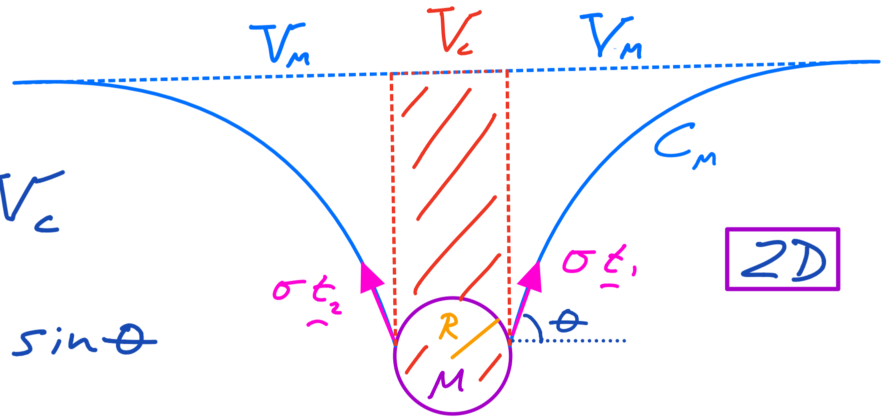
Lecture 6. Interfacial statics

Recall: for small floating bodies

Vertical Force Balance: $\boxed{Mg = F_b + F_c}$

has 2 components:



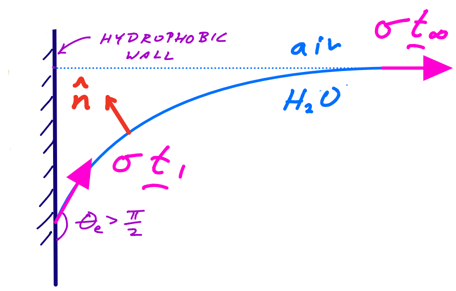
Buoyancy: $F_b = \rho g V_c$

Capillary: $F_c = 2\sigma \sin\theta$

But last class, we saw that for a simple meniscus

$\sigma \cos\theta_e$ = wt of liquid displaced above meniscus

$= \sigma \sin\theta$ above



Similarly, pressure balance along the meniscus C_M:

$$\rho g z \underline{n} = \sigma \underline{\nabla} \cdot \underline{n}\, \underline{n} \Rightarrow \text{integrate along } C_M$$

$$\Rightarrow \int_{C_M} \rho g z \underline{n}\, dl = \int_{C_M} \sigma \underbrace{\underline{\nabla} \cdot \underline{n}\, \underline{n}}_{\frac{d\underline{t}}{dl}}\, dl$$

via Frenet-Serret

$$\Rightarrow \rho g V_M = \sigma (\underline{t}_\infty - \underline{t}_1) \cdot \hat{z} = \sigma \sin\theta$$

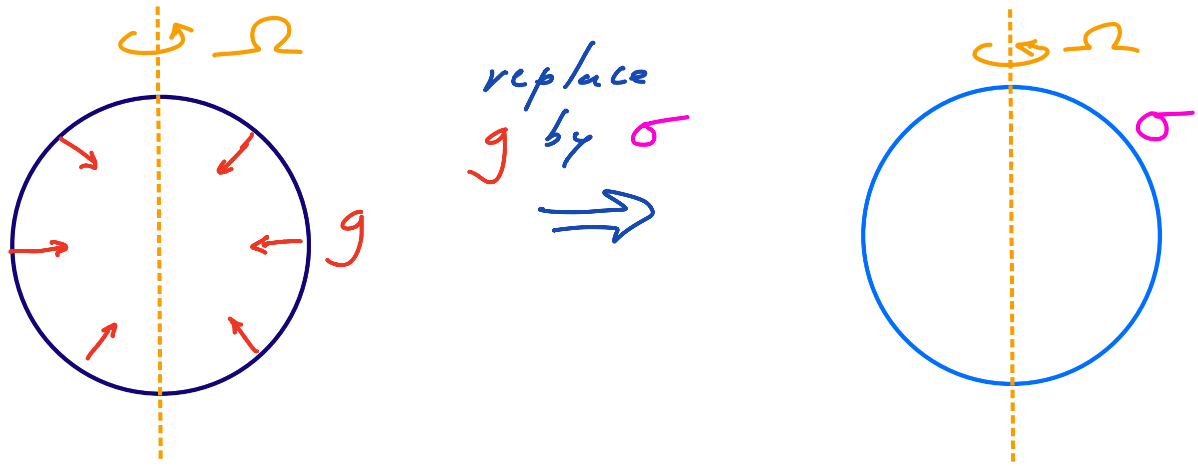
$$\Rightarrow 2\sigma \sin\theta = 2\rho g V_M$$

CAP FORCE WT DISPACED OUTSIDE CONTACT LINE by MENISCUS

∴ ⇒ Generalized Archimedes Principle:
wt of body = wt of displaced fluid

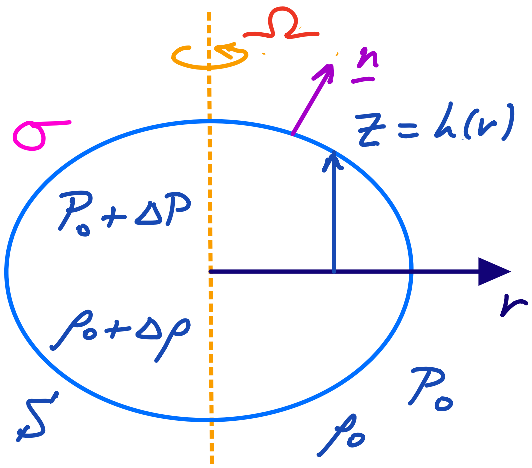
Rotating Drops

: first considered experimentally by Plateau (1863) as a model of heavenly bodies



Problem Formulation

Normal stress balance on S:



$$P_0+\Delta P+\tfrac{1}{2}\Delta\rho\,\Omega^2 r^2 = P_0+\sigma\,\underline{\nabla}\cdot\underline{n}$$

Nondimensionalize:

$$\Delta P + 4B_0\left(\frac{r}{a}\right)^2 = \underline{\nabla}\cdot\underline{n}$$

where $\Delta P = \dfrac{a\,\Delta P}{\sigma}$, $\quad B_0 = \dfrac{\Delta\rho\,\Omega^2 a^3}{8\sigma} = \dfrac{\text{centrifugal}}{\text{curvature}}$

= Rotational Bond #

For axisymmetric shapes, this may be integrated

Define: $f(r) = z - h(r) = 0$ on S

$$\underline{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{\hat{z} - h_r(r)\,\hat{r}}{(1+h_r^2)^{\frac{1}{2}}}, \qquad \underline{\nabla}\cdot\underline{n} = \frac{-r h_r - r^2 h_{rr}}{r^2(1+h_r^2)^{\frac{1}{2}}}$$

Brown + Scriven (1970): computed drop shapes and stability for $B_o > 0$

- for $B_o < 0.09$, only axisymmetric sol'ns (oblate ellipsoids)
- for $0.09 < B_o < .31$, both oblate ellipsoids and lobed sol'ns possible and stable
- for $B_o > 0.31$, no stable axisymmetric sol'ns

TEKTITES: centimetric metallic ejecta produced by meterite impact

Q1: why are they so much larger than raindrops?

From raindrop scaling, we expect drop size

$l_c \sim \sqrt{\frac{\sigma}{\Delta\rho g}}$, but both σ and $\Delta\rho$ higher by a factor of ~ 10

$\Rightarrow l_c$ unchanged

$\Rightarrow$ large tektite size suggests they are not equilibrium forms, but from into shape during flight

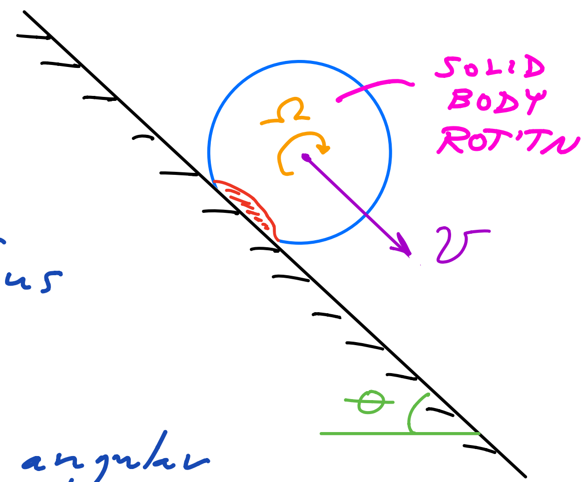
Q2: why are their shapes so different from those of raindrops?

$\Rightarrow$ owing to high ρ, the internal dynamics of the drop (esp. rotation) dominates the aerodynamic stresses

$\Rightarrow$ drop shape set by its rotation

Rolling and Tumbling Drops

- similar shapes arise : oblate ellipsoids, lobate drops.



Note: energetics : for steady descent at a speed v

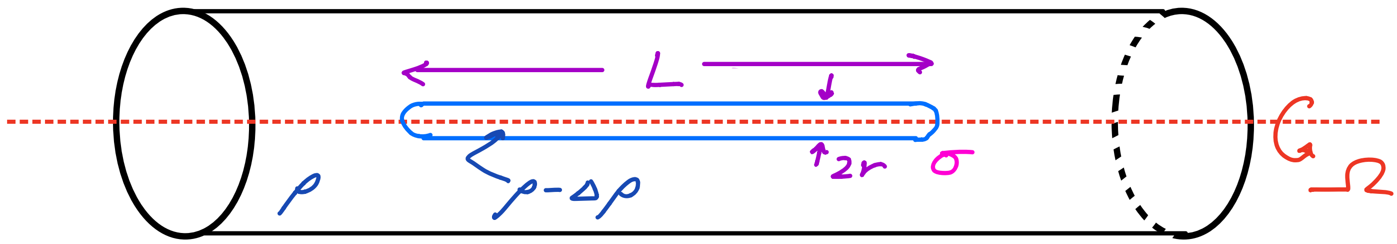
$$Mg\,v\sin\theta = \text{Rate of viscous dissipation}$$

$\Rightarrow$ sets v $\Rightarrow$ $\Omega = v/R$ is angular speed.

- stability characteristics different : dimpled "life-saver" shaped oblate forms now stable

Spinning Drop Tensiometer ($B_0 < 0$)

- insert buoyant bubble in a liquid-filled rotating cylinder



- at large deformation, drop assumes form of a cylinder with hemispherical caps
- centripetal force flings heavy liquid outwards

Drop Energy: $E = \underbrace{\frac{1}{2} J \Omega^2}_{\text{rotational KE}} + \underbrace{2\pi r L \sigma}_{\text{Surface energy (neglect ends)}}$

Use drop volume $V = \pi L r^2$ and moment of inertia $J = \dfrac{\Delta m\, r^2}{2} = \Delta\rho \dfrac{\pi}{2} L r^4$

Energy per unit volume: $\dfrac{E}{V} = \dfrac{1}{4}\Delta\rho\,\Omega^2 r^2 + \dfrac{2\sigma}{r}$

Minimize: $\dfrac{d}{dr}\left(\dfrac{E}{V}\right) = \dfrac{1}{2}\Delta\rho\,\Omega^2 r - \dfrac{2\sigma}{r^2}$

Vanishes when $\dfrac{\Delta\rho\,\Omega^2}{2} r = \dfrac{2\sigma}{r^2} \Rightarrow \boxed{r = \left(\dfrac{4\sigma}{\Delta\rho\,\Omega^2}\right)^{\frac{1}{3}}}$

Note: $r \uparrow$ with σ; $r \downarrow$ with Ω ✓

Using $V = \pi L r^2 \Rightarrow r = \left(\frac{V}{\pi L}\right)^{\frac{1}{2}} = \left(\frac{4\sigma}{\Delta\rho\Omega^2}\right)^{\frac{1}{3}}$

$$\boxed{\sigma = \frac{1}{4\pi^{3/2}} \Delta\rho\Omega^2\left(\frac{V}{L}\right)^{3/2}}$$

Vonnegut's Formula (Bernard V, MIT, 1942)

- useful technique for measuring small σ
  $\Rightarrow$ avoids difficulties associated with liquid-solid contact.

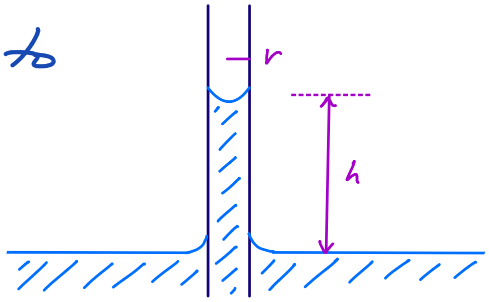
Capillary Rise in Tubes

- one of the most well-known + vivid illustrations of capillarity
- arises in nature (drinking strategies of insects birds, bats) flow in porous media

Historical Notes

- Leonardo de Vinci (1452-1519) recorded the effect in his notes, appealled to it for the origin of mountain streams (erroneously)
- Jacques Rohault (1620-1675): capillary rise due to suppression of air currents (creation of a vacuum) in narrow tubes
- Geovanni Borelli (1608-1675): showed $h \sim \frac{1}{r}$

- Geminiano Montanari (1633-1687): attributed circulation in plants to capillary rise



- Francis Hauksbee (1700s): conducted an extensive series of capillary rise expts reported by Newton (in Opticks) but unattributed

- James Jurin (1684-1750): an english physiologist who independently confirmed $h \sim \frac{1}{r}$

$\Rightarrow$ "Jurin's Law"