$$\frac{\partial u}{\partial t} = \frac{U^2}{L}\frac{\partial u'}{\partial t'}$$

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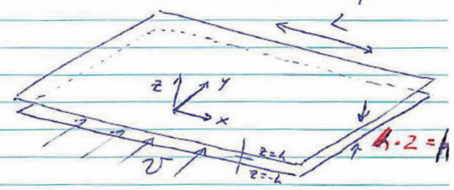
The Hele-Shaw Cell

- we shall demonstrate that viscous flows in a thin gap geometry (ie. a "Hele-Shaw Cell"), typically comprised of fluid pressed between glass plates, provide a simple means for experimental modeling of potential flows: high Re flows past obstacles, and flow in porous media.

Consider a steady flow in a thin gap (thickness d) characterized by a lengthscale L and speed U



Writing $\underline{u} = (u, v, w)$, we have N-S eqns (steady)

$$\underline{u} \cdot \underline{\nabla}\, \underline{u} = -\underline{\nabla} p + \nu \nabla^2 \underline{u}$$

$$\nabla \cdot \underline{u} = 0 \qquad t \sim \frac{L}{U} t'$$

Nondimensionalize: $(x, y) \sim L(x', y')$, $\quad z \sim \cancel{H}\, h z'$ $\cancel{\sim H z'}$

$$(u, v) \sim (u', v')\, U, \quad w \sim W w', \quad p \approx \frac{\mu U L}{\cancel{H^2}\, h^2} p' \ (\text{LUB})$$

Continuity: $\frac{U}{L}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) + \frac{W}{H}\frac{\partial w'}{\partial z'} = 0 \Rightarrow W \sim \frac{U L^{-1}}{H^{-1}}$

$H \to h$ throughout !

$\hat{x}$-mom: $Re \frac{H^2}{L^2}\left\{u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'}\right\} = -\frac{\partial p'}{\partial x'} + \frac{H^2}{L^2}\left\{\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right\} + \frac{\partial^2 u'}{\partial z'^2}$

$\hat{y}$-: $Re \frac{H^2}{L^2}\left\{u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} + w'\frac{\partial v'}{\partial z'}\right\} = -\frac{\partial p'}{\partial y'} + \left(\frac{H}{L}\right)^2\left\{\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right\} + \frac{\partial^2 v'}{\partial z'^2}$

$\hat{z}$: $Re \left(\frac{H}{L}\right)^4\left\{u'\frac{\partial w'}{\partial x'} + v'\frac{\partial w'}{\partial y'} + w'\frac{\partial w'}{\partial z'}\right\} = -\frac{\partial p'}{\partial z'} + \left(\frac{H}{L}\right)^4\left\{\frac{\partial^2 w'}{\partial x'^2} + \frac{\partial^2 w'}{\partial y'^2}\right\} + \left(\frac{H}{L}\right)^2\left\{\frac{\partial^2 w'}{\partial z'^2}\right\}^2$

If $\left(\frac{H}{L}\right) \ll 1$, and $Re\left(\frac{H}{L}\right)^2 \ll 1$ *, then we have

$$Re = \frac{UL}{\nu}$$

$$0 = -\frac{\partial p'}{\partial x'} + \frac{\partial^2 u'}{\partial z'^2}$$

$$0 = -\frac{\partial p'}{\partial y'} + \frac{\partial^2 v'}{\partial z'^2}$$

$$0 = -\frac{\partial p'}{\partial z'}$$

Solve w/ B.C.s

1. $u' = v' = 0$ at $z' = \pm 1$
2. $\frac{\partial}{\partial z'}(u', v') = 0$ at $z' = 0$

*

Note: we see the natural appearance of the "Reduced Re":

$$Re\left(\frac{h}{L}\right)^2 = \frac{UL}{\nu}\frac{H^2}{L^2} \sim \frac{\underline{U}\cdot\nabla\underline{U}}{\nu\nabla^2\underline{u}}$$

$$= \frac{H^2/\nu}{L/U} = \frac{\text{TIMESCALE OF DIFFUSION ACROSS GAP}}{\text{CONVECTIVE TIMESCALE OF MEAN FLOW}}$$

Integration yields: (since p' indep of z) DIMENSIONAL

$$u' = \frac{1}{2}\left(-\frac{\partial p'}{\partial x'}\right)(1-z'^2) \quad \Rightarrow \quad u = \frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left[1-\left(\frac{z}{h}\right)^2\right]$$

$$v' = \frac{1}{2}\left(-\frac{\partial p'}{\partial y'}\right)(1-z'^2) \quad \Rightarrow \quad v = \frac{h^2}{2\mu}\left(-\frac{\partial p}{\partial y}\right)\left[1-\left(\frac{z}{h}\right)^2\right]$$

$$w' = 0 \quad \Rightarrow \quad w = 0$$

Our flow to leading order has the 2-D description

$$\underline{u} = \frac{-h^2}{2\mu}\underline{\nabla}p\underbrace{\left[1-\left(\frac{z}{h}\right)^2\right]}_{\text{PARABOLIC}}$$

We define the mean velocity as

$$\bar{\underline{u}}(x,y) = \frac{2}{2h}\int_0^h \underline{u}(x,y)\,dz = -\frac{h2}{4\mu}\underline{\nabla}p\int_0^h\left[1-\left(\frac{z}{h}\right)^2\right]dz$$

$$= -\frac{h2}{4\mu}\underline{\nabla}p\left[z-\frac{z^3}{3h^2}\right]_0^h = -\frac{2h^2}{6\mu}\underline{\nabla}p$$

$$\Rightarrow \quad \bar{\underline{u}} = -\frac{K_{eff}}{\mu}\underline{\nabla}p \qquad \text{where } K_{eff} = \frac{h^2 2}{6} = \frac{h^2}{3}$$

$\Rightarrow$ The depth-averaged velocity field is thus a potential flow; specifically, satisfies Darcy's eqn with $K_{eff} = \frac{2h^2}{6} = \frac{h^2}{3}$

$\Rightarrow$ the pressure field $p(x,y)$ resulting from Stokes flow can be considered to be the velocity potential for an inviscid irrotational mean flow

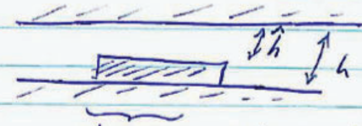
$\Rightarrow$ the H-S cell is often used to study flow in porous media

$$\underline{\nabla}\cdot\bar{\underline{u}} = 0 \quad \Rightarrow \quad \nabla^2 p = 0$$

NOTE: $\underline{u}(z=0) = -\frac{h^2}{2\mu}\underline{\nabla}p$ is also a potential/Darcy flow

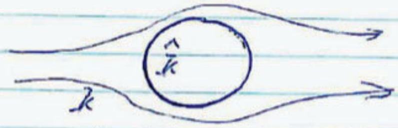
Eg. 3 Flow past porous inclusion

$\Rightarrow$



$$\underline{k}_{eff} = \frac{h^2(\underline{x})}{3}$$

One may thus model experimentally the problem considered last class, flow past a cylindrical porosity inclusion:

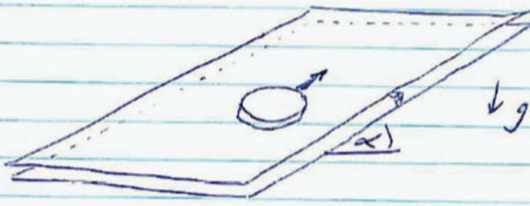


$$\hat{\underline{k}} = \frac{\hat{h}^2}{3}$$

$$\underline{k} = \frac{h^2}{3}$$

Note 1: in the $\hat{h} \to 0$ limit, $\hat{k} \to 0$ and we retain the sol'n for inviscid irrotational flow past a cylinder

A bubble rising in a H-S cell:



- to calculate drag on bubble, D, one must consider dissipation associated with the channel flow in the thin gap

$$\frac{\text{DISSIPATION}}{\text{AREA}} = \underline{\Phi}_{/AREA} = 2\mu \int_{-h}^{h} \left(\frac{\partial u}{\partial z}\right)^2 dz$$

where $\underset{\sim}{u} = \frac{h^2}{2\mu}(-\underline{\nabla} p)\left[1 - \left(\frac{z}{h}\right)^2\right]$

$$\Rightarrow \frac{\partial u}{\partial z} = +\frac{h^2}{2\mu}(\underline{\nabla} p)\left[\frac{2z}{h^2}\right] = \frac{1}{\mu}(\underline{\nabla} p) z$$

$$\Rightarrow \underline{\Phi}_{/area} = 2\mu \int_{-h}^{h} \left(\frac{\underline{\nabla} p}{\mu}\right)^2 z^2 \, dz$$

$$\frac{\Phi}{\text{area}} = \frac{2}{\mu}\left(\underset{\sim}{\nabla} p\right)^2 \left(\frac{z^3}{3}\right)\Bigg|_{-h}^{h} = \frac{4}{3\mu}\left(\underset{\sim}{\nabla} p\right)^2 h^3$$

indep of z

but we know $\quad \bar{\underset{\sim}{u}} = -\frac{k_{eff}}{\mu}\underset{\sim}{\nabla} p = -\frac{h^2}{3\mu}\underset{\sim}{\nabla} p$

$$\Rightarrow \quad \underset{\sim}{\nabla} p = -\frac{3\mu}{h^2}\bar{\underset{\sim}{u}}$$

$$\Phi/_{\text{area}} = \frac{4}{3\mu}\left(\frac{9\mu^2}{h^4}\right)h^3\left(\bar{\underset{\sim}{u}}\cdot\bar{\underset{\sim}{u}}\right)$$

$$= \frac{12\mu}{h}\left(\bar{u}_r^{\,2} + \bar{u}_\theta^{\,2}\right)$$

But $\bar{u}_r$ and $\bar{u}_\theta$ are known from potential flow generated by a uniformly translating cylinder; specifically, (see PS #5)

$$\bar{u}_r = \frac{a^2}{r^2}\,\mathcal{V}\cos\theta$$

$$\bar{u}_\theta = \frac{a^2}{r^2}\,\mathcal{V}\sin\theta$$

$$\bar{u}_r^{\,2} + \bar{u}_\theta^{\,2} = \frac{a^4}{r^4}\mathcal{V}^2$$

$$\therefore \quad \Phi_{\text{TOTAL}} = \int_0^{\pi}\int_a^{\infty} \Phi \, r\, dr\, d\theta$$

$$= 12\pi\mu\,\mathcal{V}^2 a^2\,\frac{1}{2h}$$

$$\int_a^{\infty}\frac{1}{r^3}\,dr = -\frac{1}{2}\frac{1}{r^2}\Bigg|_a^{\infty} = \frac{1}{2a^2}$$

We now equate the total viscous dissipation with the rate of work done by buoyancy:

$$6\pi\mu\,\mathcal{V}^2\frac{a^2}{h} = \pi a^2\cdot 2h\,\mathcal{V}\rho g\sin\alpha$$

$$\Rightarrow \mathcal{V} = \frac{g h^2}{3\nu}\sin\alpha = \frac{g\,d^2}{12\nu}\sin\alpha \qquad \text{where } d = 2h$$

Note: $\mathcal{V}$ indep of a!