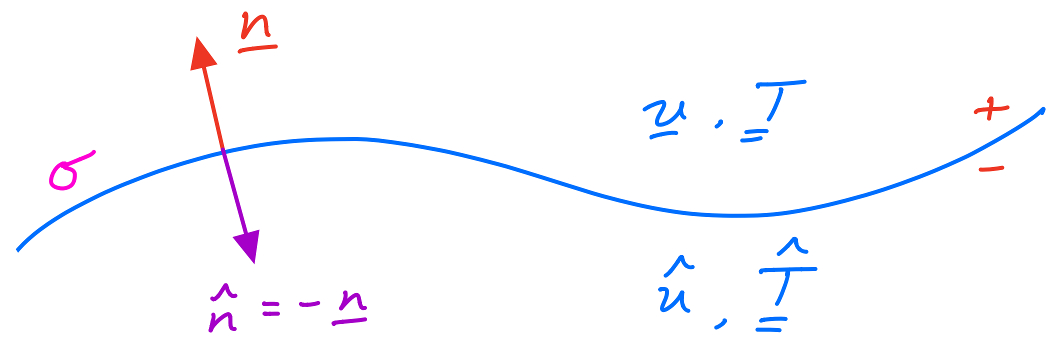
Lecture 5. Interfacial boundary conditions, Statics



⇒ Interfacial Stress Balance Equation

$$\boxed{\underline{n} \cdot \underline{\underline{T}} - \underline{n} \cdot \hat{\underline{\underline{T}}} = \sigma \, \underline{n} \, \underline{\nabla} \cdot \underline{n} - \underline{\nabla} \sigma} \quad ☆$$

stress applied by + on interface

stress applied by − on interface

NORMAL CURVATURE PRESSURE ASSOCIATED WITH $\underline{\nabla} \cdot \underline{n} = 0$

TANGENTIAL/ MARANGONI STRESS

Normal Stress Balance

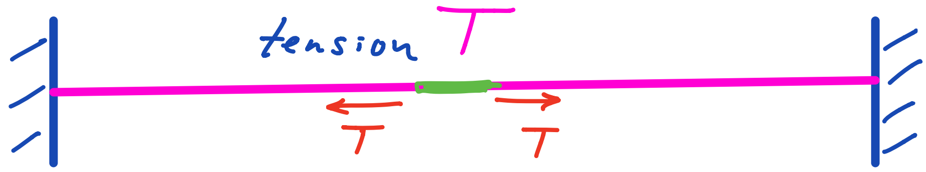
Taking $\underline{n} \cdot$ ☆ yields

$$\underbrace{\underline{n} \cdot \underline{\underline{T}} \cdot \underline{n} - \underline{n} \cdot \hat{\underline{\underline{T}}} \cdot \underline{n}}_{\text{JUMP IN NORMAL STRESS}} = \underbrace{\sigma \, \underline{\nabla} \cdot \underline{n}}_{\text{LAPLACE PRESSURE}}$$

⇒ DEPENDS ON BOTH $\underline{u}, p$

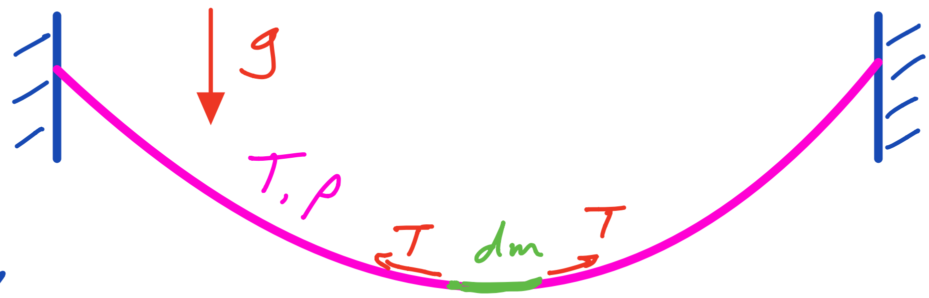
Curvature Forces : An aside ...

String under tension



String under tension sags due to gravity

$\Rightarrow$ wt supported by curvature force



Tangential Stress Balance

Taking $\underline{t}\cdot \star$, where $\underline{t}$ is ANY vector tangent to the interface :

$$\underbrace{\underline{n}\cdot\underline{\underline{T}}\cdot\underline{t} - \underline{n}\cdot\hat{\underline{\underline{T}}}\cdot\underline{t}}_{\text{JUMP IN TANGENTIAL STRESS}} = \underbrace{\underline{\nabla}\sigma\cdot\underline{t}}_{\substack{\text{MARANGONI STRESS} \\ \text{ASSOC. w } \underline{\nabla}\sigma}}$$

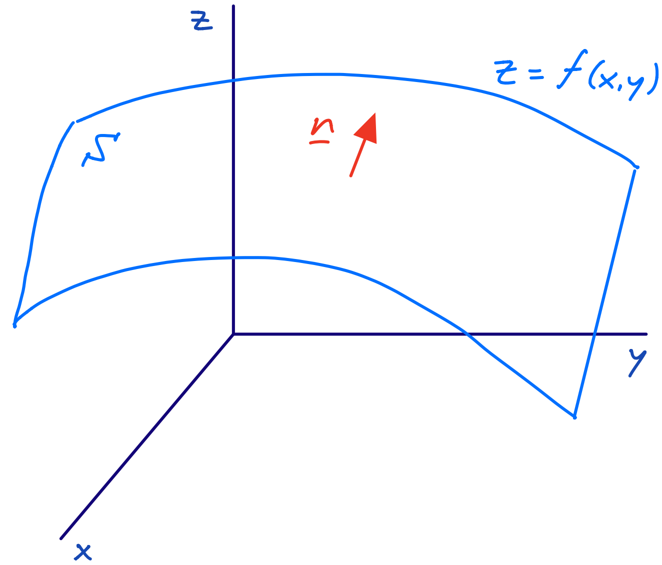
Note : 1. $\underline{\nabla}\sigma$ may arise since $\sigma(C, T)$

C: chemistry; T: temperature

$\Rightarrow$ Marangoni stresses arise from $\vec{\nabla}C$ or $\vec{\nabla}T$

2. the LHS contains only velocity gradients, not pressure ; therefore, a non-zero $\underline{\nabla}\sigma$ at at a fluid interface ALWAYS drives motion.

Quick Review on Computing Normals



Normal to $z = f(x,y)$

Define functional :

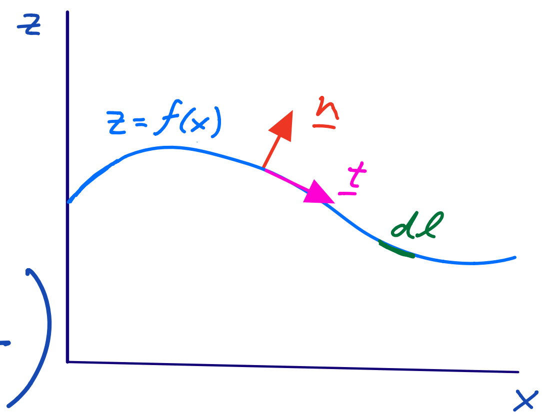
$$g(x,y,z) = z - f(x,y)$$
$$= 0 \text{ on } S$$

$$\therefore \quad \vec{\nabla} g = \vec{\nabla}(z - f(x,y))$$
$$= -f_x \hat{i} - f_y \hat{j} + \hat{k}$$

and $$\hat{n} = \frac{\vec{\nabla} g}{|\vec{\nabla} g|} = \frac{\hat{k} - f_x \hat{i} - f_y \hat{j}}{\sqrt{1 + f_x^2 + f_y^2}}$$

and curvature deduced from $\underline{\nabla} \cdot \underline{n}$.

Special Case : normal to curve $z = f(x)$



$$\Rightarrow \quad \underline{n} = \frac{\hat{k} - f_x \hat{i}}{\sqrt{1 + f_x^2}}$$

$$\underline{\nabla} \cdot \underline{n} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{k} \frac{\partial}{\partial z} \right) \cdot \left(\frac{\hat{k} - f_x \hat{i}}{\sqrt{1 + f_x^2}} \right)$$
$$= \frac{-f_{xx}}{(1 + f_x^2)^{3/2}}$$

An Aside : differential geometry provides some useful relations between $\underline{n}$ and $\underline{t}$.

$$(\underline{\nabla}\cdot\underline{n})\,\underline{n} = \frac{d\underline{t}}{dl}$$

PROPORTIONAL TO CURVATURE PRESSURE AT INTERFACE

$$-(\underline{\nabla}\cdot\underline{n})\,\underline{t} = \frac{d\underline{n}}{dl}$$

Fluid Statics

- the stress tensor reduces to $\underline{\underline{T}} = -p\,\underline{\underline{I}}$, so that $\underline{n}\cdot\underline{\underline{T}}\cdot n = -p$ and the normal stress balance :

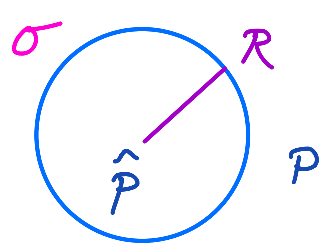
$$\underbrace{\hat{p} - p}_{\Delta p} = \underbrace{\sigma\, \underline{\nabla}\cdot\underline{n}}_{\text{LAPLACE PRESSURE}}$$

- tangential stress must take the form $\underline{\nabla}\,\underline{\sigma} = 0$.

  $\Rightarrow$ THERE CAN'T BE A STATIC SYSTEM in the presence of $\vec{\nabla}\sigma \neq 0$

Let's consider now a couple of static systems.

I. Stationary Bubble



Recall in spherical coords

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 F_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}F_\phi$$

Here $\vec{F} = \underline{n} = (1,0,0)$ in (r,θ,ϕ) coords

$$\Rightarrow \underline{\nabla}\cdot\underline{n} = \frac{1}{r^2}\frac{d}{dr}(r^2) = \frac{2}{R}$$

$$\Rightarrow \quad \Delta p = \hat{p} - p = \frac{2\sigma}{R} \quad \text{as previously}$$

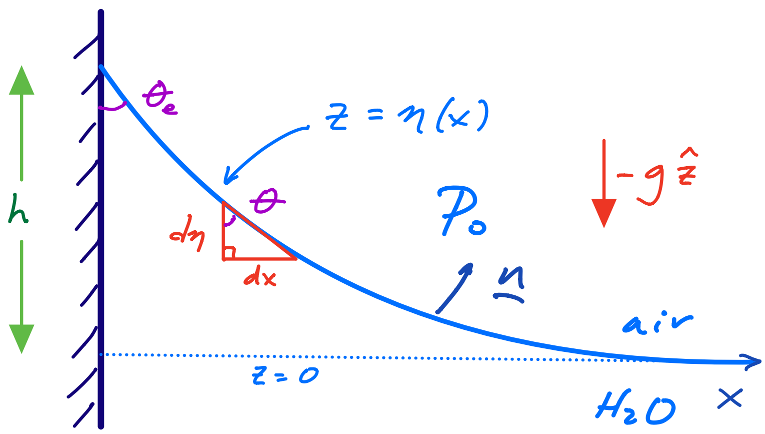
II. The Static Meniscus

Pressure varies according to $p(z) = p_0 + \rho g z$.

Normal stress balance:

$$p_0 - \rho g \eta(x) = p_0 + \sigma \underline{\nabla} \cdot \underline{n}$$

$$\Rightarrow \quad -\rho g \eta(x) = \sigma \underline{\nabla} \cdot \underline{n} \quad \boxed{\times}$$



Now $\underline{\nabla} \cdot \underline{n} = \dfrac{-\eta_{xx}}{(1+\eta_x^2)^{3/2}}$

Approximate sol'n: if slope is everywhere small ($\eta_x \ll 1$), we obtain $(1+\eta_x^2)^{3/2} \approx 1$

$$\rho g \eta = \sigma \eta_{xx}$$

Apply BCs: $\eta(\infty) = 0$ and $\eta_x(0) = -\cot\theta_e$

Solving: $$\boxed{\eta(x) = l_c \cot\theta_e \, e^{-x/l_c}}$$

where $l_c = \sqrt{\sigma/\rho g}$ is capillary note

Note: meniscus drops off on a scale of l_c.

But let's now solve $\boxed{\times}$ exactly.

We can integrate directly

$$\rho g \eta \eta_x = \sigma \frac{\eta_x \eta_{xx}}{(1+\eta_x^2)^{3/2}}$$

$$\frac{d}{dx}\frac{\eta^2}{2} = \ell_c^2 \frac{d}{dx}\frac{1}{(1+\eta_x^2)^{\frac{1}{2}}}$$

Integrate from x to ∞ :

$$LHS = \int_x^\infty \frac{d}{dx}\frac{\eta^2}{2}\,dx = \frac{1}{2}\left[\eta^2(\infty) - \eta^2(x)\right] = -\frac{\eta^2}{2}$$

$$RHS = \int_x^\infty \frac{d}{dx}\frac{1}{(1+\eta_x^2)^{\frac{1}{2}}}\,dx = \frac{1}{(1+0)^{\frac{1}{2}}} - \frac{1}{(1+\eta_x^2)^{\frac{1}{2}}} = 1-\sin\theta$$

$$\Rightarrow \boxed{\sigma \sin\theta + \frac{1}{2}\rho g \eta^2 = \sigma}$$

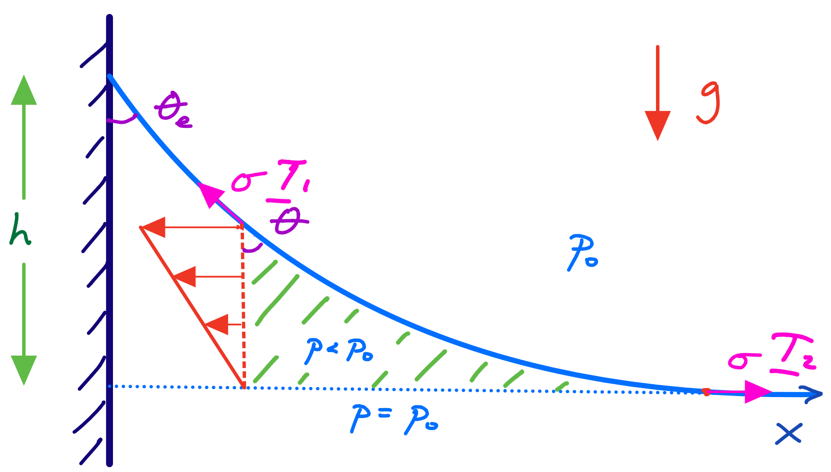
Note: at $z = h$, $\theta = \theta_e \Rightarrow \frac{1}{2}\rho g h^2 = \sigma(1-\sin\theta_e)$

$$\Rightarrow \boxed{h = \sqrt{2}\,\ell_c\,(1-\sin\theta_e)^{\frac{1}{2}}}$$

is the maximum rise height

Consider the force balance on a portion of the meniscus.

Horizontal Force Balance



$$\underbrace{\sigma \sin\theta}_{\text{horizontal projection of } \sigma \underline{T}_1} + \underbrace{\tfrac{1}{2}\rho g \eta^2}_{\text{hydrostatic suction}} = \underbrace{\sigma}_{\text{tensile stress from } \sigma \underline{T}_2} \quad \star$$

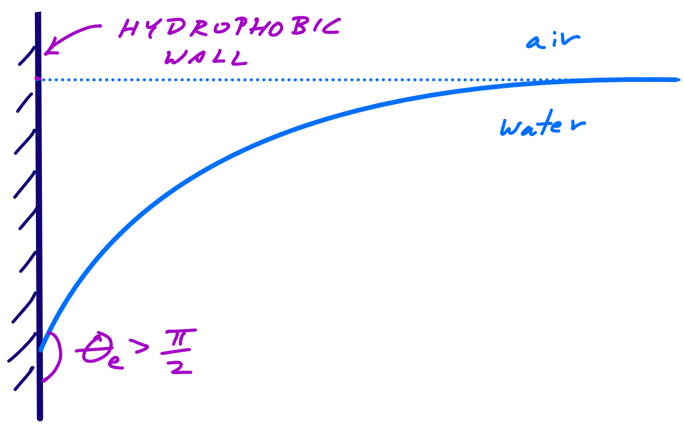
Vertical Force Balance

$$\underbrace{\sigma \cos\theta}_{\text{vertical proj. of } \sigma \underline{T}_1} = \underbrace{\int_x^\infty \rho g z \, dx}_{\text{wt of the fluid}}$$

The latter makes clear that, setting $x = 0$ (where $\theta = \theta_e$)

$\Rightarrow \sigma \cos\theta_e =$ wt of fluid displaced above $z = 0$

$\Rightarrow$ changes sign according to whether $\theta_e > \frac{\pi}{2}$ or $\theta_e < \frac{\pi}{2}$



Floating Bodies

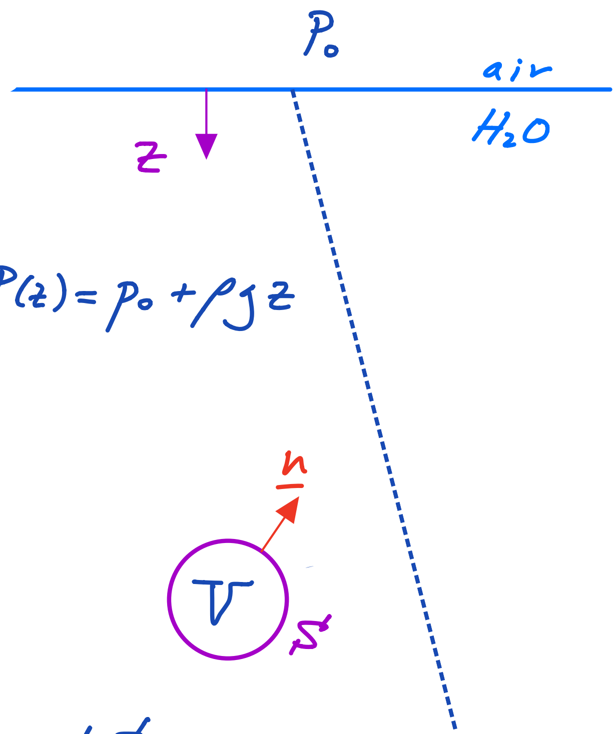
"Heavy things sink. Light things float."

$\Rightarrow$ Not really true for small objects.

Recall: Archimedean force on a submerged body.

In general, the hydrodynamic force acting on a body in a fluid:

$$\boxed{\underline{F}_h = \int_S \underline{\underline{T}} \cdot \underline{n} \, dS}$$



$$P(z) = p_0 + \rho g z$$

where $\underline{\underline{T}} = -p\underline{\underline{I}} + 2\mu \underline{\underline{E}}$

$= -p\underline{\underline{I}}$ for statics

$$\Rightarrow \underline{F}_h = -\int_S p \, \underline{n} \, dS = -\int_S \rho g z \, \underline{n} \, dS$$

$$= -\rho g \int_V \nabla z \, dV \quad \text{by Gen Div Thm}$$

$$= -\rho g \int_V dV \, \hat{z} = -\rho g V \hat{z}$$

$$= \text{wt of displaced fluid}$$

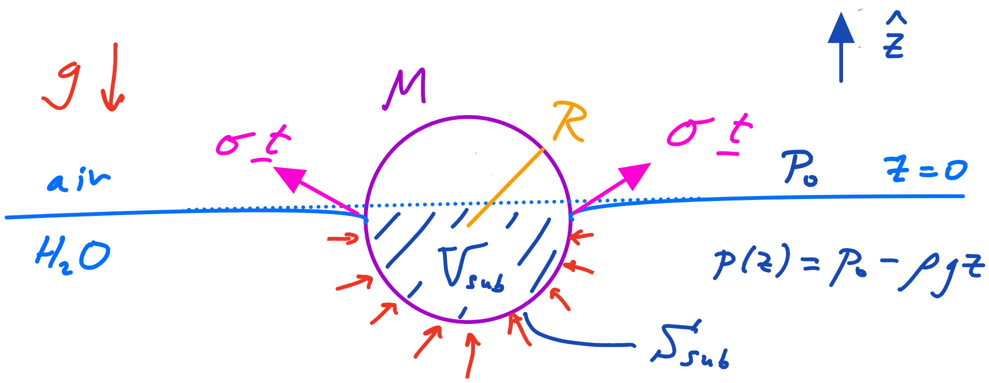
Archimedes Principle: buoyant force is equal to the wt of the displaced fluid

Generalization to Floating Bodies

Case 1: Large bodies ($R \gg l_c$)

Force balance $\quad Mg = \int_{S_{sub}} \rho g \, \underline{n} \cdot \hat{z} \, dS + 2\pi R \sigma \, \underline{t} \cdot \hat{z}$

BUOYANCY $\qquad$ SURFACE TENSION



$$\Rightarrow \quad Mg = \rho g V_{sub} + 2\pi R \, \sigma \, \underline{t} \cdot \hat{z}$$

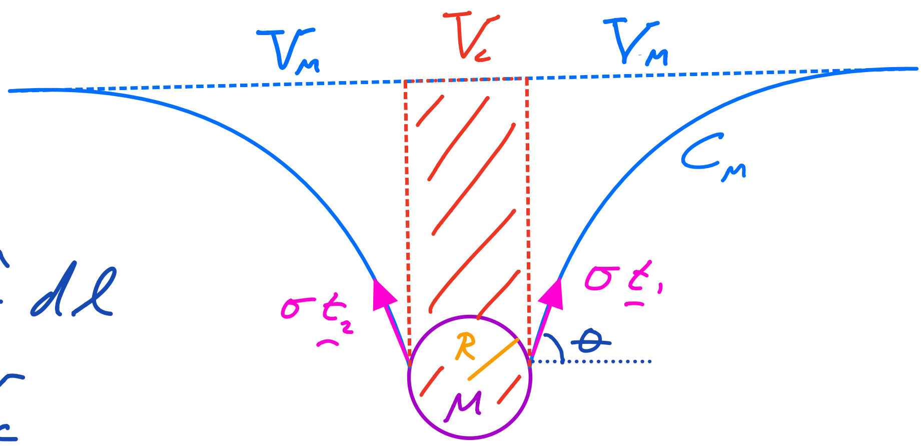
$$\frac{\text{BUOYANCY}}{\text{CAPILLARITY}} \sim \frac{\rho g V_{sub}}{2\pi R \sigma} \sim \frac{\rho g R^3}{R \sigma} \sim \frac{R^2}{\ell_c^2} = B_o$$

where Bond number $B_o = \dfrac{\rho g R^2}{\sigma}$

$\Rightarrow$ σ negligible provided $R \gg \ell_c$

Case 2 : Small bodies ($R \leq \ell_c$)

Vertical force balance : $Mg = F_b + F_c$



Buoyancy :

$$F_b = \int_c \rho g z \, \underline{n} \cdot \hat{z} \, d\ell$$
$$= \rho g V_c$$

where V_c = volume above body, but inside the contact line

Capillary : $F_c = 2\sigma \sin\theta$