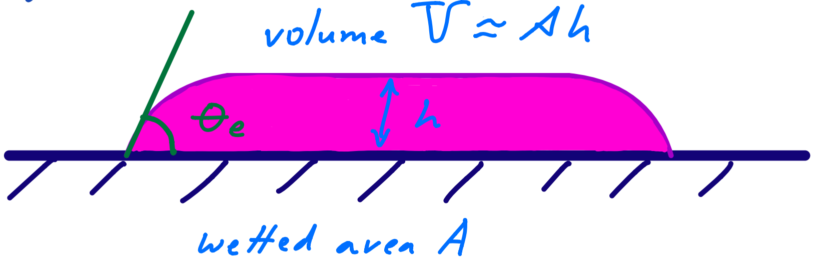
Lecture 4. Interfacial boundary conditions

But first, back to puddles ...

Total Energy :

$$E = (\gamma_{SL} - \gamma_{SV}) A + \gamma_{LV} A + \frac{1}{2}\rho g h^2 A$$

$$= - S \frac{V}{h} + \frac{1}{2}\rho g V h$$

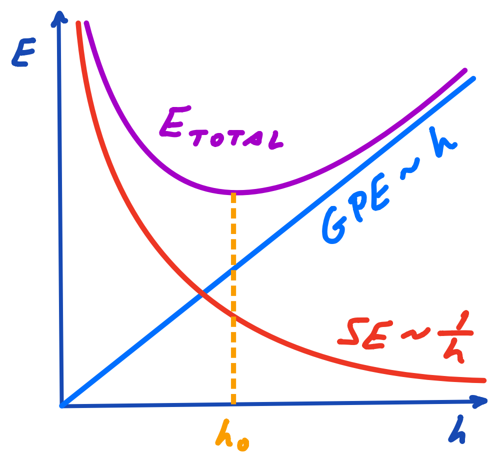


Minimum Energy :

$$\frac{dE}{dh} = 0 = S V \frac{1}{h^2} + \frac{1}{2}\rho g V$$

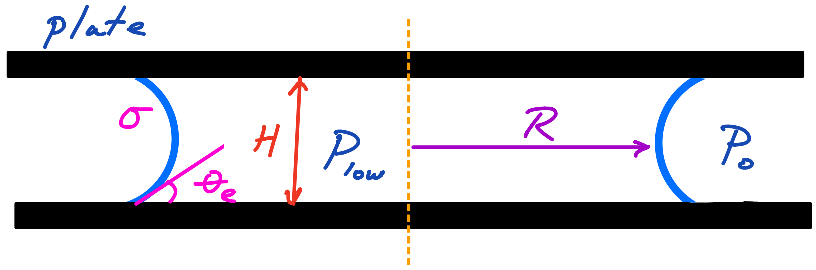
when $- S/h^2 = \frac{1}{2}\rho g$

$$\Rightarrow \boxed{h_0 = \left(\frac{-2S}{\rho g}\right)^{\frac{1}{2}} = 2 l_c \sin\frac{\theta_e}{2}}$$



Capillary Adhesion :

two wetted surfaces can stick together with great strength if $\theta_e < \frac{\pi}{2}$



e.g. two glass plates with Si oil between them

Laplace Pressure :

$$\Delta P = \sigma \left(\frac{1}{R} - \frac{\cos\theta_e}{H/2} \right) \simeq -\frac{2\sigma\cos\theta_e}{H}$$

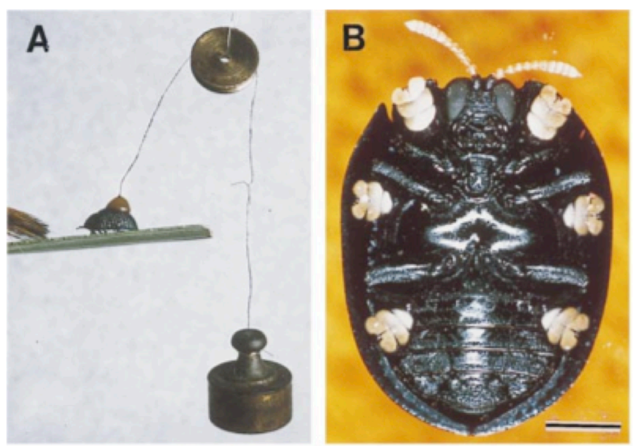
i.e. low P inside film provided $\theta_e < \frac{\pi}{2}$

If $H << R$, $\quad F = \pi R^2 \dfrac{2\sigma\cos\theta_e}{H}$

is the attractive force between the two plates

e.g. for H_2O, with $R = 1\,cm$, $H = 5\,\mu m$ and $\theta_e = 0$, one finds $\Delta P \sim \frac{1}{3}$ atm and adhesive force $F \sim 10\,N$, the wt of $1\,\ell$ of H_2O

$\Rightarrow$ this is used by beetles in nature



Q: how do they deadhere?

Interfacial Fluid Mechanics

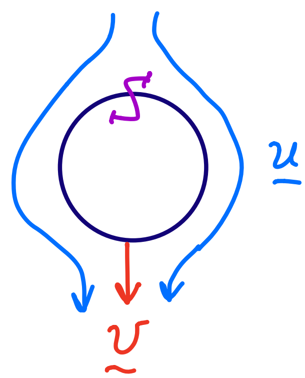
Governing Eqns : Navier Stokes

For an incompressible, homogeneous fluid of density ρ and viscosity $\mu = \rho\nu$, acted upon by an external force per unit volume $\underset{\sim}{f}$. Its velocity $\underline{u}$ and p fields evolve according to :

$$\rho\underbrace{\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\underline{\nabla}\,\underline{u}\right)} = -\vec{\nabla}p + \underset{\sim}{f} + \mu\nabla^2\underline{u}$$

$$\vec{\nabla}\cdot\vec{u} = 0 \qquad \frac{Du}{Dt} : \text{MATERIAL/LAGRANGIAN DERIVATIVE}$$

This system of 4 eqns in 4 unknowns (u_1, u_2, u_3, p) must be solved subject to appropriate BCs.

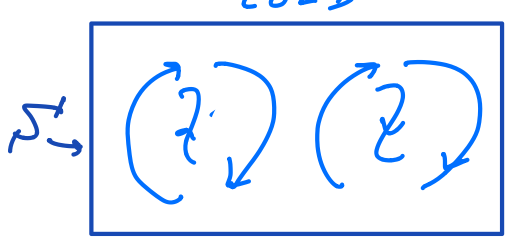


Fluid - Solid BCs : no-slip : $\underset{\sim}{u} = \underline{V}_{\text{SOLID}}$

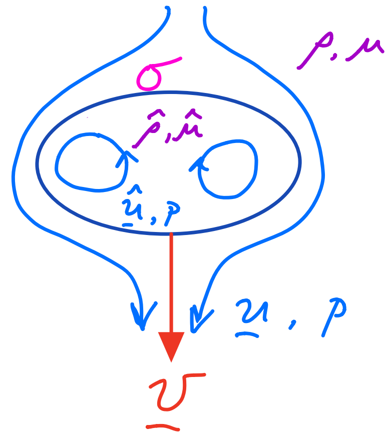
Eg. 1 Falling sphere : $\underline{u} = \underline{V}$ on S

Eg. 2 Convection in a box :

$$\underline{u} = 0 \text{ on } S$$

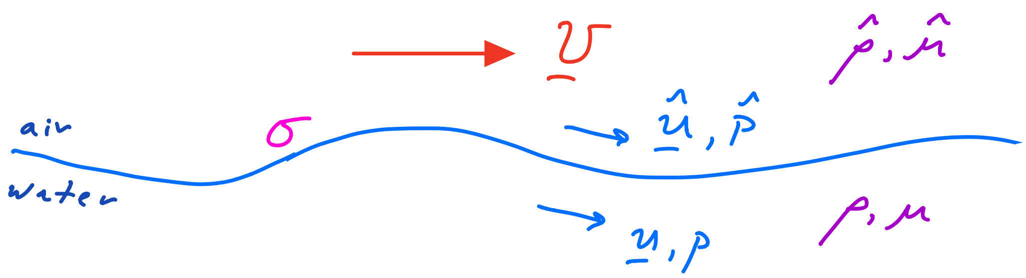


In this course, we are concerned with flows dominated by interfacial effects,



e.g. drop motion

e.g. water waves



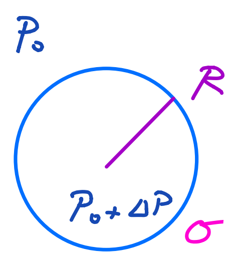
Note: these interfaces are free to move; thus, this of problems are known as FREE BOUNDARY PROBLEMS

Continuity of Velocity at an interface requires that

$$\underline{u} = \hat{\underline{u}} .$$

And what about pressure?

We've seen that $\Delta P \sim \sigma / R$ for a static bubble/drop. But to answer this question in general, we must develop stress conditions at a fluid-fluid interface.



Recall: Stress Tensor

For an incompressible Newtonian fluid, the state of stress within a fluid is described by the stress tensor:

$$\underline{\underline{T}} = -p\,\underline{\underline{I}} + \underbrace{2\mu\,\underline{\underline{E}}}_{\underline{\underline{\tau}}\, = \text{ viscous stress}} \qquad \text{where } \underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\underline{\underline{E}} = \frac{1}{2}\left[\nabla \underline{u} + (\nabla \underline{u})^T\right]$ is the deviatoric stress tensor

In Cartesian coords (x, y, z), we can expand as

$$\underline{\underline{T}} = \begin{pmatrix} -p + 2\mu \frac{\partial u_1}{\partial x_1} & \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \mu\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & -p + 2\mu \frac{\partial u_2}{\partial x_2} & \mu\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \mu\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) & \mu\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) & -p + 2\mu \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

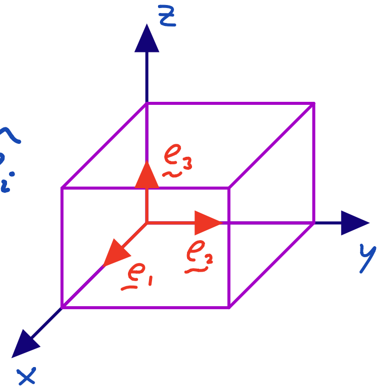
The associated hydrodynamic force per unit volume (force density) within a fluid is $\vec{\nabla} \cdot \underline{\underline{T}}$.

Navier-Stokes:

$$\rho \frac{D\underline{u}}{Dt} = \underline{\nabla} \cdot \underline{\underline{T}} + \underline{f}$$

$$= -\underline{\nabla} p + \mu \nabla^2 \underline{u} + \underline{f}$$

Now $T_{ij} = \dfrac{\text{force}}{\text{area}}$ acting in the $\hat{e}_j$ direction on a surface whose normal is $\hat{e}_i$



Note: 1. normal stresses (diagonals) T_{11}, T_{22}, T_{33} involve both p and u_i

2. tangential stresses (off-diagonals) T_{12}, T_{13}, T_{23}, etc. involve only velocity gradients, i.e. viscous stresses

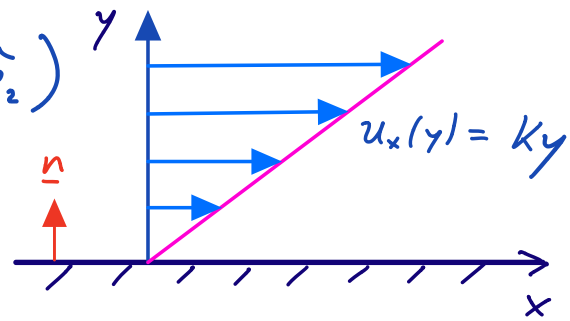
3. T_{ij} is symmetric.

4. $\underset{\sim}{t}(\underset{\sim}{n}) = \underset{\sim}{n} \cdot \underset{=}{T}$ = stress vector acting on surface with normal $\underset{\sim}{n}$.

Eg: Boundary stress from a shear flow is tangential

Force/area on lower surface (in x-direction on surface with $\underset{\sim}{n} = \hat{e}_2$)

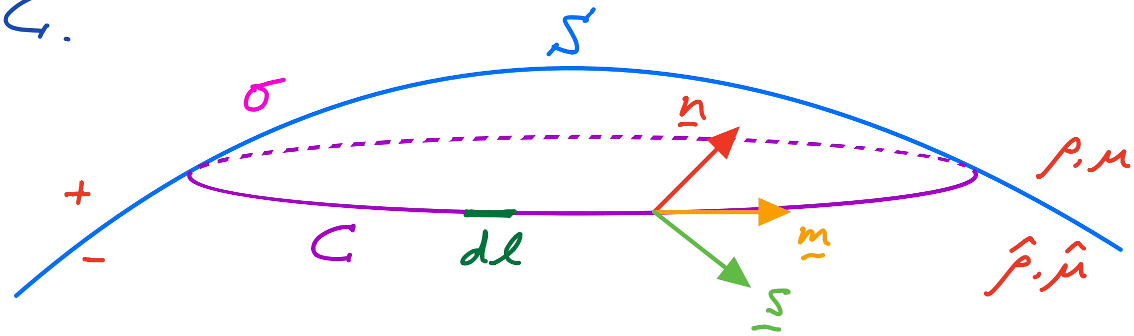
$$\tau_{yx} = \mu \frac{\partial u_x}{\partial y} = \mu k$$



Note: form of $\underset{=}{T}$ in arbitrary curvilinear coordinates is given in the Appendix of Batchelor.

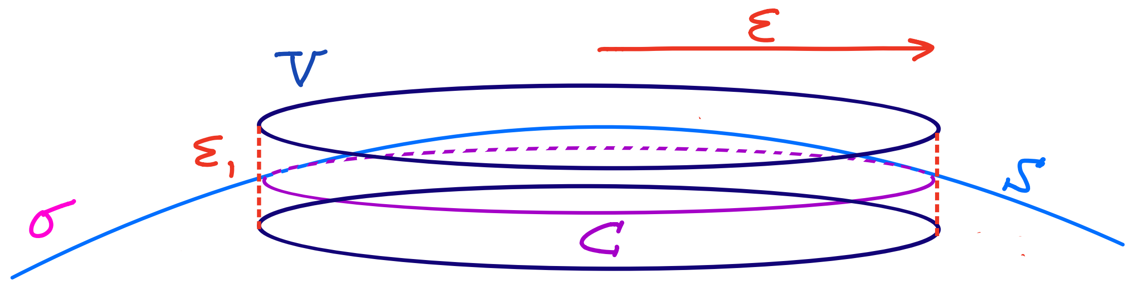
Interfacial Boundary Conditions

Consider an interfacial surface bound by a closed curve C.



where $\underline{n}$ is unit normal to S, C
$\underline{s}$ is unit normal to C, tangent to S
$\underline{m}$ is unit tangent to S, C

Consider an infinitesimal cylindrical pillbox V of radius ε and height ε_1 (s.t. $\varepsilon_1 \ll \varepsilon$) that intersects C.



For a force balance on V, we must consider:

$\underline{t}(\underline{n}) = \underline{n} \cdot \underline{\underline{T}}$ is the force/area exerted by upper fluid (+)

$\hat{\underline{t}}(\hat{\underline{n}}) = \hat{\underline{n}} \cdot \hat{\underline{\underline{T}}}$ is " " lower fluid (−)

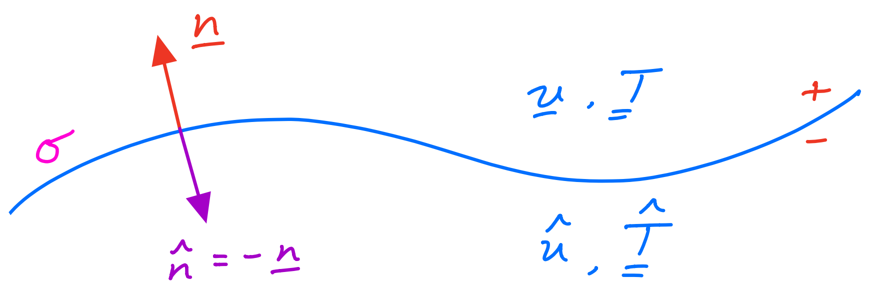
Force balance :

$$\underbrace{\int_V \rho \frac{D\underline{u}}{Dt}\, dV}_{\text{inertial force}} = \underbrace{\int_V \underline{f}\, dV}_{\text{body forces}} + \int_S \underbrace{\left[\underline{t}(\underline{n}) + \hat{t}(\hat{n})\right] dS}_{\text{hydrodynamic forces acting on upper + lower surfaces}} + \underbrace{\int_C \sigma \underline{s}\, dl}_{\text{surface tension}}$$

Now, the inertial + body forces must scale as $\varepsilon^2 \varepsilon_1$, while the surface forces scale as ε^2. Hence, in the limit of $\varepsilon_1 \to 0$, surface fores must balance:

$$\int_S \underline{t}(\underline{n}) + \hat{t}(\hat{n})\, dS + \int_C \sigma \underline{s}\, dl = 0$$

Now, we have $\underline{t}(\underline{n}) = \underline{n} \cdot \underline{\underline{T}}$, $\quad \hat{t}(\hat{n}) = \hat{n} \cdot \hat{\underline{\underline{T}}}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -\underline{n} \cdot \hat{\underline{\underline{T}}}$



Moreover, application of Stokes Thm (see Handout 1) allows us to write:

$$\int_C \sigma \underline{s}\, dl = \int_S \underline{\nabla}_s \sigma - \sigma \underline{n} (\underline{\nabla}_s \cdot \underline{n})\, dS$$

where $\underline{\nabla}_s \equiv (\underline{\underline{I}} - \underline{n}\,\underline{n}) \cdot \underline{\nabla} = \underline{\nabla} - \underline{n} \frac{\partial}{\partial n}$

is the tangential gradient operator,

required because σ and $\underline{n}$ are only defined on the interface. We proceed by dropping the subscript, $\underline{\nabla}_s \rightarrow \underline{\nabla}$, with this understanding.

The surface force balance:

$$\int_{S'} \left(\underline{n} \cdot \underline{\underline{T}} - \underline{n} \cdot \hat{\underline{\underline{T}}} \right) dS' = \int_{S'} \sigma \underline{n} (\underline{\nabla} \cdot \underline{n}) - \underline{\nabla} \sigma \, dS'$$

Now since the surface element S' is arbitrary, the integrand must vanish identically.

$\Rightarrow$ Interfacial Stress Balance Equation

$$\boxed{\underline{n} \cdot \underline{\underline{T}} - \underline{n} \cdot \hat{\underline{\underline{T}}} = \sigma \underline{n} \, \underline{\nabla} \cdot \underline{n} - \underline{\nabla} \sigma}$$

STRESS JUMP AT INTERFACE

NORMAL LAPLACE/ CURVATURE PRESSURE

MARANGONI STRESS