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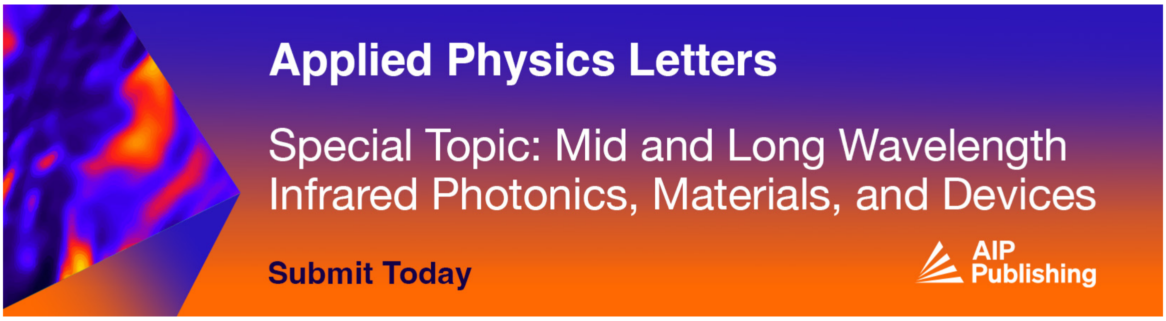

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ABSTRACT

We present a number of fresh perspectives on pilot-wave hydrodynamics, the field initiated in 2005 by Couder and Fort's discovery that millimetric droplets self-propelling along the surface of a vibrating bath can capture certain features of quantum systems. A recurring theme will be that pilot-wave hydrodynamics furnishes a classical framework for reproducing many quantum phenomena and allows one to rationalize such phenomena mechanistically, from a local realist perspective, obviating the need to appeal to quantum nonlocality. The distinction is drawn between hydrodynamic pilot-wave theory and its quantum counterparts, Bohmian mechanics, the Bohm–Vigier stochastic pilot-wave theory, and de Broglie's theory of the double-solution. Each of these quantum predecessors provide a valuable touchstone as we take the physical picture engendered in the walking droplets and extend it into the quantum realm via theoretical modeling. Emphasis is given to recent developments in the field, both experimental and conceptual, and to forecasting potentially fruitful new directions.

I. INTRODUCTION

The walking-droplet system[1] represents a macroscopic realization of wave-particle duality and has furnished a platform for exploring the boundary between classical and quantum behavior.[2,3] The physical picture suggested is one rooted in classical mechanics and so local realism. Some hydrodynamic quantum analogs (HQAs) achieve statistical behavior that is strikingly similar to their quantum counterparts, provided the Faraday wavelength is identified with the de Broglie wavelength. For example, the emergent statistics in the walking droplet experiments are virtually identical to those in Friedel oscillations[4] and the quantum corral.[5,6] In other HQAs, including single-particle diffraction and interference[7–9] and orbital dynamics,[10–12] the emergent statistics is only similar in form, the connection to quantum mechanics only qualitative. Whether the match is quantitative or qualitative, when a hydrodynamic quantum analog is achieved, it demonstrates that one can grasp the essential physics of the quantum system of interest, rationalize the phenomenon, from a local realist perspective.

The HQA venture was launched in 2006 by Couder and Fort's study of walker diffraction.[7] While these results were hotly contested,[13,14] they have since been reproduced repeatedly in experiments of increasing precision and sophistication.[8,9,15] In particular, Pucci *et al.*[8] characterized the dependence of the diffraction on drop size and vibrational acceleration, while Ellegaard and Levinsen[9] imposed thermal control on the bath in order to eliminate the slow drift of the Faraday threshold during the course of experiments. The conclusion is now clear: even though the particles pass through the slits one at a time, the emergent statistics of the diffraction angle take a wave-like form. The diffraction pattern does not conform to the Fraunhofer pattern that arises in certain limits of the quantum problem, so one must concede that the analog is strictly qualitative. Nevertheless, the emergent wave-like statistics and single-particle interference are plainly apparent. After familiarizing oneself with the diffraction of walking drops, one can only puzzle over Feynman's insistence on the absolute inscrutability of the electron doubles-slit experiment,[16] particularly when the basic physical picture needed to understand the effect was furnished in the 1920s by Louis de Broglie in his double-solution theory.[17,18]

The traditional view of quantum mechanics, also known as the Copenhagen interpretation, provides no description of particles traveling along well-defined paths. A quantum system is described completely by its wave function, which evolves deterministically according to the Schrödinger equation. When a measurement is made, the system collapses to a particular outcome through a process that is non-unitary and nonlocal, giving rise to the notorious measurement problem. It has, thus, been said that, according to the Copenhagen view, particles are nothing more than a measurement of them, and without measurement there is nothing but the wave function.[19] Heisenberg's uncertainty principle suggests that one may think of particles as localized wave packets that represent the probability of detecting particles spread out in space. However, a measurement may result in the materialization of a particle at an appreciable distance from the center of the wave packet that represents it. Thus, at best, one can associate the trajectory of a quantum wave packet with the average trajectory of the quantum particle, but not with its actual trajectory. In what follows, we argue that this representation of quantum particles in terms of their associated wave packets is responsible for a number of the conceptual difficulties arising in the foundations of quantum mechanics.

We proceed by considering a number of key hydrodynamic quantum analogs. Recent experimental advances are highlighted, including HQAs of spin lattices,[20] Anderson localization,[21] and new directions in hydrodynamic interferometry.[22,23] Conceptual advances are also enumerated, with particular attention given to the relation between the guiding wave field in the hydrodynamic system and the quantum potential in Bohmian mechanics. Potentially fruitful directions for future work are discussed.

II. QUANTUM PILOT-WAVE THEORIES

Prior to the advent of the Copenhagen Interpretation or to the notion of impossibility proofs, de Broglie proposed a model of quantum dynamics, his theory of the double solution, in which microscopic quantum particles have an internal vibration at the Compton frequency, $\omega_c = mc^2/\hbar$. This vibration excites Klein–Gordon waves that propel the particle along in such a way that the particle momentum is related to the wave number of the pilot wave through the de Broglie relation $\mathbf{p} = \hbar \mathbf{k_B}$. De Broglie proposed, but never proved, that the resulting dynamics could give rise to emergent statistics described by the standard quantum wavefunction Ψ. The pilot-wave hydrodynamic system represents a macroscopic realization of this physical picture,[2] with the Faraday frequency playing the role of the Compton frequency, the Faraday wavelength that of the de Broglie wavelength $\lambda_B = 2\pi/k_B$. Its success in achieving several HQAs would seem to speak in favor of de Broglie's original proposal.

Bohmian mechanics[24] is a realist theory of the quantum realm that restores the notion of particle trajectories[25] and has provided a valuable touchstone in connecting pilot-wave hydrodynamics to quantum mechanics.[2,3] The theory is rooted in Madelung's hydrodynamic interpretation of the linear Schrödinger equation, which emerges from a polar transformation of the wavefunction.[26] If one assumes that the particle velocity, $\dot{\mathbf{x}}_\mathbf{p}$ is equal to the quantum velocity of probability, then one obtains a trajectory equation that describes a classical particle moving in response to both classical and quantum potentials, V and Q, respectively: $m\ddot{\mathbf{x}}_\mathbf{p} = -\nabla V - \nabla Q$, where Q is prescribed by the wavefunction Ψ. Alternatively, the first order formulation of Bohmian mechanics allows one to write the trajectory equation in the form $\frac{d\mathbf{x_p}(t)}{dt} = \frac{\hbar}{m}\mathrm{Im}\left[\frac{\nabla\psi(\dot{\mathbf{x}}_\mathbf{p}(t),t)}{\psi(\dot{\mathbf{x}}_\mathbf{p}(t),t)}\right]$. While local in the sense that the particle responds only to the local form of the wavefunction, Bohmian mechanics is nonlocal in that the wavefunction depends on the system geometry and configuration of all the particles in the system, and reacts to any change thereof instantaneously. This nonlocality represents the most important distinction between Bohmian mechanics and pilot-wave hydrodynamics. In the former, the guiding wave is imposed by fiat; in the latter, it is created by the particle itself, rendering the theory local. In this regard, pilot-wave hydrodynamics is more closely aligned with de Broglie's original conception, his theory of the double solution, which was rooted in local realism.

In response to a number of criticisms leveled at Bohm's theory,[27] only 2 years after its initial launch, Bohm was joined by Vigier[28] in proposing that his trajectory equation be augmented by a stochastic forcing component. It was, thus, posited that quantum particles move in an erratic fashion about the Bohmian trajectories, just as dust particles mean in a gas flow jiggle around streamlines in response to Brownian motion. The precise form of this stochastic motion was not specified and so must be considered *ad hoc*. The particle motion may, thus, be decomposed into a mean component guided by a nonlocal field, plus an *ad hoc* stochastic component. A similar decomposition arises in consideration of the HQA of the quantum corral.

III. HYDRODYNAMIC QUANTUM ANALOGS

A. Corrals

Hydrodynamic analogs of the quantum corral[5,6,29] are among the most compelling in that the emergent statistics are strikingly similar to those in their quantum counterparts.[30–32] When a walker explores a circular corral of characteristic radius 2 cm, it executes a structured random walk over its self-induced wave potential. After approximately 1 h, the resulting correlation between position and speed gives rise to a statistical signature that is remarkably similar in form to that arising in the quantum corral and may be expressed in terms of the wave modes of the cavity. If one assumes that the emergent statistical form is a complete description of the system, then one must contend with its instantaneous collapse to a point following the act of observing the droplet, a nonlocal effect in the quantum measurement problem that is seen here to be an unnecessary conceit.[3]

In an elliptical corral (Fig. 1), the emergent statistics may likewise be expressed as the sum of the eigenmodes of the cavity. Moreover, the introduction of topographic inhomogeneities allowed for alteration of the emergent statistics through favoring one cavity mode over the others. For example, introduction of a well at one of the foci of the ellipse favored a mode with maxima at both foci, giving rise to a statistical projection effect analogous to the quantum mirage.[32,35] It was further noted that the mean pilot-wave $\bar{\eta}$ was comparable in form to that of the particle's emergent probability distribution function, $\mu(\mathbf{x})$.[6] This observation motivated the theoretical deduction of the relation between the two,[36,37] for both periodic and ergodic droplet motion. Specifically, the mean-pilot-wave potential may be expressed as the convolution,

$$\bar{\eta}(x) = \int_{-\infty}^{\infty} \eta_B(x-y)\mu(y)\,dy, \tag{1}$$

where $\eta_B(x,y)$ is the wave field of a stationary bouncer located at position y. Notably, this mean pilot-wave potential is nonlocal in the same sense as the quantum potential in Bohmian mechanics, that is, specified by the system statistics. Decomposition of the instantaneous pilot-wave field into its mean and fluctuating components, $\eta(\mathbf{x},t) = \bar{\eta}(\mathbf{x}) + \eta'(\mathbf{x},t)$ suggests that the former could be seen as being analogous to the quantum potential, the latter to the *ad hoc* stochastic dynamics proposed by Bohm and Vigier.[28] Exploring and clarifying the extent to which the motion of walking droplets may be meaningfully decomposed into mean and stochastic components is the focus of ongoing investigations in HQA. Particular attention is being given to

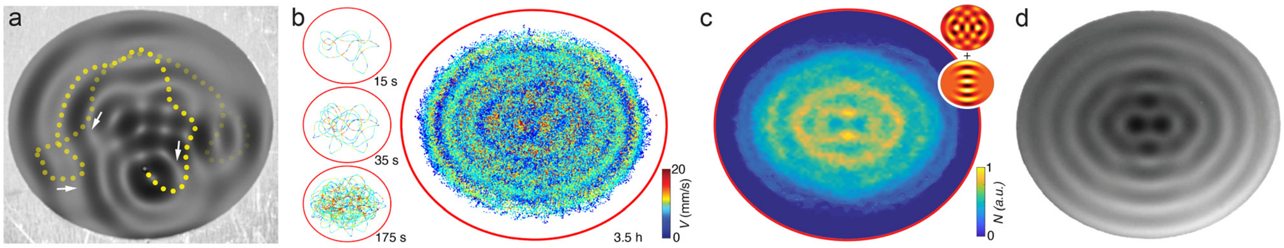

FIG. 1. The wave field and emergent statistics in the elliptical corral.[6] (a) The instantaneous wave field and recent droplet trajectory. (b) The build-up of the droplet trajectory over 3.5 h eventually reveals a correlation between droplet position and speed. (c) The result is the emergent statistical form that may be expressed in terms of two elliptical cavity modes. (d) Time-averaging the instantaneous wave field reveals that the mean pilot-wave field has the same form as the emergent statistics. Reprinted with permission from Saenz *et al.*, Nat. Phys. **14**, 315–319 (2018).

elucidating the manner in which the breaking of synchrony between particle and wave induces additional forces not considered in theoretical models based on the assumption of wave-particle resonance.[37,38]

B. Orbital dynamics

Orbital quantization is also a robust feature of pilot-wave hydrodynamics and has been demonstrated to arise for walker motion in a rotating frame[10,11,33] [Figs. 2(a)–2(c)] and a walker subjected to a simple harmonic potential, specifically a radial spring force.[12,39] Orbital quantization represents a clear example of a particle exciting, then being constrained by, its own wave potential. Recent theoretical work has provided new insight into the stability and origins of the quantized orbits.

A hierarchy of theoretical models with different degrees of complexity have been successful in capturing the vast majority of phenomena observed in experiments.[3] In the so-called "stroboscopic" model,[38] perfect resonance between the bouncing droplet and its guiding wave is assumed, and the droplet is approximated as a continuous source of waves. The resulting stroboscopic model has been particularly important in the theoretical description of the walking droplets owing to its analytical tractability. Oza *et al.*[33] demonstrated that it rationalizes the emergent orbital quantization of walkers in a rotating frame.[10,11] Specifically, they used it as the basis of a stability analysis to show that the discrete orbits observed in experiments correspond to stable circular solutions of the walker's equation of motion [Fig. 2(b)]. In a recent theoretical study, Liu *et al.*[40] demonstrated that the quantized orbits arise at local extrema of the mean-pilot-wave potential; moreover, the form of instability that sets in as the memory increases is prescribed by the local form of the mean pilot-wave potential, $\bar{\eta}$. Specifically, when the drop orbits at local maxima or minima of $\bar{\eta}$, the onset of instability is, respectively, monotonic or wobbling. The manner in which the mean pilot-wave potential influences the dynamics and emergent statistics is an area of focus as we explore its relation to the quantum potential.

Elucidating the mechanism responsible for the walker's orbital quantization has also received significant attention. In the first study of orbital quantization, Fort *et al.*[10] introduced the notion of a "virtual walker" orbiting at a position diametrically opposite to the walker's position. Bush *et al.*[41] derived a reduction of the stroboscopic model, valid in the weak-acceleration limit, which captures analytically the wave force due to the recent past. This so-called "boost" model

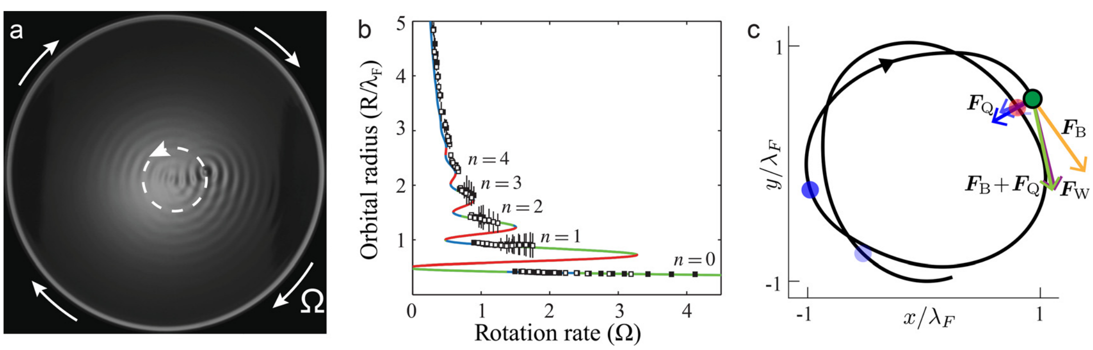

FIG. 2. Orbital quantization in a rotating frame. (a) The trajectory of a walker becomes circular (dashed line) in a rotating bath due to the Coriolis force. As the vibrational forcing increases ($\gamma/\gamma_F = 0.971$), the wave decay time eventually exceeds the orbital period. The walker then interacts with its own wake, leading to the emergence of (b) quantized circular orbits.[10,11,33] The orbits are color-coded according to their stability: blue indicates stable orbits and green and red indicate orbits that destabilize via oscillatory and non-oscillatory instabilities, respectively. (c) Blitstein *et al.*[34] have recently demonstrated that, owing to wave interference, the force responsible for the orbital quantization originates at the stationary points along the past trajectory, specifically points at which the distance to the droplet is not instantaneously changing. The full pilot-wave force, $\boldsymbol{F}_W$, may, thus, be approximated by the sum of long-range quantization forces, $\boldsymbol{F}_Q$, emanating from stationary points (blue), and a local "boost" force, $\boldsymbol{F}_B$, that sets the walker speed. Reprinted with permission.

demonstrates that the most recent waves composing the pilot-wave field act to (i) increase the walker mass relative to the droplet mass and (ii) produce a non-linear friction force that always drive the droplet toward its free walking speed. The boost model was successful in rationalizing the offset between the orbital radii of walkers at low memory[10,11] and those expected in the absence of the pilot-wave field.

Blitstein *et al.*[34] have recently provided new insight into the origin and form of the wave-mediated forces responsible for orbital quantization. The authors demonstrated that, owing to wave interference, the force responsible for orbital quantization originates from waves excited near specific stationary points along the walker's path [Fig. 2(c)], which constructively interfere at the droplet position. Waves excited elsewhere along the walker's path interfere destructively at the droplet position and may, thus, be safely neglected. Based on this insight, the authors were able to approximate the quantizing force, $\boldsymbol{F}_{\mathrm{Q}}$, in terms of a sinusoidal wave potential, projected from the stationary points toward the droplet position, that restricts the allowed radii. The authors, thus, derived a model with the minimal ingredients required to capture the origins of quantization in pilot-wave orbital dynamics, as well as the quasi-periodic and chaotic orbits with preferred radii arising at higher memory. This minimal model makes clear the distinction between the forces captured by the boost model,[41] which prescribe the walker's wave-induced added mass and nonlinear drag, and the forces responsible for quantization, as are non-local in the sense that they originate at locations along the walker's past trajectory. The quantizing force elucidated by Blitstein *et al.*[34] arises not only in orbital dynamics, but in other settings where walkers depart from rectilinear motion. Notably, this force serves to drive the droplet along paths with preferred radii of curvature, a feature that has been noted in a number of HQAs.[4,5,11]

C. Spin lattices

Saenz *et al.*[20] introduced hydrodynamic spin lattices (HSLs) as an analog system that allows one to investigate the wave-mediated interactions and collective dynamics of hydrodynamic spin states[42,43] in both stationary and rotating frames. Using submerged circular wells at the bottom of the fluid bath,[4,6] the authors stabilized walker spin states, leading to a clockwise or counterclockwise angular motion centered on each well [Fig. 3(a)]. When a collection of such spin states is arranged in a 1D or 2D lattice geometry, a thin fluid layer between wells enables wave-mediated interactions between neighboring droplets [Fig. 3(a), inset]. For sufficiently strong pair-coupling, these interactions may induce local spin flips, leading to preferred collective states of analog antiferromagnetic [Fig. 3(b)] or ferromagnetic order [Fig. 3(c)] depending on the lattice spacing. Moreover, the authors used the mathematical equivalence between the Lorentz force acting on a moving charge and the Coriolis force acting on a moving mass to demonstrate a transition from anti-ferromagnetic to ferromagnetic states when the spin lattice is subject to uniform rotation at constant frequency [Fig. 3(d)]. Inverting the sense of bath rotation inverts the sign of the magnetization, analogous to the polarization observed in antiferromagnetic materials subject to external magnetic fields. Saenz *et al.* also investigated 2D lattices to demonstrate that square HSLs can sustain antiferromagnetic order in the absence of rotation [Fig. 3(e)] and undergo a polarization transition as the Coriolis force is increased [Fig. 3(f)].

To rationalize the collective order observed in HSLs, Saenz *et al.*[20] examined the wave-mediated coupling between two spinning walkers, which revealed four distinct modes of pairwise symmetry breaking. Specifically, there are two ferromagnetic phases and two antiferromagnetic phases, distinguished by preferential in-phase and out-of-phase rotation, respectively, that may be adjusted by changing the lattice spacing. The authors derived a generalized Kuramoto phase model to demonstrate that walkers self-organize in such a way as to minimize the coupling potential.

HSLs present a range of interesting new directions, in particular, in two dimensions wherein classical and quantum spin lattices display features that are absent in their 1D counterparts. Of particular interest is exploring the ability of HSLs to solve combinatorial optimization problems,[44] wherein NP (nondeterministic polynomial time) problems

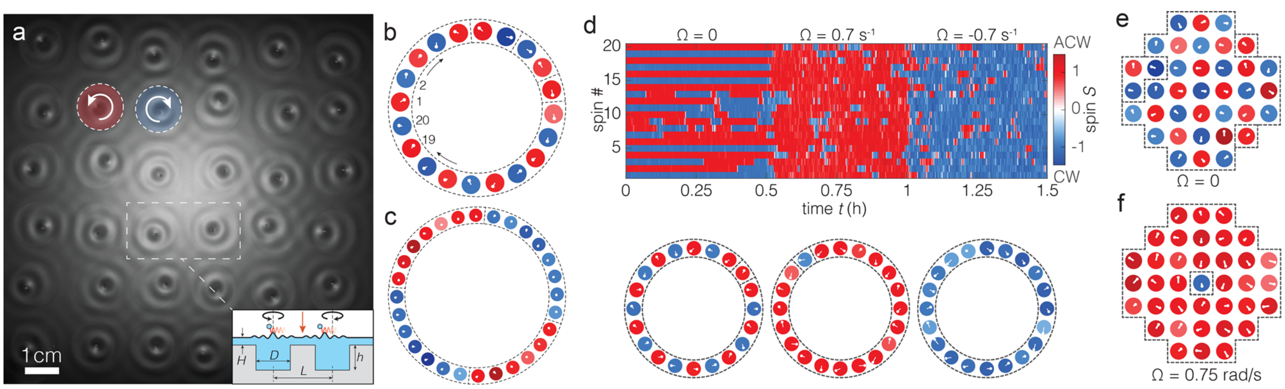

FIG. 3. Hydrodynamic spin lattices (HSLs). (a) Top view and schematic cross section of square lattice of hydrodynamic spin states. Owing to the influence of submerged circular wells, the walkers propel themselves along circular orbits in the clockwise (blue) or anti-clockwise (red) directions. A thin liquid layer between wells enables wave-mediated interactions that may prompt spin flips, leading to coherent collective order. Depending on the lattice spacing, collective (b) anti-ferromagnetic and (c) ferromagnetic order may spontaneously emerge, here illustrated in one-dimensional lattices with periodic boundary conditions. (d) Rotating the system about the vertical axis with angular frequency Ω prompts a transition from anti-ferromagnetic to ferromagnetic states, analogous to the global polarization produced when an anti-ferromagnetic material is subject to an external magnetic field. The same (e) collective order and (f) rotation-induced polarization was observed in two-dimensional square HSLs. The emergent collective order oscillates with the lattice separation between four different modes of pairwise symmetry breaking. Reprinted with permission from Saenz *et al.*, Nature **596**, 58–62 (2021).

are formulated as an Ising problem that is more easily solved. The ability to fine-tune the inter-spin correlations in the HSL system by varying its geometry may be translated to optimization algorithms inspired by certain approaches to quantum computing, such as quantum annealing.[45] This approach would require the formulation of a Hamiltonian for a simple HSL configuration (i.e., one for which the ground state is well known), then to slowly vary the geometry until a different Hamiltonian is obtained, the ground state of which would constitute a solution to a given optimization problem. Another interesting future direction would be to explore the collective order under frustrated 2D lattice geometries, such as triangular or hexagonal spin lattices. In such lattices, competing interactions may not be simultaneously satisfied, giving rise to a large degeneracy of the system ground state and interesting frustrated dynamics such those exhibited by spin liquids[46] and spin ice.[47] Additional areas for new research with HSLs include the development of wave-coupled metamaterials[48] and spin waves such as those used for the development of spintronics[49] and modern magnonics.[50]

D. Anderson localization

A growing number of experiments[4,6,20,51] have demonstrated that pilot-wave hydrodynamics is also viable in relatively shallow liquid layers, wherein the lower boundary influences the walking droplet's dynamics without entirely suppressing the guiding wave field.[6] Notably, variations in bottom topography lead to spatial gradients in memory that may be harnessed to subject walkers to spatial potentials.[4,6,52] A walker is, thus, repelled from pillars[53] and drawn toward submerged wells,[4] the modeling of which requires more sophisticated pilot-wave models[37,54] to capture the effect of the submerged topography on the pilot-wave.

Abraham *et al.*[21] have recently exploited variable bottom topography to realize a hydrodynamic analog of Anderson localization.[56,57] The fundamental problem underlying Anderson localization is the transport of an electron in a metal with random impurities, which may be modeled through a random potential, $U(\mathbf{x})$, in the limit when the particle's kinetic energy, K, is much larger than the characteristic energy of the background potential, $U_0 \ll K$. A classical particle evolving under such conditions is able to move easily through the disordered landscape, however, its trajectory is gradually deflected by small random forces. After a sufficiently long period of time, the particle will, thus, exhibit zigzag motion, effectively evolving diffusively in two or higher dimensions[58] (ballistic in one dimension). The motion of a quantum particle in the same settings is fundamentally different and particularly counter-intuitive; an electron will effectively come to a halt, or "localize," beyond a critical amount of disorder even though the particle's energy is large relative to that of the underlying potential.[57,59] The inability of classical particle systems to exhibit localization similar to that of quantum particles has, thus, been regarded as another fundamental limit of classical mechanics. Since Anderson's work,[56] we have known that the reason for localization is rooted in the wave-like nature of quantum particles; specifically, the disordered potential causes the quantum states given by Schrödinger's equation to be exponentially localized in space. Subsequent studies have proven noise-induced localization to be a generic wave effect that may be observed in various systems, including Bose–Einstein condensates, microwaves, light waves, and ultrasound.[60] However, one should recall that, while these more recent examples have been realized with purely classical waves, the original quantum localization is fundamentally different as it includes the notion of a particle.

To demonstrate that pilot-wave hydrodynamics may likewise exhibit a *dual* particle-wave localization analogous to that of quantum particles, Abraham *et al.*[21] considered the erratic motion of a walker above submerged random topography [Fig. 4(a)]. The walker is subject to a random potential through the influence of the bottom topography of the bath, which is composed of square tiles, each with a random height drawn from a uniform distribution. Combining experiments, simulations, and theory, Abraham *et al.* demonstrate that after a sufficiently long period of time, the erratic walker motion at high memory gives rise to coherent statistics wherein the histogram of the droplet position features a prominent localization region [Fig. 4(b)]. Notably, the authors examined the first eigenmode of Schrödinger's equation with a potential of the same form as the bottom topography and

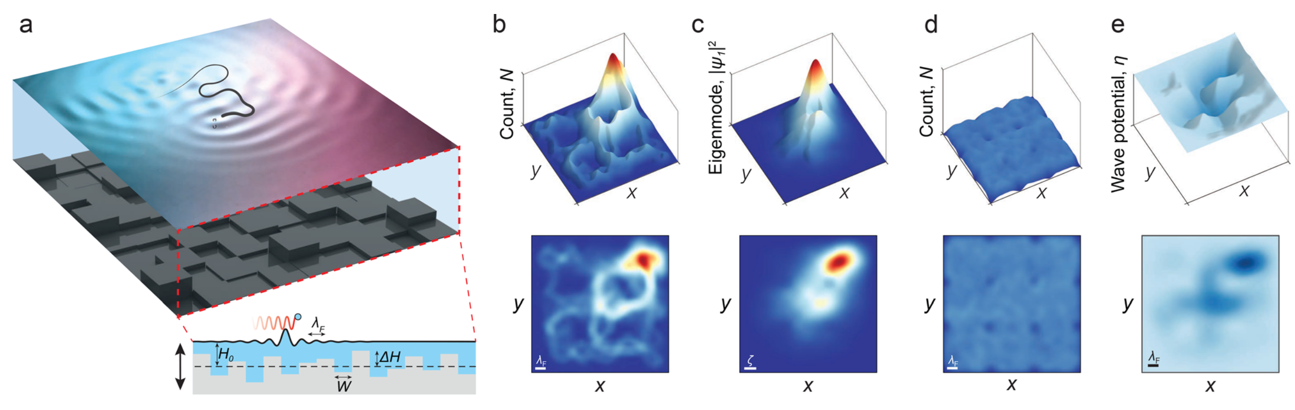

FIG. 4. Anderson localization of walking droplets. (a) A walking droplet may move erratically along the surface of fluid bath due to the influence of submerged bottom topography, which acts as a disordered potential. In the study by Abraham *et al.*,[21] the bath bottom profile is composed of square tiles of width W, each with a random height $\pm\Delta H$ about a base depth H_0 drawn from a uniform distribution. (b) Once the system reaches a statistically steady state, the histogram of the walker position exhibits a localization region, which bears a strong resemblance to the first eigenmode of Schrödinger's equation for the same potential in the weak regime (c). In contrast, (d) the position histogram of a classical particle without the wave field in the same high kinetic energy regime is uniform, indicating a homogeneous exploration of the random domain. (e) Effective mean-wave potential about the localization region. Reprinted with permission.

observed striking similarities [Fig. 4(c)]. To highlight the peculiar behavior of walking droplets, Abraham *et al.*[21] also considered the motion of a classical particle without an accompanying wave field in the same potential to demonstrate its diffusive motion, leading to a relatively homogeneous position histogram [Fig. 4(d)]. Numerical simulations further demonstrated the walker localization in domains much larger than those accessible in the laboratory. To rationalize the walker localization, Abraham *et al.*[21] investigated the mean wave field, $\bar{\eta}(\mathbf{x})$ where they noted a large-scale envelope acting as an effective potential about the localization region [Fig. 4(e)], which may be computed by convolving the position histogram with the long-wave modes with the slowest decay rates excited during the impacts. This new HQA demonstrates that Anderson localization, widely believed to be purely a wave phenomenon, also arises for classical particles propelled by self-excited waves, motivating its further investigation in pursuit of connections to landscape theory.[60]

E. Surreal trajectories

Pilot-wave hydrodynamics furnishes a physical picture that would provide rationale for several notable quantum oddities. One such example is the notion of "surreal trajectories" put forth by Englert, Scully, Süssman, and Walther (ESSW).[55] The authors proposed an interference experiment intended to expose the shortcomings of the predictions of Bohmian mechanics. The arrangement considered by ESSW is depicted in Fig. 5(a). A single particle is directed toward a beam splitter *B*, where, according to the Copenhagen interpretation, it is split into an equal superposition of packets ψ_1 and ψ_2 that propagate, respectively, toward the mirrors M_1 and M_2. After being reflected, the packets will recombine at the interference region *I* before eventually reaching the detectors D_1 and D_2, at which point the particle will be detected at D_1 or D_2 with equal probability. However, a Bohmian particle will travel either left or right, with equal probability. If, for example, the particle goes to the right, it will be reflected by M_2 and will continue toward the interference region *I*. ESSW showed in their analysis that the Bohmian particle, instead of proceeding toward D_1, changes course in region *I* owing to the influence of the quantum potential, and so proceeds toward D_2. The resulting trajectories, thus, never cross the centerline of the interferometer [the dashed vertical line in Fig. 5(a)]. ESSW concluded that "the Bohm trajectory is here macroscopically at variance with the actual, that is: observed, track. Tersely: Bohm trajectories are not realistic, they are surrealistic."[55] Despite surreal trajectories being confirmed experimentally by Mahler *et al.*[61] using weak measurements, ESSW declared "the Bohmian picture to be at variance with common sense."[62] Notably, Mahler *et al.* invoked nonlocality to rationalize their experimental results, stating that "the trajectories seem surreal only if one ignores their manifest nonlocality."[61]

Recently, Frumkin *et al.*[22] demonstrated a variant of the thought experiment proposed by ESSW, in the hydrodynamic pilot-wave system (Fig. 5). They constructed a hydrodynamic interferometer consisting of a submerged rhombus that forces a droplet with an initial random orientation, toward one of two submerged barriers with equal probability. The rhombus, thus, acted as a beam splitter, and the submerged barriers as reflectors. In the case of a symmetric setup [Fig. 5(b)], specifically when both barriers are present, the droplet passed the beam splitter and was reflected away from the adjacent barrier. However, once it approached the centerline, it changed direction again and so followed a "surreal" trajectory. The red curves in Fig. 5(b) depict 20 such trajectories, obtained in a continuous experiment with a single droplet. In the second experiment [Fig. 5(c)], one of the barriers was removed, resulting in an asymmetric configuration. In this case, after the droplet was reflected from the barrier, it maintained its course, and surreal trajectories were suppressed, as is the case in Bohmian mechanics when mirror *M*1 is removed.

The origin of surreal trajectories in pilot-wave hydrodynamics is that the spatially extended pilot wave interacts with the system's topography so as to ensure that droplet never crosses the system's centerline and instead follows a surreal path. Figure 5(d) shows the mean wave field that results from numerically calculating the weighted average of the pilot wave forms arising over all droplet trajectories for a symmetric setup. The nature of a specific droplet trajectory is determined by its initial condition and the topography of the experimental setup. However, consideration of an ensemble of initial conditions would produce mean trajectories guided by the mean wave field. The mean wave field should, thus, act as a potential that is effectively statistical in that it guides the mean trajectories, just as the quantum potential in

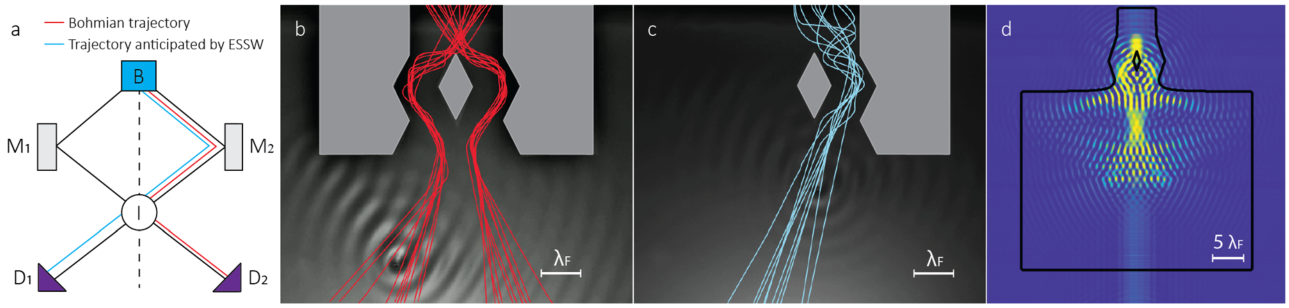

FIG. 5. Surreal trajectories in pilot-wave hydrodynamics. (a) A variant of the interferometer setup considered by ESSW.[55] An incoming wave packet is split by a beam splitter *B* and reflected by the mirrors M_1 and M_2. The wave packets interfere in the region *I* and then move toward the detectors D_1 and D_2. The blue path represents the particle trajectory anticipated by ESSW, while the red path is that predicted by Bohmian mechanics. (b) In a symmetric setup, the droplet enters the right or left channel with equal probability, after which it is deflected away from the system centerline, resulting in a "surreal" trajectory. (c) When one of the barriers is removed, the symmetry of the system is broken. The walking droplet is then reflected away from the remaining barrier, resulting in the trajectory that one might expect. (d) The mean wave-field generated by averaging simulated droplet trajectories with a Gaussian distribution of initial impact parameters, with a standard deviation of $1.4\lambda_F$. Reprinted with permission from Frumkin *et al.*, Phys. Rev. A **106**, L010203 (2022).

Bohmian mechanics guides the quantum velocity of probability. Notably, in contrast to Bohmian mechanics, where nonlocality is required to rationalize surreal trajectories, in pilot-wave hydrodynamics, such trajectories result from a wave potential generated locally by the droplet.

F. The quantum bomb tester

Interaction-free measurement is a peculiarity of quantum mechanics that seemingly allows one to obtain information about the quantum state of an object without its being "disturbed" by the measurement process. The most famous example of this phenomenon is the so-called Elitzur–Vaidman bomb tester,[63] according to which, 25% of the time, a photon can detect a bomb present in an interferometer 50% of the time without directly interacting with it. A variant of this thought experiment is depicted in Fig. 6(a). A photon is emitted from a source S into a Mach–Zehnder interferometer with arms of equal length. At the first beam splitter B_1, the photon's wave function is split in two, with each part traveling along one arm of the interferometer, and then recombined at the second beam splitter B_2. At this point, the wave function returns to its original state, thus the photon will be detected at detector D_1 with probability 1. Now consider a highly sensitive bomb that is placed along path 1 in the interferometer, so that any photon encountering it will detonate it. In this case, 50% of the time the particle follows path 1 and detonates the bomb. Otherwise, the particle follows path 2, while path 1 remains blocked by the bomb, preventing interference between the two wave packets at B_2. As a result, the particle has a 50% chance of being detected at D_2, thus providing the experimenter information about the presence of a bomb positioned along a path that it never traveled. Overall, if the bomb is present in the interferometer 50% of the time, the experimenter has a 25% chance of detecting the particle at D_2 and so the bomb along path 1. Note that the experiment can be considered "interaction-free" only in the sense that the particle does not interact with the bomb: the wave form propagating along path 1 interacts with and is altered by the bomb without detonating it.

As in the case of surreal trajectories, the Elitzur–Vaidman bomb tester is a result of quantum particles exhibiting wave-like statistics, causing their detection probability to be governed by constructive and destructive interference effects. Building on the similarity between these two systems, Frumkin and Bush[23] recently demonstrated that such a detection of an object by a particle that interacts with the object only through its associated waveform may likewise be achieved in a hydrodynamic pilot-wave system. Their setup consisted of two configurations of a hydrodynamic interferometer: one symmetric [Fig. 6(b)], identical to that used to realize surreal trajectories,[22] and another asymmetric [Fig. 6(c)], with the upper part of the left reflector removed. The upper part of the left reflector effectively served as the "bomb," an object whose presence one seeks to detect without direct interaction. In the symmetric case [Fig. 6(b)], if the droplet took the channel which contained the "bomb" (left), it would effectively detonate it, and the droplet's trajectory was considered to be terminated. Conversely, if the droplet took the right channel, its pilot-wave would deflect it away from the centerline, along a surreal trajectory. Thus, if the "bomb" is present, the droplet would either detonate it, or be detected on the right side of the setup. In the absence of the "bomb" [Fig. 6(c)], the symmetry was broken, preventing the droplet from following a surreal trajectory. In this case, if the droplet took the right path, it was reflected, and continued in a straight line toward the left. If the droplet took the left path, no reflection occurred, and it again went to the left following a straight path. Thus, in the absence of the "bomb," the droplet was always detected on the left side of the setup. After many realizations of the experiment in which the bomb was present 50% of the time, the experimenter has a 25% chance of detecting the droplet on the right side. Such a detection indicated that the "bomb" was present in the left channel, even though the droplet took the right path and so never interacted with it directly.

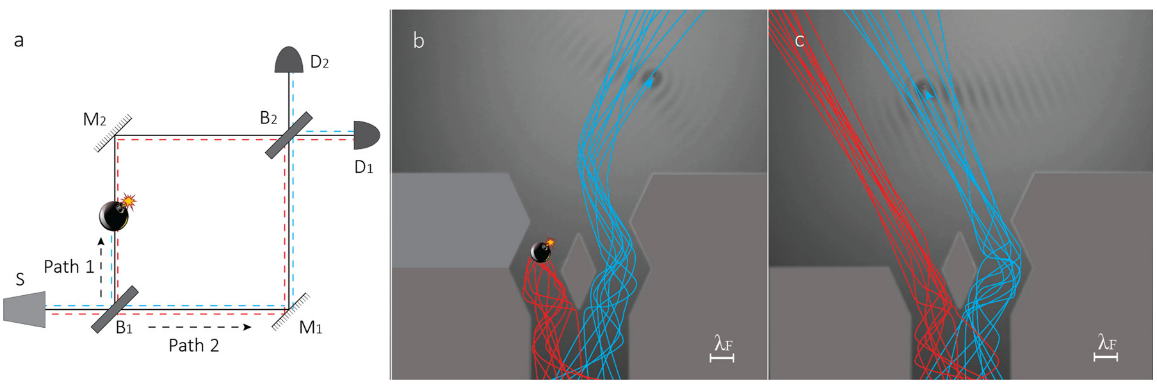

FIG. 6. Interaction-free measurement in pilot-wave hydrodynamics. (a) A schematic of the Elitzur–Vaidman bomb experiment. A particle emitted from a source S passes through a beam splitter B_1, at which point its associated wave is split in two. The wave is then recombined at a beam splitter B_2, and the particle continues toward the detectors. In the absence of a bomb, the particle will be detected at D_1 100% of the time, while in its presence, the particle will be detected at D_2 25% of the time. A detection event at D_2 indicates the presence of a live bomb along a path the particle never took. Red and blue dashed lines indicate possible paths taken by a particle emitted from S in the absence and presence of the bomb, respectively. (b) In a symmetric hydrodynamic setup, the droplet enters the right or left channel with equal probability. If the droplet goes to the left channel, it effectively detonates the "bomb," and if it goes to the right, it is deflected away from the system centerline, resulting in a "surreal" trajectory.[22] Thus, the droplet will always be detected on the right side of the setup. Twenty such trajectories are shown. (c) If the "bomb" is removed, the symmetry of the system is broken, the droplet's pilot-wave does not interact with the bomb, and "surreal" trajectories are suppressed. Thus, the droplet will always be detected on the left side of the setup. If the bomb is present 50% of the time, there is 25% chance of the droplet being detected on the right, and so the bomb on the left. The scale bar represents the Faraday wavelength $\lambda_F = 4.84$ mm. Reprinted with permission from Frumkin and Bush, Phys. Rev. A **108**, L060201 (2023).

Like a Bohmian particle, the droplet is localized at all times, while its pilot-wave is spatially extended and, thus, influenced by the geometry of its environment. The similarity of the statistical behavior between this hydrodynamic system and its quantum counterpart may again be rationalized in terms of the influence of the droplet's delocalized pilot-wave field. As was the case with surreal trajectories, the origin of the pilot wave in the particle vibration allows one to rationalize the phenomenon without appealing to nonlocal effects.

G. Pair–pair correlations

Nachbin[64,65] has examined the extent to which wave-mediated coupling can give rise to long-range correlations in pairs of bouncing droplets. Papatryfonos *et al.*[66] adapted the numerical methodology of Nachbin and co-workers,[67,68] in order to perform the first hydrodynamic Bell tests. Specifically, they assessed the correlations of two droplets, each confined to wave-coupled tunneling subsystems. In this analog system, the cavity location (inner or outer) played the role of the dichotomic variable (spin up or down) in the quantum Bell tests, and the system geometry the role of the measurement settings (polarizer angles). For specific combinations of the measurement settings, Papatryfonos *et al.*[66] reported violations of Bell's inequality, as may be rationalized in terms of the system memory.

Violations of Bell's inequality have also been reported in both classical electromagnetic[69–71] and acoustic wave systems.[72] However, the hydrodynamic pilot-wave system has yielded the first such violations with a spatially separated bipartite classical system. The result is comparable to the violation achieved in the "static" Bell tests of Aspect,[73,74] in which the analyzer settings were unaltered during the course of the experiment. In the hydrodynamic pilot-wave system, the observed violations may be rationalized in terms of the wave-mediated communication between the two subsystems. Eliminating communication between the two subsystems would require the implementation of tests with dynamic topography. Specifically, a wall could be imposed between the two subsystems during the course of the experiment, prior to the alteration of the measurement settings.[65] Successfully violating Bell's inequalities with such dynamic tests would rely on the violations achieved with the static test surviving the isolation of the two subsystems. However unlikely, such a result is not entirely inconceivable given that the bath serves as the memory of the system, and so stores information concerning the initial interaction between the droplet pair, even after isolation of the two subsystems.

IV. DISCUSSION

Purely wave-based hydrodynamic quantum analogs include the ripple-tank diffraction experiments of Young,[75] and Berry's analog of the Aharanov–Bohm effect.[76] Another such analog has recently provided an elegant means of connecting the dynamics of surface wave packets to Bohmian mechanics.[77] The walking-droplet system has provided the first class of particle-based hydrodynamic quantum analogs. As such, it has furnished a platform for exploring the boundary between classical and quantum effects; moreover, it has inspired investigations of other macroscopic realizations of wave-particle duality in which oscillators interact with their suspending fluids. Le Gal *et al.*[78] investigated the motion of a neutrally buoyant object suspended in a stably stratified fluid. By driving the mass distribution within the object pneumatically, they excited vertical oscillations. When the object was excited at the Brunt–Vaisala frequency, the object attained resonance with its ambient wave field and so transitioned into a self-propagating state. It has been known for some time that acoustically forced bubbles may be induced to propel in an erratic fashion.[79] Baudoin and co-workers[80] has recently been exploring the possibility of bubbles self-propelling in response to the pressure field associated with their own vibration. Harris and co-workers[81] have investigated the "surfer-bot," a floating object with an onboard oscillator that excites capillary waves that propel the surfer-bot forward. Finally, Neufeld and co-workers[82] have demonstrated that when a canoeist is literally up a creek with no paddle, her best bet is "gunwale bobbing," standing on the gunwales of her canoe and bobbing up and down, then surfing on the resulting waveform.

The theory of quantum mechanics is undoubtedly nonlocal. HQA is questioning the extent to which quantum physics need be nonlocal. Quantum nonlocality would appear to be ubiquitous, manifest everywhere from wave-function collapse to statistical projection effects to slit diffraction, from surreal trajectories to the Elitzur–Vaidman bomb tester. HQA has made clear that all these phenomena, at least, can be understood from a local realist perspective. The experimental violation of Bell's inequality is generally taken as proof that quantum mechanics is nonlocal; thus, one can feel justified in not seeking a mechanistic explanation of quantum phenomena, in appealing to nonlocality as needed. Bohmian mechanics provides a viable realist, nonlocal theory of quantum dynamics. HQA suggests the possibility of going one step further in developing a local realist theory of quantum dynamics, in which the particle is the source of its own guiding wave. A number of such theories of quantum dynamics, informed by the walker system, are currently being developed and explored.[83–88] Such theories would follow the prescription of Einstein in providing a dynamical underpinning for the theory of quantum statistics, just as statistical mechanics underlies thermodynamics. For those insistent on local realism being a basic tenet of science, the physical picture imagined by de Broglie,[18] manifest in stochastic electrodynamics[89,90] and engendered in pilot-wave hydrodynamics is an appealing one.

A number of important questions remain to be addressed by the HQA community. What is the analog of the linear Schrödinger equation in the hydrodynamic system? This question motivated the recent study of Kutz *et al.*,[91] who used dynamic mode decomposition to characterize the evolution of the walker wave field. Their study demonstrated that, as the memory is increased progressively, new wave modes enter discretely until the wave dynamics becomes chaotic. Using the convolution result of Durey *et al.*,[36,37] they were then able to infer the statistical behavior of the droplet without ever measuring the particle position. Does the Faraday system have the potential to capture aspects of particle creation and annihilation? Frumkin *et al.*[92] took the first step in this direction by presenting a hydrodynamic analog of superradiance from atomic pairs, which involved the correlations of droplet ejection events from neighboring deep wells in the hydrodynamic system. Finally, the wave-particle nature of the walking droplets gives them unique properties that may be harnessed to process information in an unorthodox way. For example, information may be encoded in a variety of hydrodynamic bound states, processed through statistical interference, and read out directly via droplet position measurements. Thus, given that implementations of quantum algorithms do not rely on quantum nonlocality, only on quantum statistics, it is only natural to ask whether the hydrodynamic pilot-wave system may provide a new platform for quantum-inspired classical computing.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

John W. M. Bush: Conceptualization (equal); Funding acquisition (equal); Project administration (equal); Writing – original draft (equal); Writing – review & editing (equal). **Valeri Frumkin:** Conceptualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Pedro J. Sáenz:** Conceptualization (equal); Funding acquisition (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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