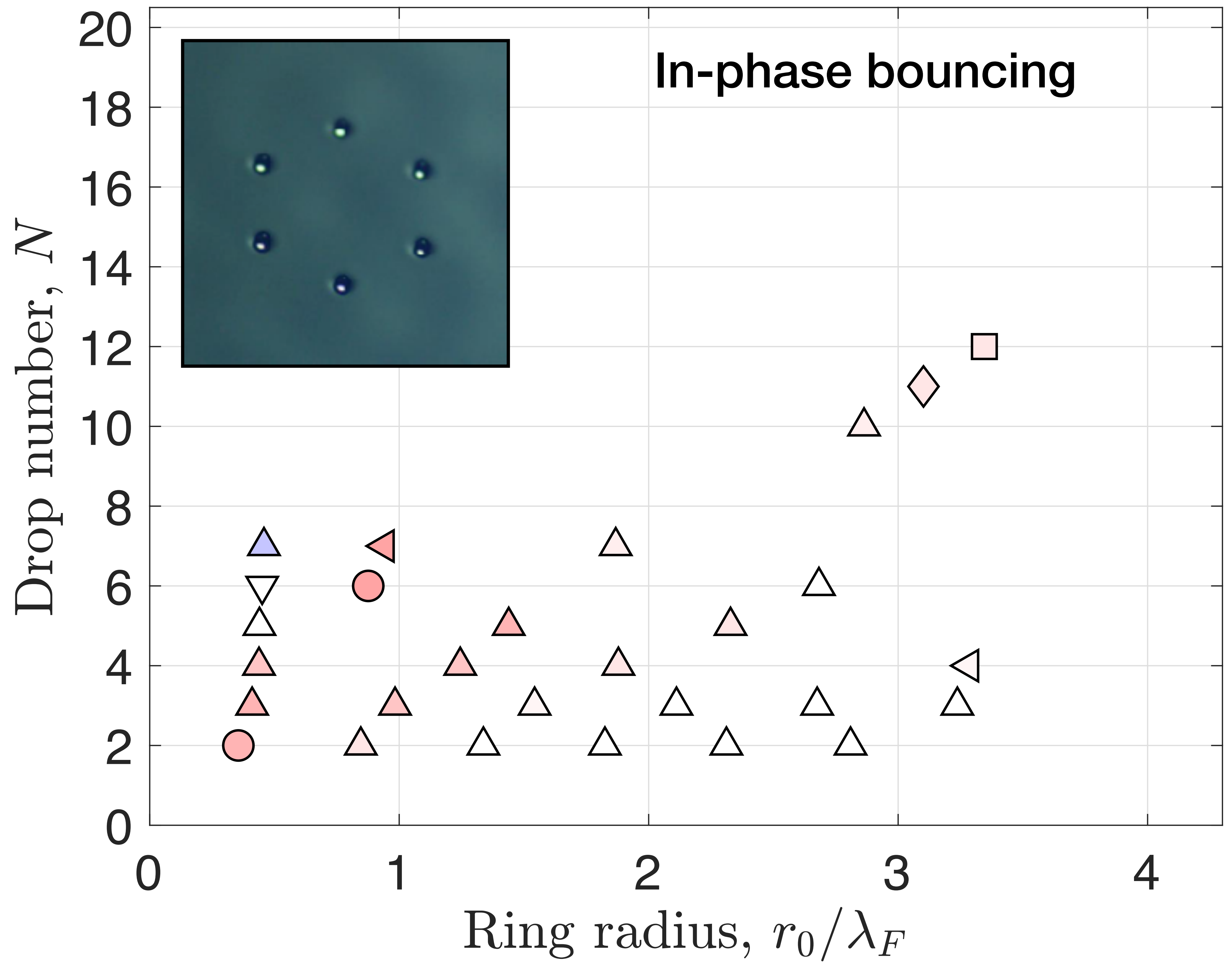
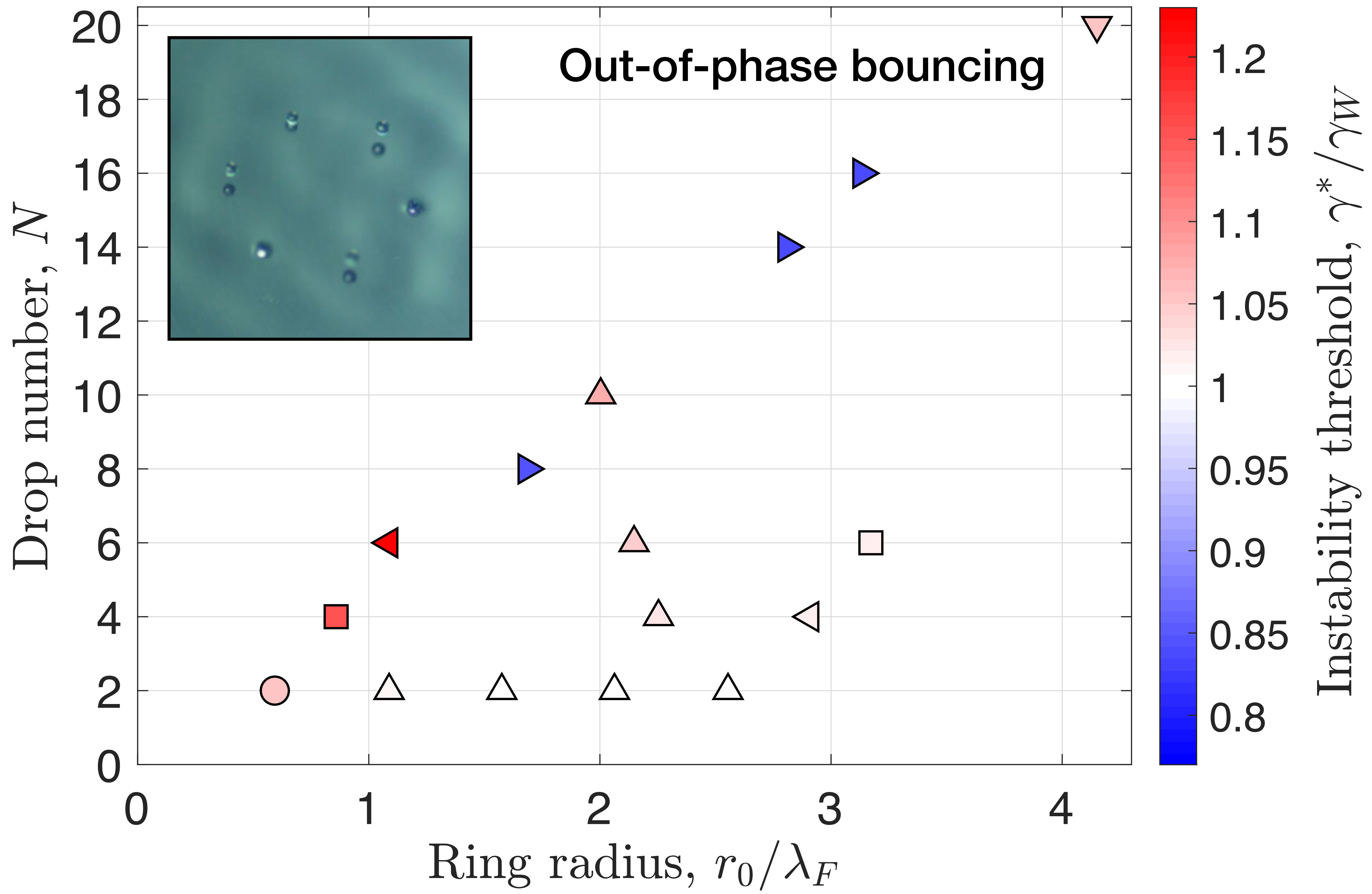
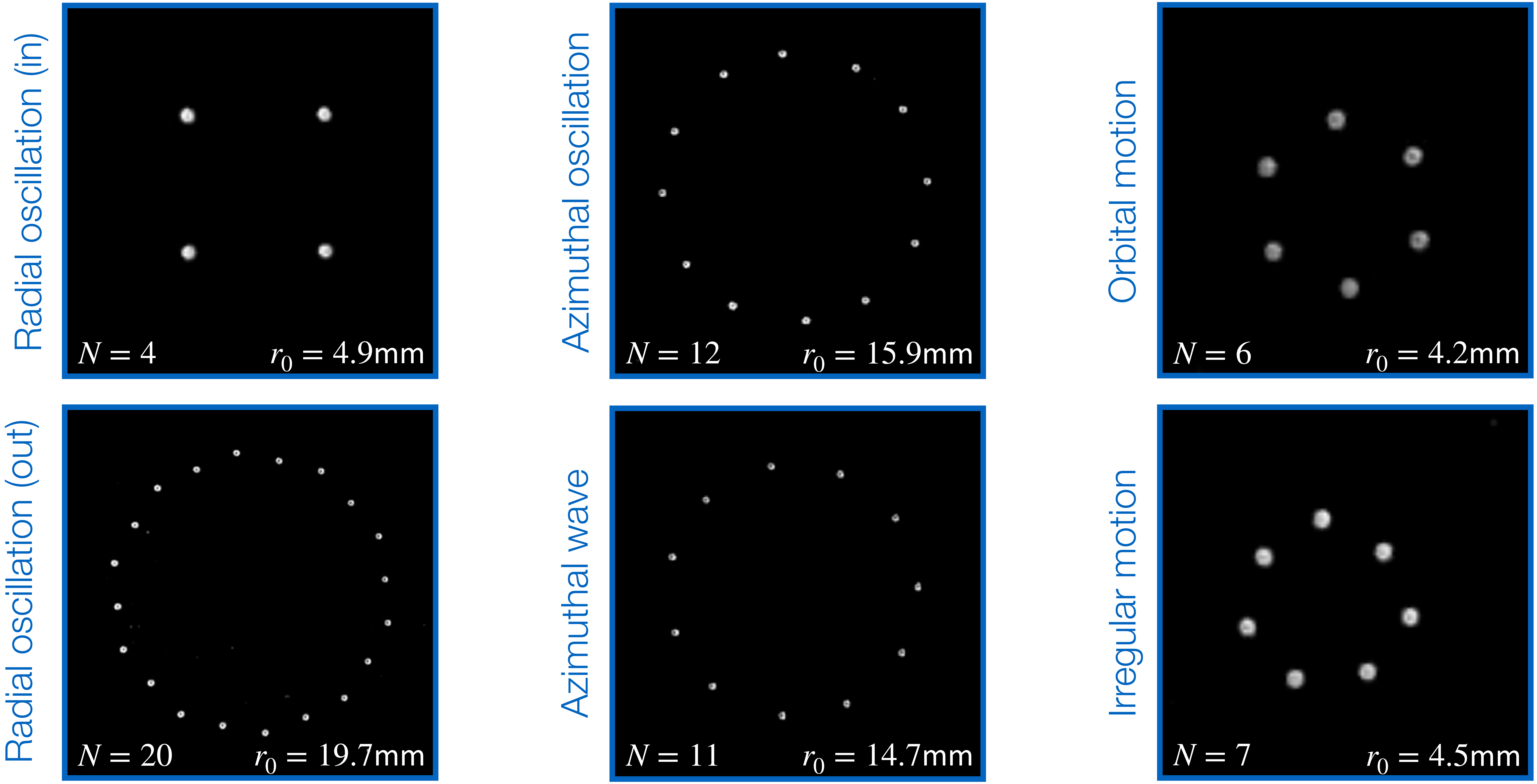
Discrete set of possible rings

- for given drop number, N, discrete set of possible radii, r_0
- each drop must bounce in a minimum of net wavefield produced by other drops



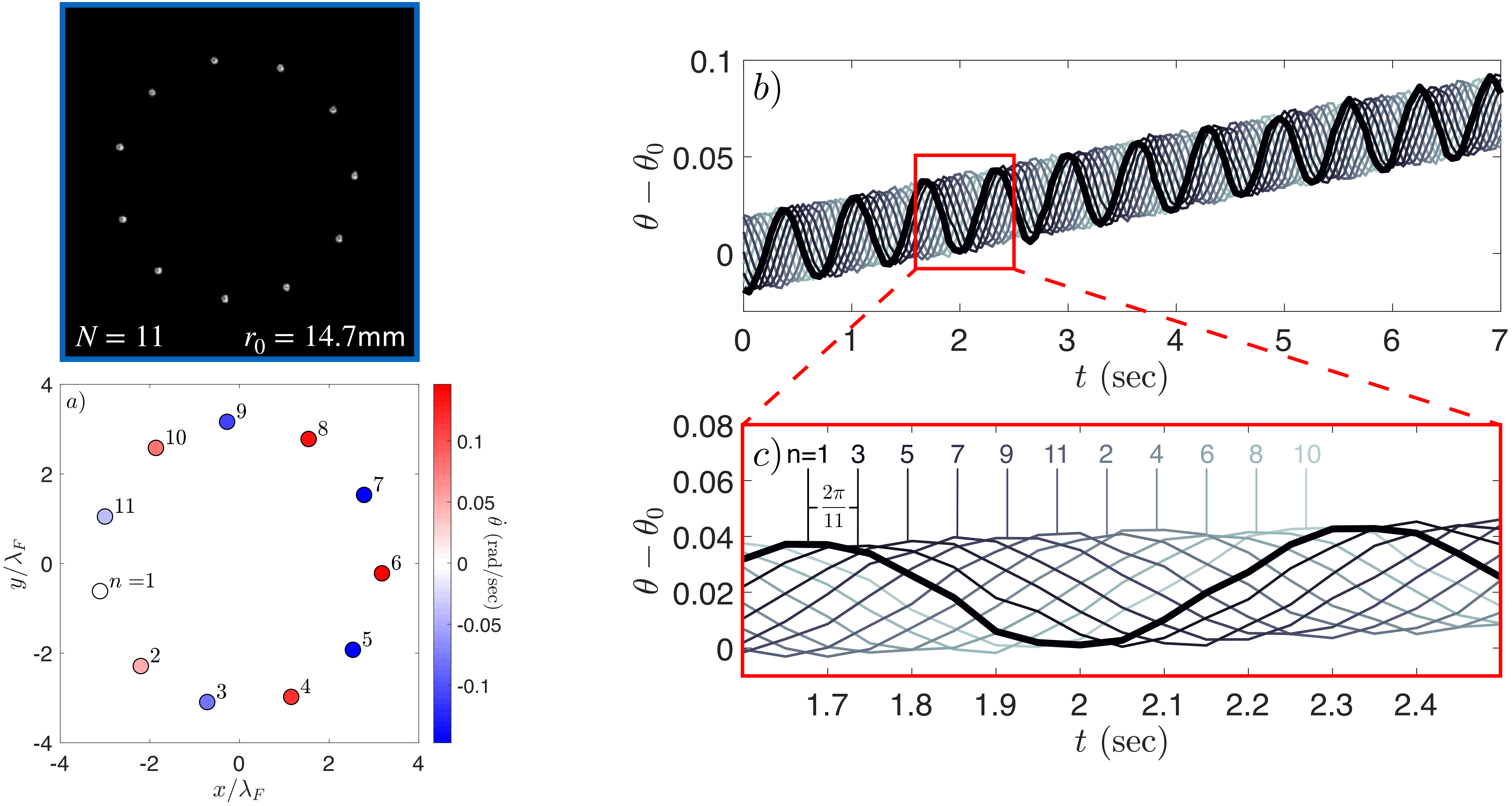


Types of instability



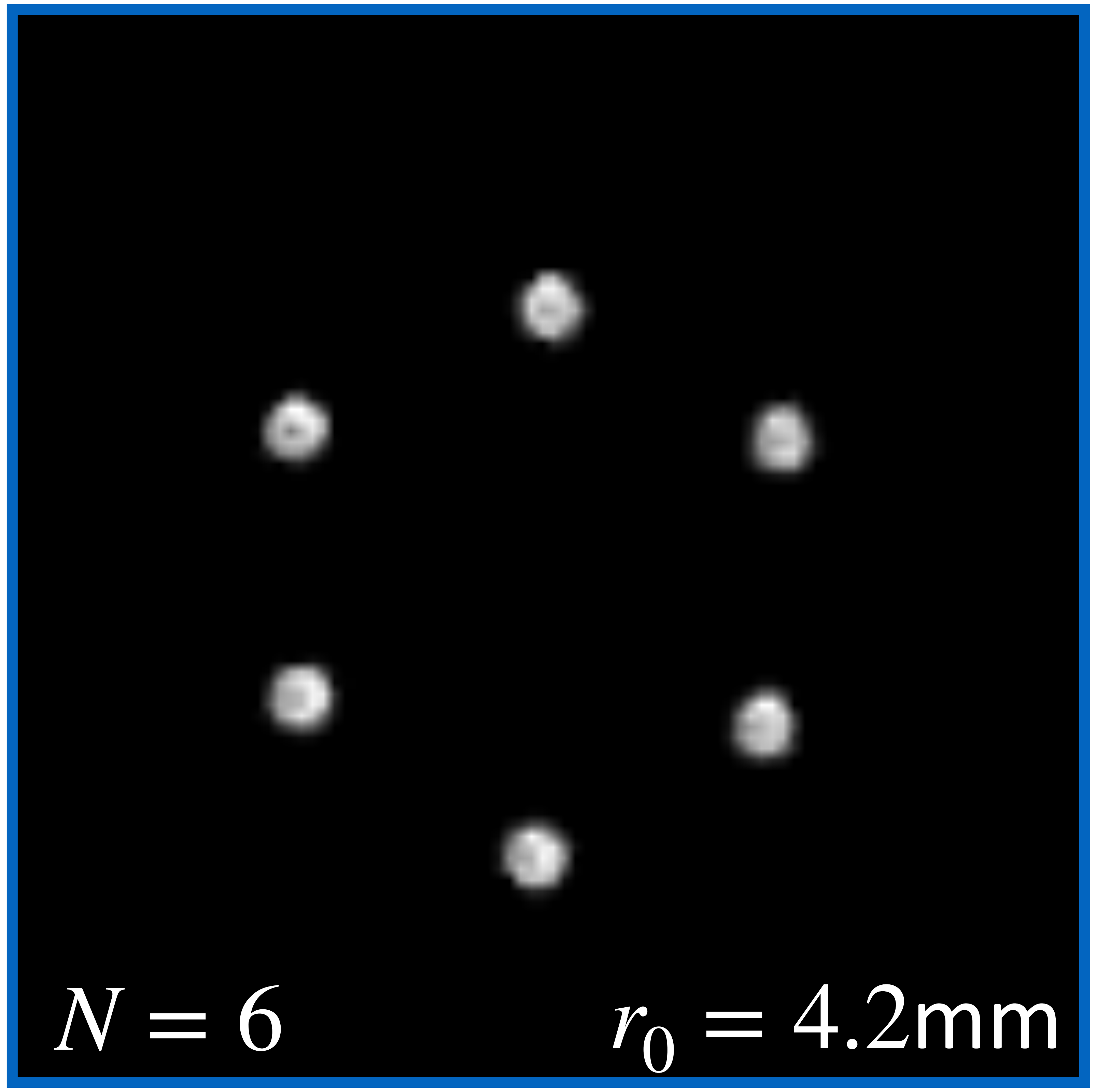
Free ring of 10 bouncing droplets: binary oscillations

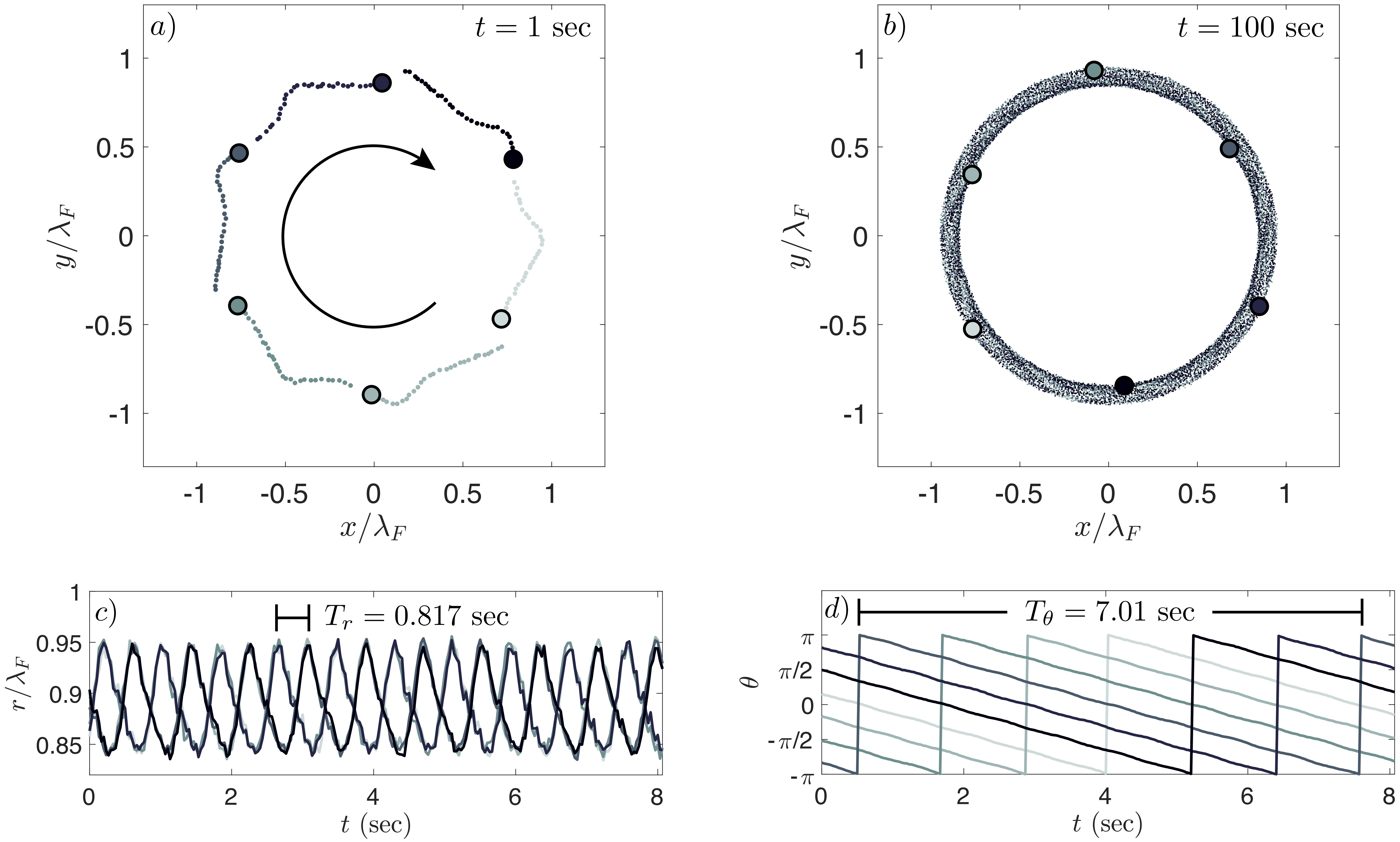
Azimuthal wave



Quasi-periodic motion

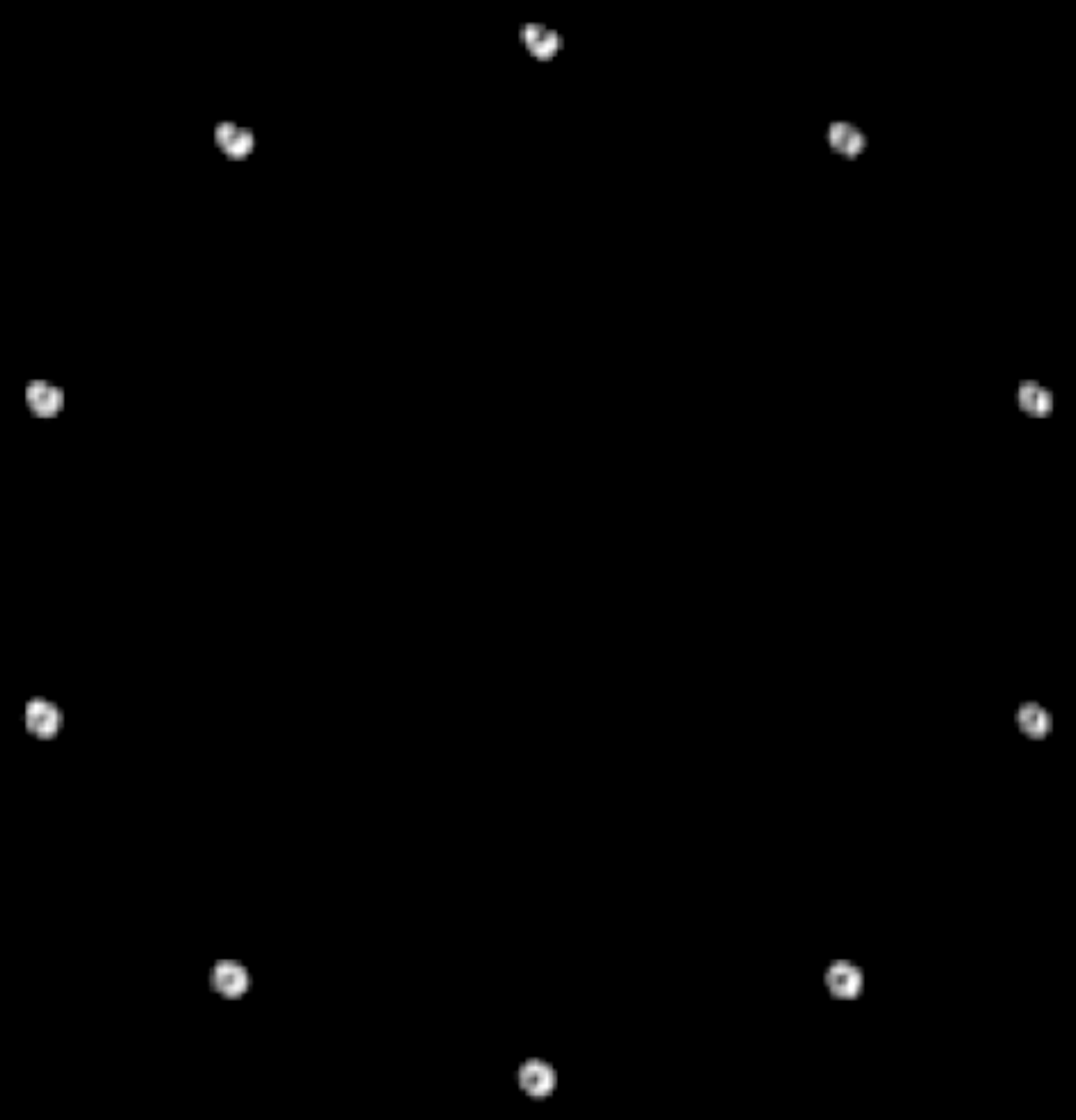
- as vibrational acceleration further increased beyond instability threshold, exotic states emerge



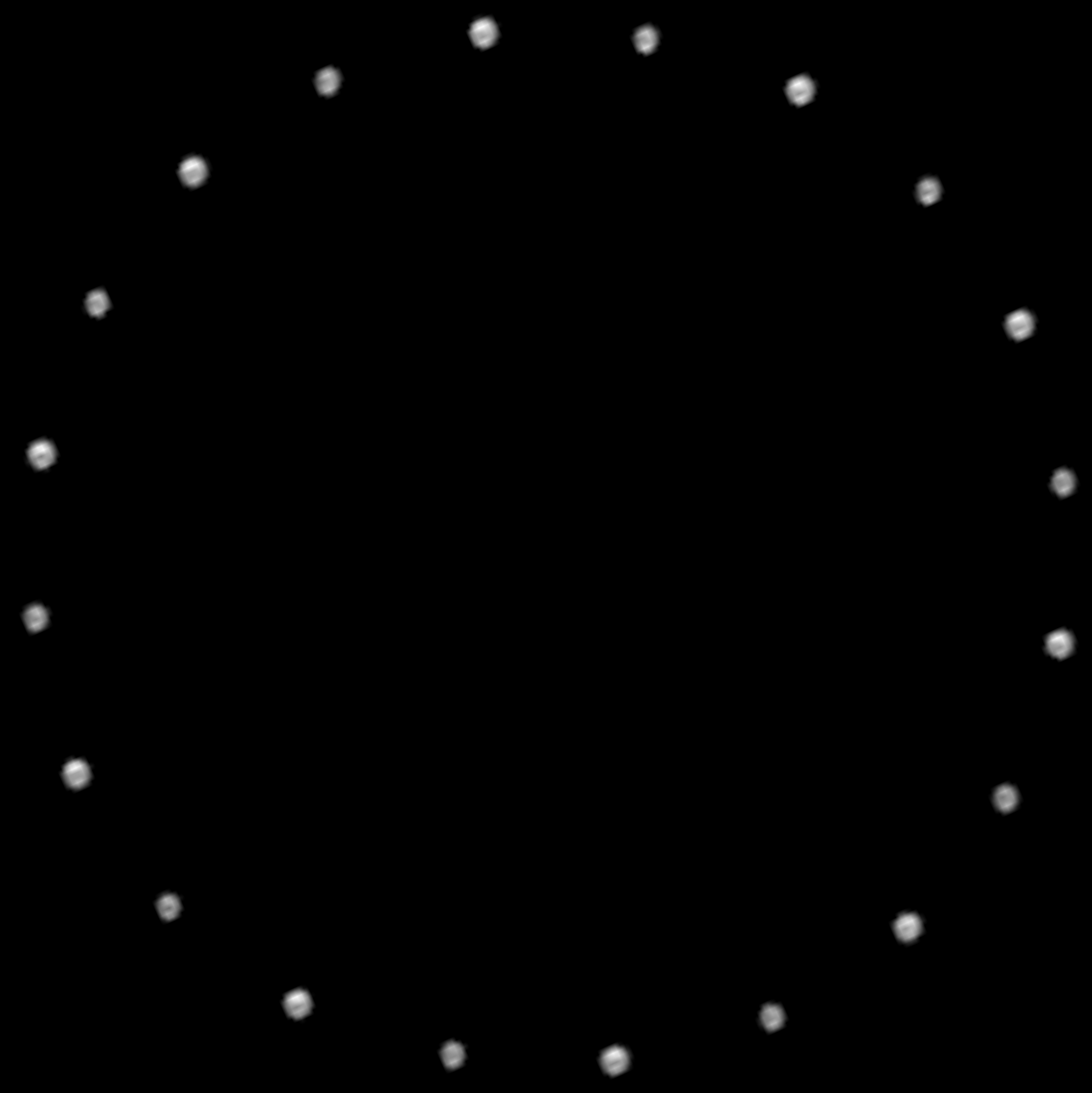


Exotic transitions

$N = 10,\ r_0 = 9.5\text{mm}$
In-phase



$N = 20,\ r_0 = 19.5\text{mm}$
Out-of-phase



Equations of Motion

- Stroboscopic, integro-differential equations describe motion of N coupled drops

  Molacek and Bush (2013), Oza *et al*. (2013), Turton *et al*. (2018), Couchman *et al*. (2019)

$$\kappa \ddot{\boldsymbol{x}}_m + \dot{\boldsymbol{x}}_m = -\beta \sigma_m \boxed{\mathscr{C}_m \left[h\left(\boldsymbol{x}_m, t\right), \gamma \right]} \nabla h\left(\boldsymbol{x}_m, t\right)$$

Trajectory equation for m^{th} drop

$$h(\boldsymbol{x}, t) = \sum_{n=1}^{N} \sigma_n \int_{-\infty}^{t} f\left(\left| \boldsymbol{x} - \boldsymbol{x}_n(s) \right| \right) e^{-(t-s)} ds$$

Wavefield strobed at bouncing period

$$f(r) = J_0(r) K_1\left(\xi r\right) \xi r$$

Wave kernel

- impact-phase parameter $\mathscr{C}$ accounts for variations in a drop's vertical dynamics due to changes in vibrational acceleration, γ, and local wave amplitude, $h\left(\boldsymbol{x}_m, t\right)$
- modulations in impact phase play critical role in drop-drop interactions

  (see Couchman *et al*., JFM, 2019)

Linear Stability Analysis For Ring of N drops

- Stroboscopic equations of motion in polar coordinates:

$$\kappa\begin{pmatrix} \ddot{r}_i - r_i\dot{\theta}_i^2 \\ r_i\ddot{\theta}_i + 2\dot{r}_i\dot{\theta}_i \end{pmatrix} + \begin{pmatrix} \dot{r}_i \\ r_i\dot{\theta}_i \end{pmatrix} = -\beta\mathscr{C}_i\sigma_i\sum_{j=1}^{N}\sigma_j\int_{-\infty}^{t}\mathcal{S}_j\frac{f'\left(d_{ij}\right)}{d_{ij}}\begin{pmatrix} r_i(t) - r_j(s)\cos\left(\theta_i(t) - \theta_j(s)\right) \\ r_j(s)\sin\left(\theta_i(t) - \theta_j(s)\right) \end{pmatrix} e^{-(t-s)}ds, \begin{pmatrix} \hat{r}_i \\ \hat{\theta}_i \end{pmatrix}$$

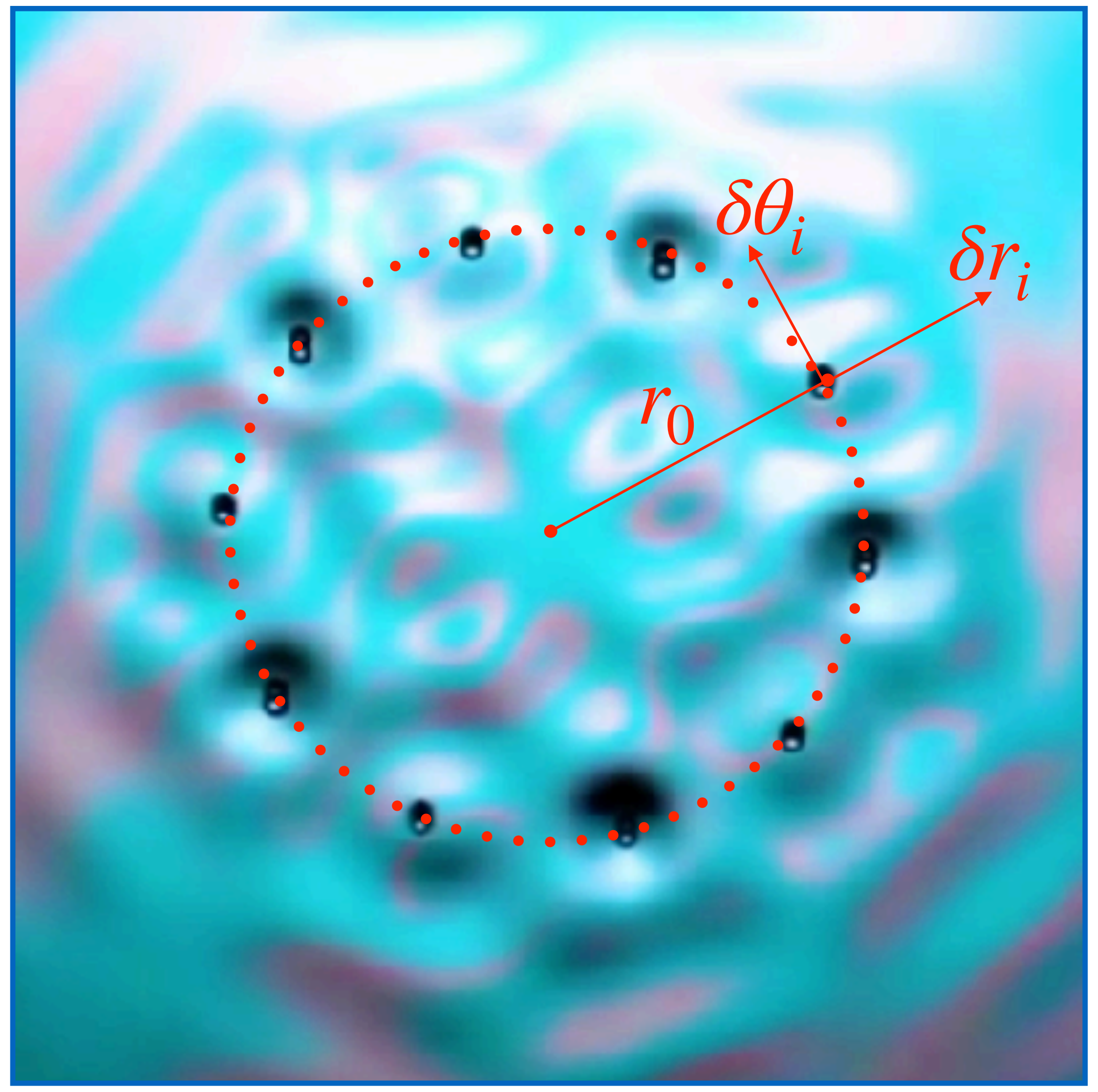
$$d_{ij} = \sqrt{r_i^2(t) + r_j^2(s) - 2r_i(t)\,r_j(s)\cos\left(\theta_i(t) - \theta_j(s)\right)}$$

- Consider arbitrary perturbations of each drop:

$$r_i(t) = r_0 + \epsilon\delta r_i(t)$$

$$\theta_i(t) = \frac{2\pi(i-1)}{N} + \epsilon\delta\theta_i(t), \quad i = 1,\ldots,N$$

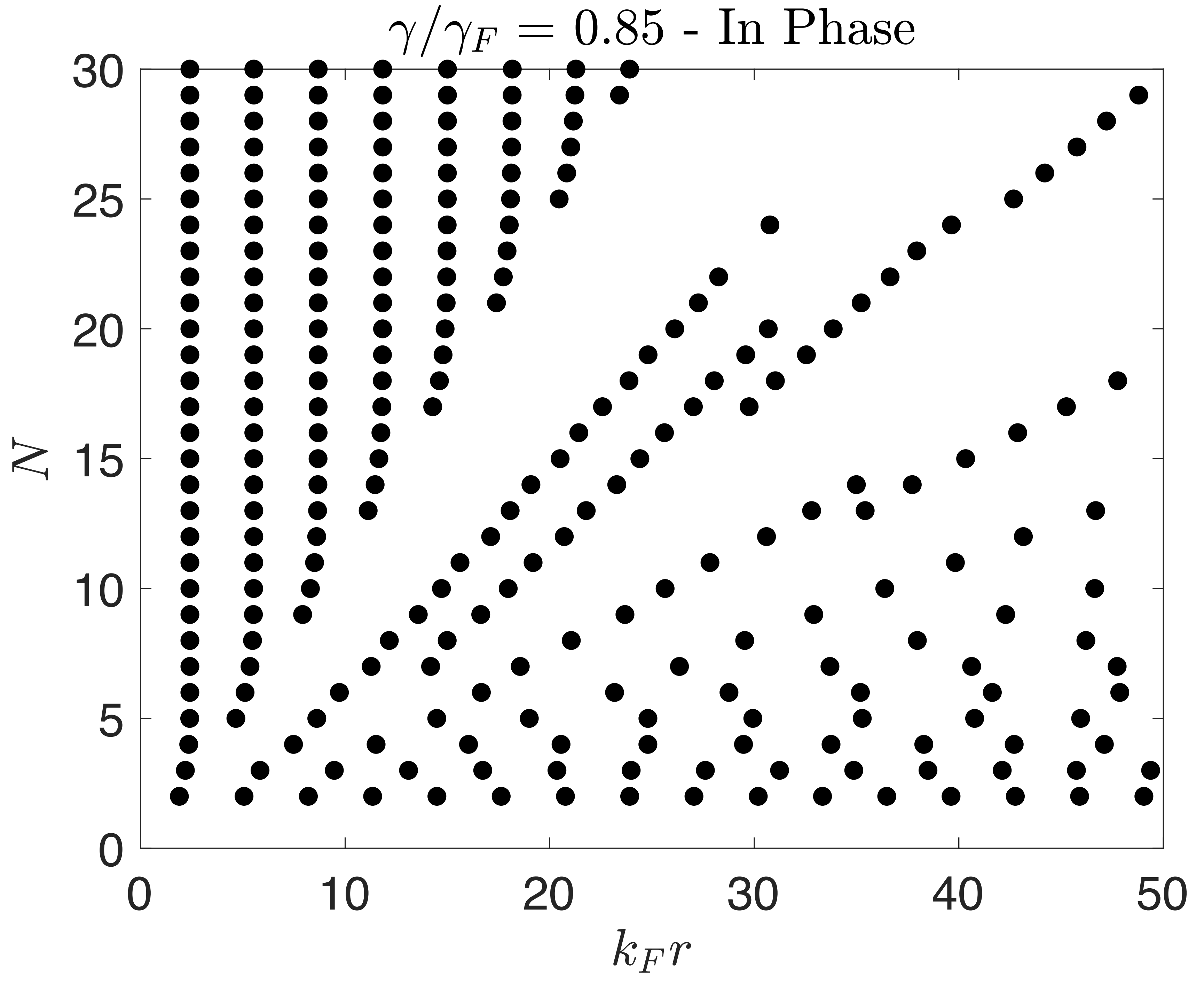
- Expand in orders of … ϵ

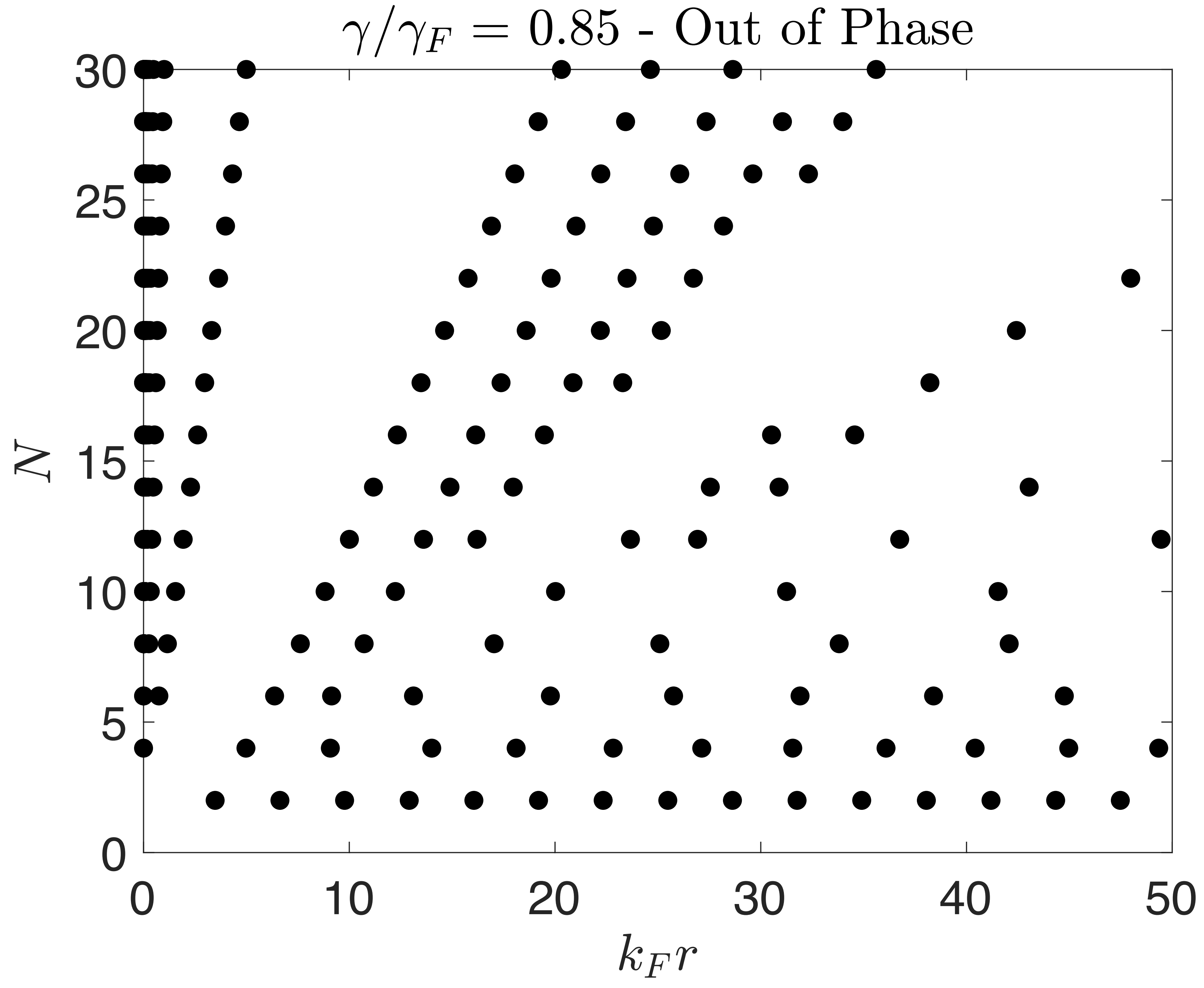


$\mathcal{O}(0)$: Stationary Configurations

$$0 = \sum_{n=0}^{N-1} \sigma_n f'\left(2r_0 \sin\left(\frac{\pi n}{N}\right)\right) \sin\left(\frac{\pi n}{N}\right)$$

- determines possible r_0 where ring will remain stationary
  - i.e. gradient beneath each drop is zero: $\nabla h(\vec{x}_i) = 0$



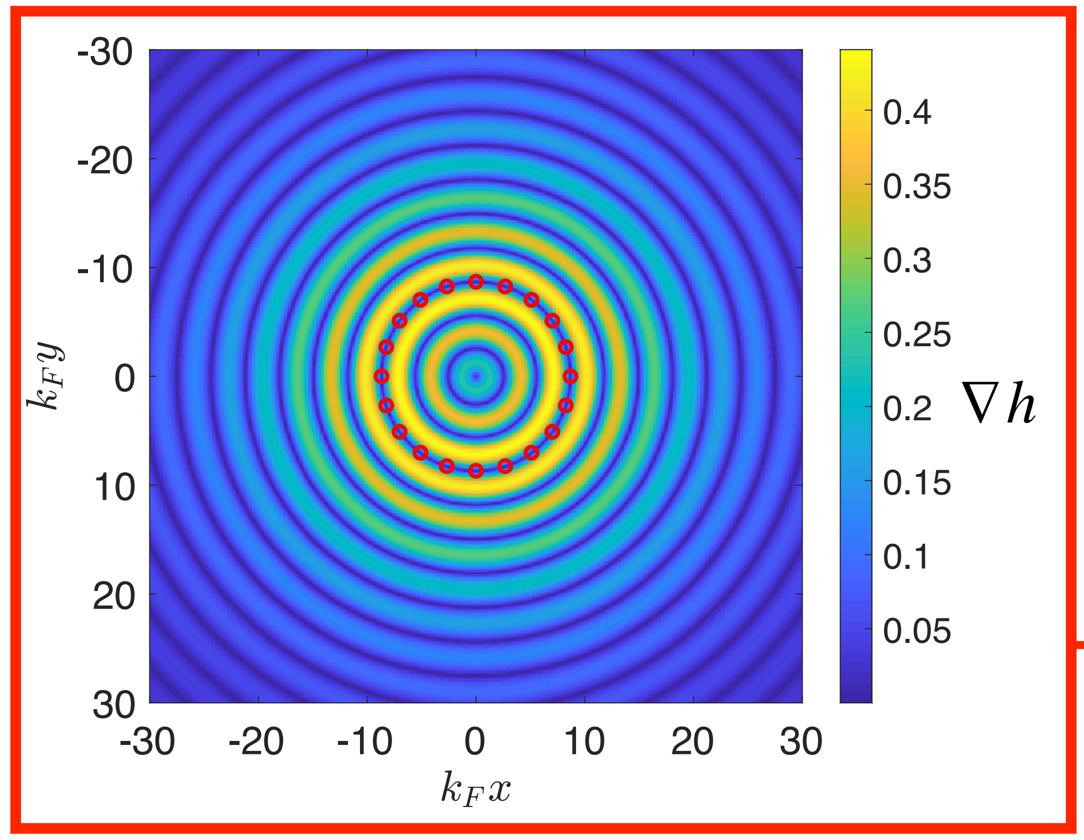


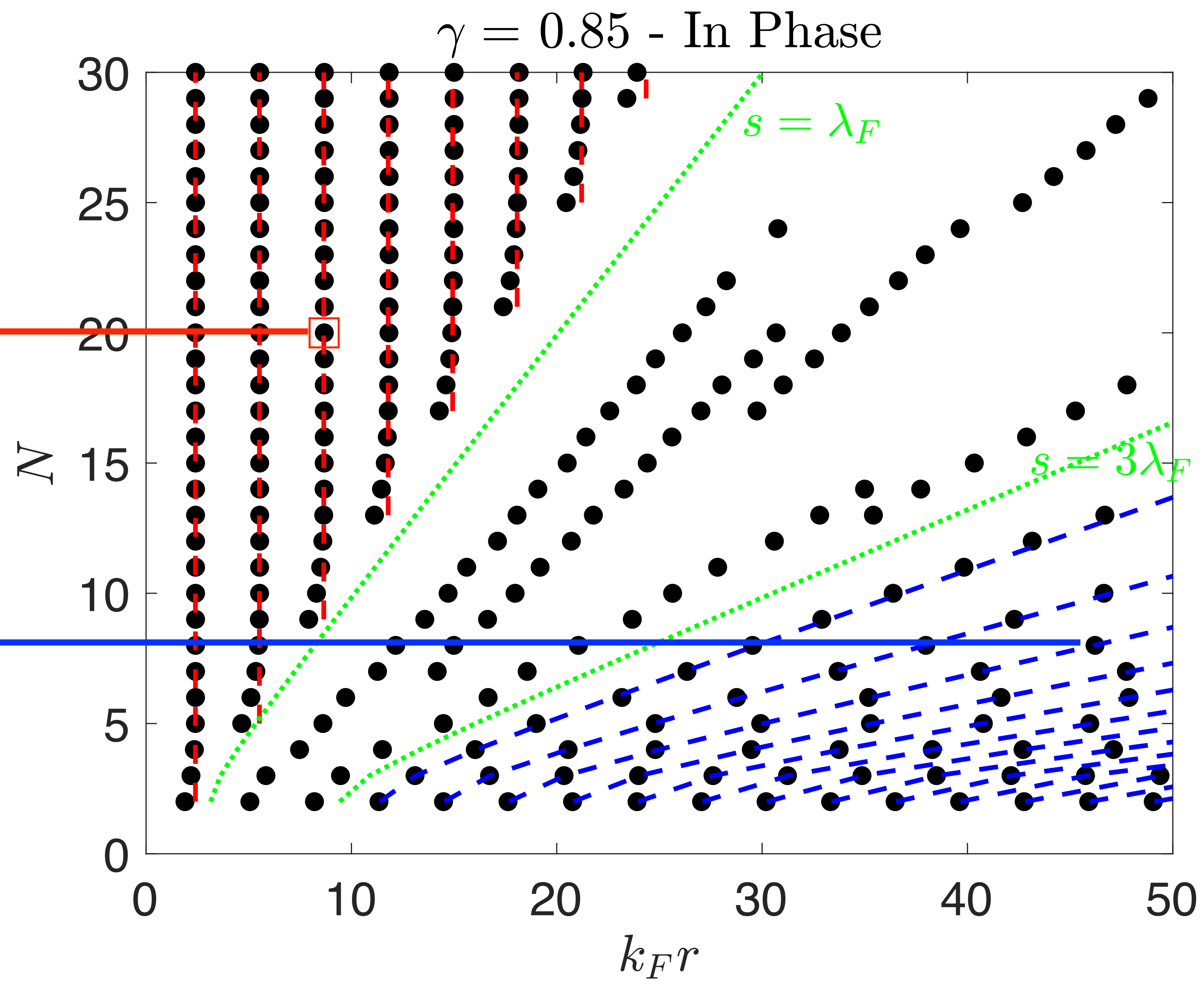
$\mathcal{O}(0)$: Stationary Configurations

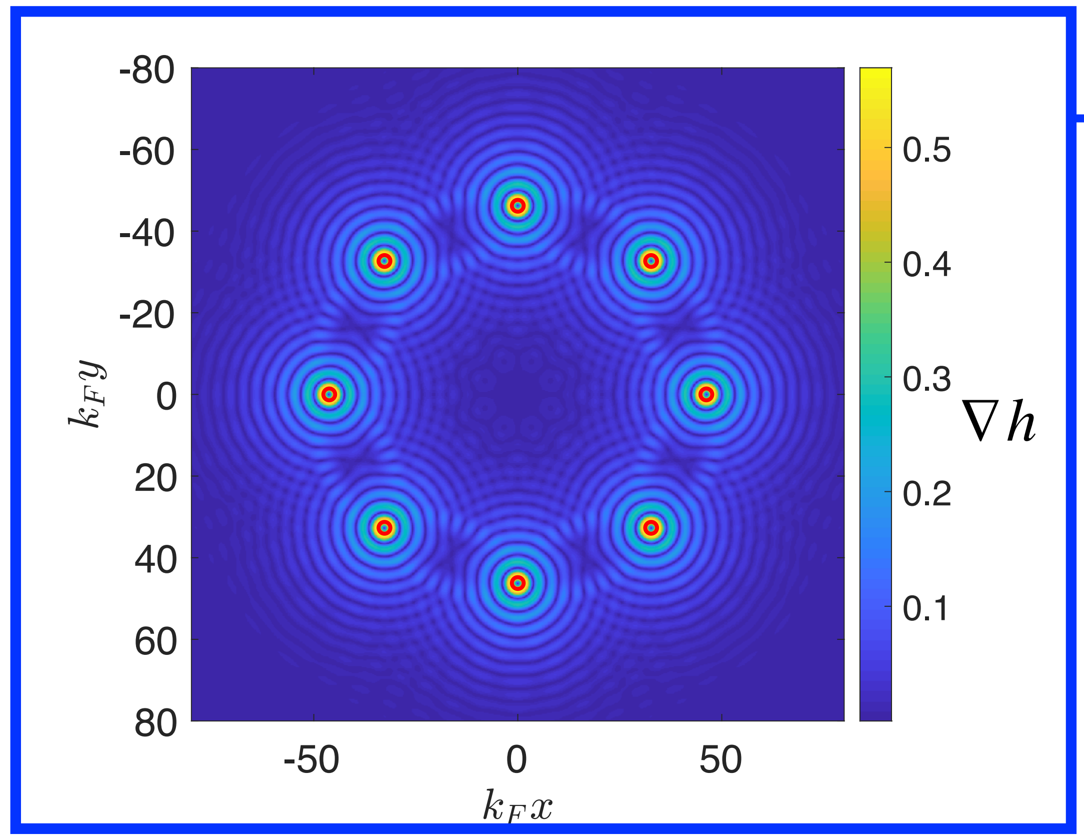
2 limiting cases (s = edge length of ring)

- approximately continuous source of Bessel functions in circle
- wavefield is proportional to Bessel function centered at origin
- r_0 determined by minima of $h(r) \propto J_0(k_F r_0) J_0(k_F r)$

$s < \lambda_F$: tightly packed ring







$s > \lambda_F$: loosely packed ring

- drops only interact with nearest neighbors
- s_0 determined by minima of $J_0(r)$
- $$r_0 = \frac{s_0}{2\sin(\pi/N)}$$

$\mathcal{O}(\epsilon)$: Instability Type and Threshold

$$\kappa\delta\ddot{r}_m + \delta\dot{r}_m = -\frac{\beta_{eff}}{2}\sum_{n=1}^{N}\sigma_{mn}\left[\left(M1_{mn} + \frac{M2_{mn}}{r_0}\right)\delta r_m + \left(M1_{mn} - \frac{M2_{mn}}{r_0}\right)\delta R_n + \left(r_0 M3_{mn}\right)\delta\theta_m + \left(-r_0 M3_{mn} - M4_{mn}\right)\delta\Phi_n\right]$$

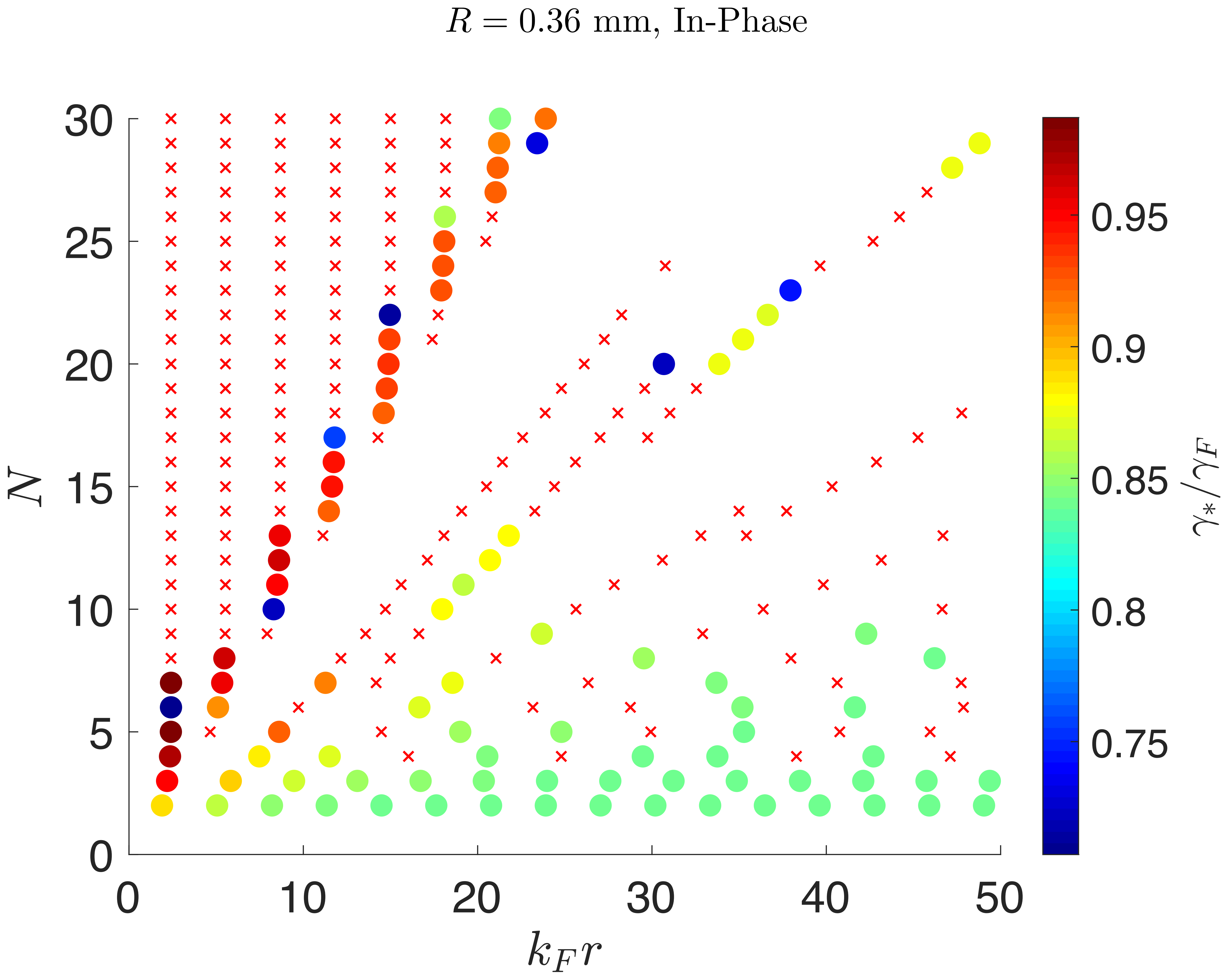
$$\kappa\delta\ddot{\theta}_m + \delta\dot{\theta}_m = -\frac{\beta_{eff}}{2r_0}\sum_{n=1}^{N}\sigma_{mn}\left[\left(M3_{mn}\right)\delta r_m + \left(M3_{mn} + \frac{M4_{mn}}{r_0}\right)\delta R_n + \left(r_0 M6_{mn}\right)\delta\theta_m + \left(-r_0 M6_{mn} + M5_{mn}\right)\delta\Phi_n\right]$$

σ_{mn}	$\sigma_m\sigma_n$
β_{eff}	$\frac{AM_e\beta}{R}\mathcal{S}_0\mathcal{C}_0$
$\delta\Phi_n$	$\int_{-\infty}^{t}\delta\theta_n(s)\,e^{-(t-s)}ds,\ \ \delta\dot{\Phi}_n = \delta\phi_n - \delta\Phi_n$
δR_n	$\int_{-\infty}^{t}\delta r_n(s)\,e^{-(t-s)}ds,\ \ \delta\dot{R}_n = \delta r_n - \delta R_n$
$M1_{mn}$	$f''\left(2r_0\left\|\sin\left(\frac{\pi(m-n)}{N}\right)\right\|\right)\left(1-\cos\left(\frac{2\pi(m-n)}{N}\right)\right)$
$M2_{mn}$	$\frac{f'\left(2r_0\left\|\sin\left(\frac{\pi(m-n)}{N}\right)\right\|\right)\cos^2\left(\frac{\pi(m-n)}{N}\right)}{\left\|\sin\left(\frac{\pi(m-n)}{N}\right)\right\|}$
$M3_{mn}$	$f''\left(2r_0\left\|\sin\left(\frac{\pi(m-n)}{N}\right)\right\|\right)\sin\left(\frac{2\pi(m-n)}{N}\right)$
$M4_{mn}$	$\mathrm{sign}\left(\sin\left(\frac{\pi(m-n)}{N}\right)\right)\cos\left(\frac{\pi(m-n)}{N}\right)f'\left(2r_0\left\|\sin\left(\frac{\pi(m-n)}{N}\right)\right\|\right)$
$M5_{mn}$	$\left\|\sin\left(\frac{\pi(m-n)}{N}\right)\right\| f'\left(2r_0\left\|\sin\left(\frac{\pi(m-n)}{N}\right)\right\|\right)$
$M6_{mn}$	$2\cos^2\left(\frac{\pi(m-n)}{N}\right)f''\left(2r_0\left\|\sin\left(\frac{\pi(m-n)}{N}\right)\right\|\right)$

$\mathcal{O}(\epsilon)$: Instability Type and Threshold

× : unstable at all γ

● : stable until γ_*



Summary: Linear stability analysis

- linearize trajectory equation by considering small perturbations away from base state:

$$r_m(t) = r_0 + \epsilon \delta r_m(t)$$

$$\theta_m(t) = 2\pi m/N + \epsilon \delta \theta_m(t)$$

- possible base states:

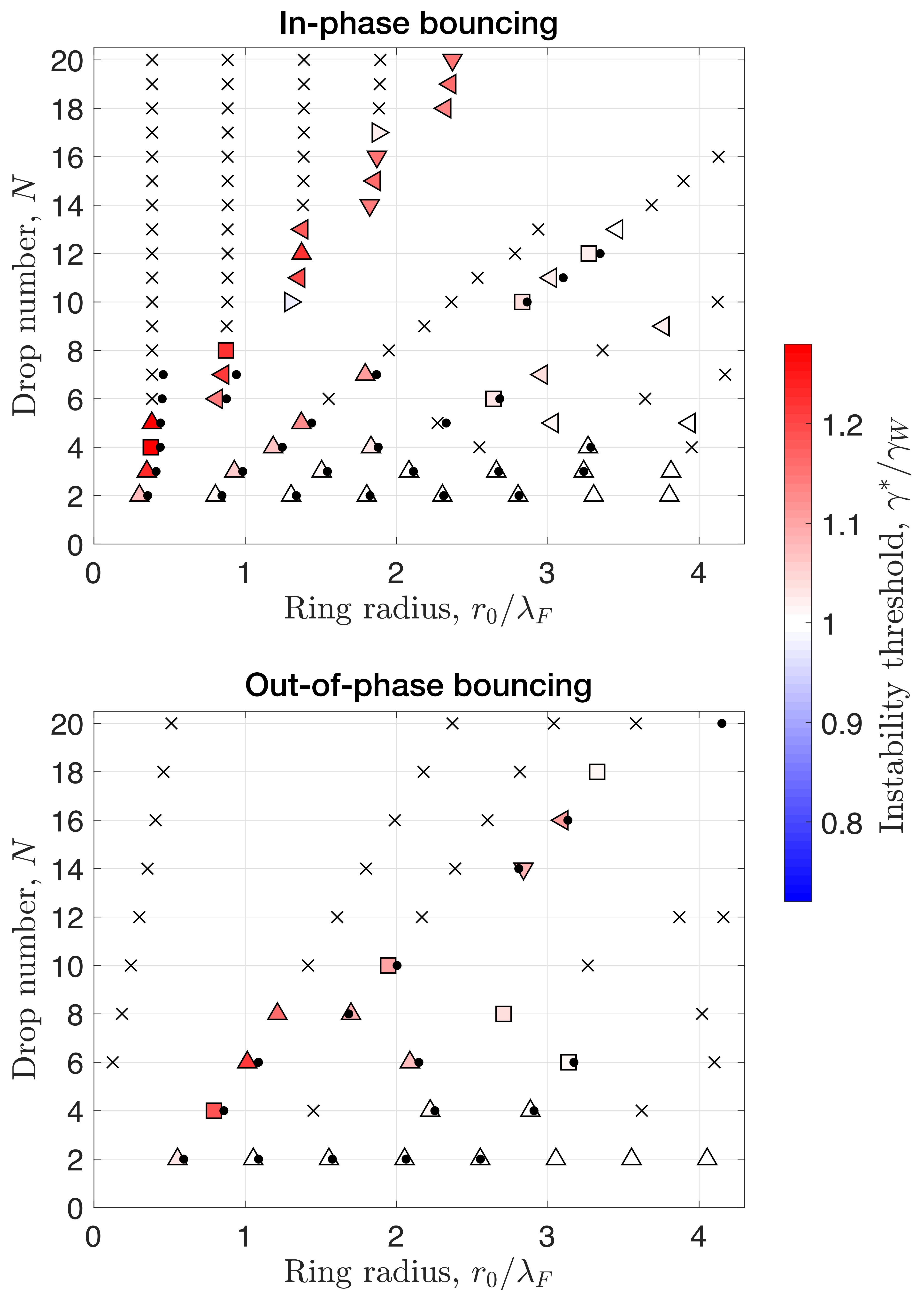
$$0 = \sum_{n=1}^{N} \zeta^n f'\left(2r_0 \sin\left(\pi n/N\right)\right) \sin\left(\pi n/N\right)$$

($\zeta = 1$: in-phase, $\zeta = -1$: out-of-phase)

- 6N-dimensional linear system determines stability of each state

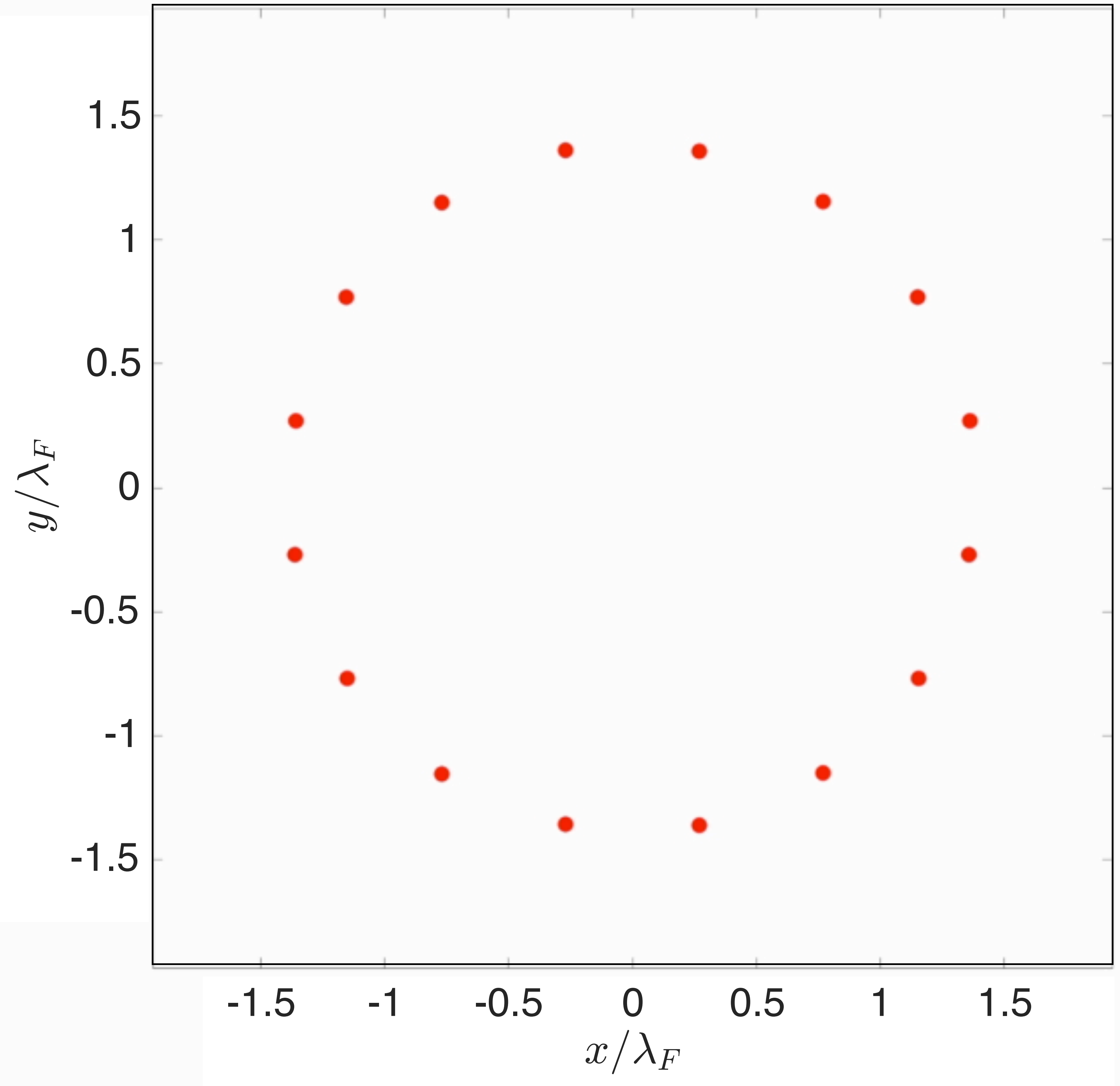
× Theoretically unstable at all γ

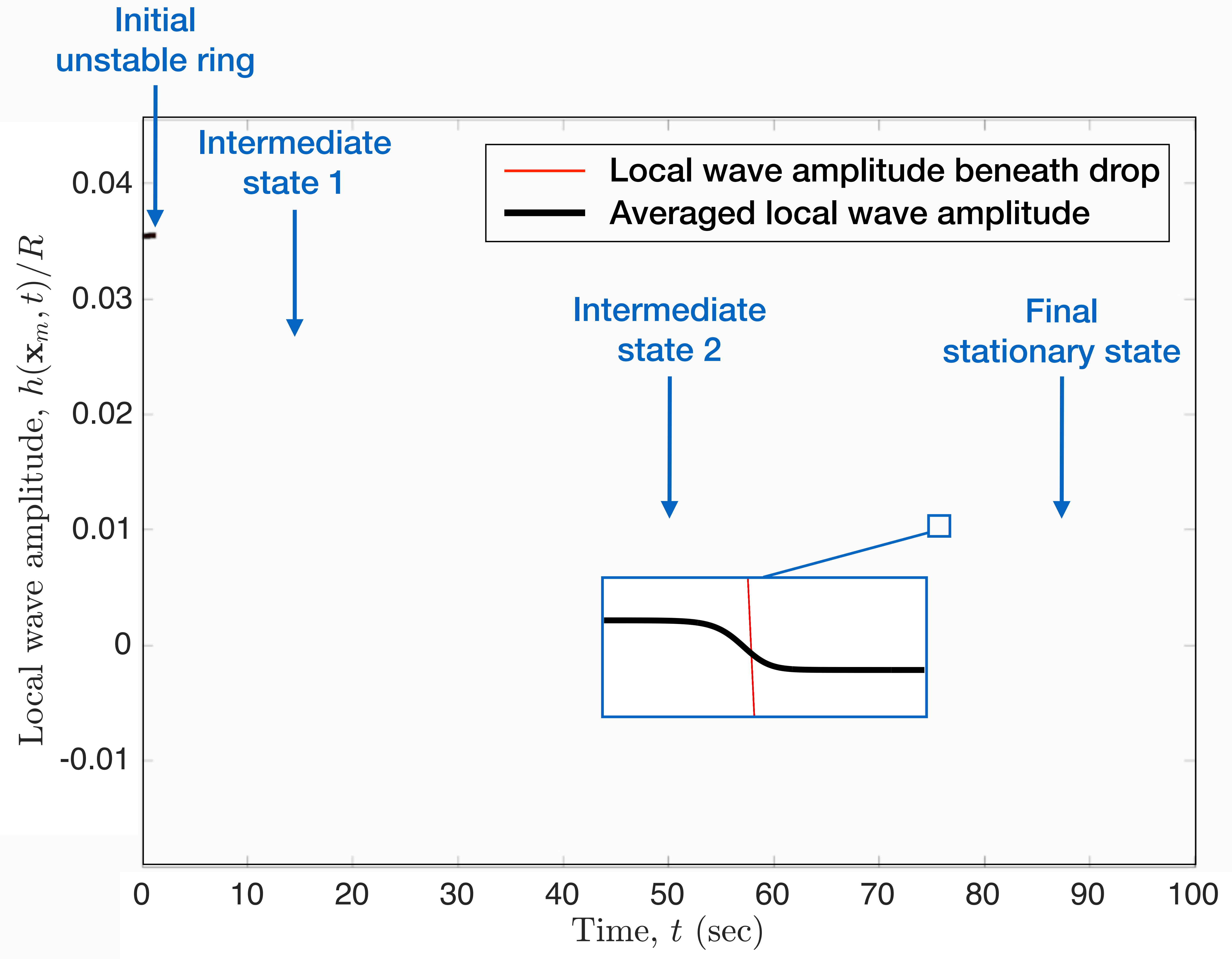
● Rings observed experimentally



Energetics of unstable rings: simulations

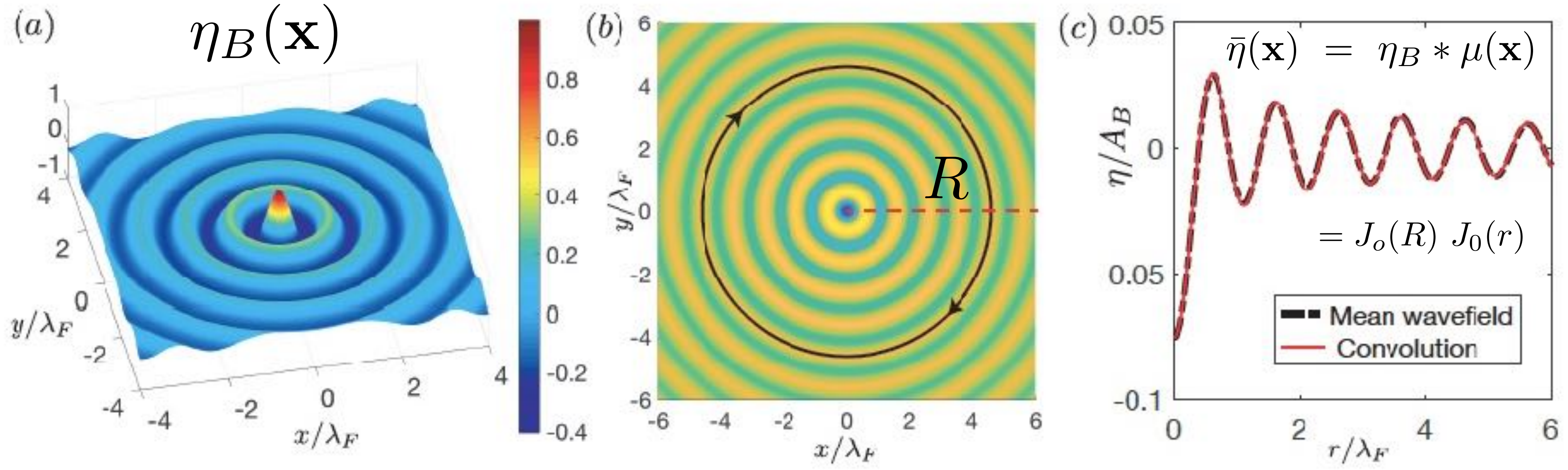
- droplets evolve toward configuration that minimizes averaged local wave amplitude



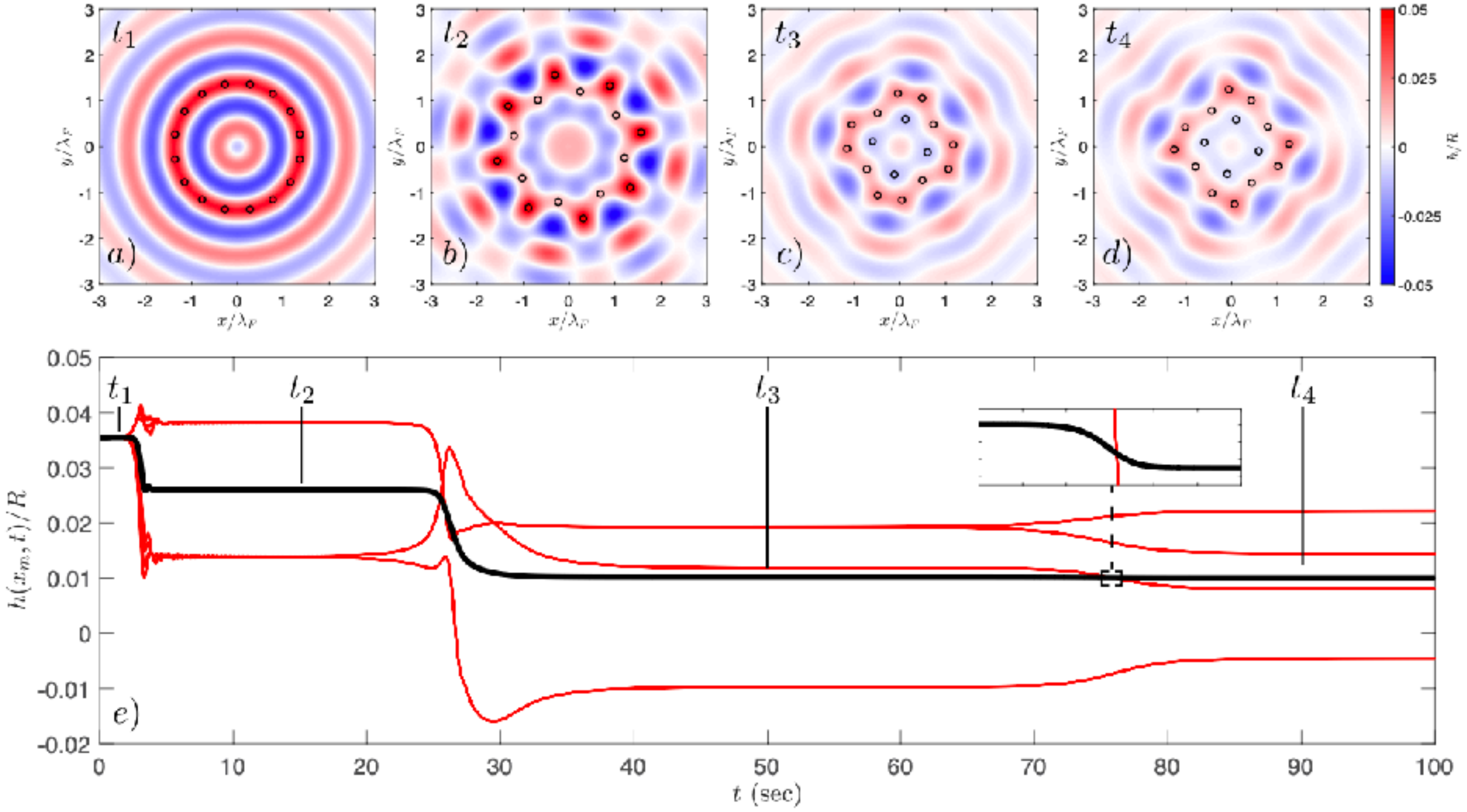


- prevalence of orbits with radii at zeros of $J_0(r)$ suggests dominance of wave energy

Energetics



- ditto for transition from bouncing to walking (c.f. Durey & Milewski, 2017)
- Miles' study of ring lattices suggests energetic dominance of drop's GPE



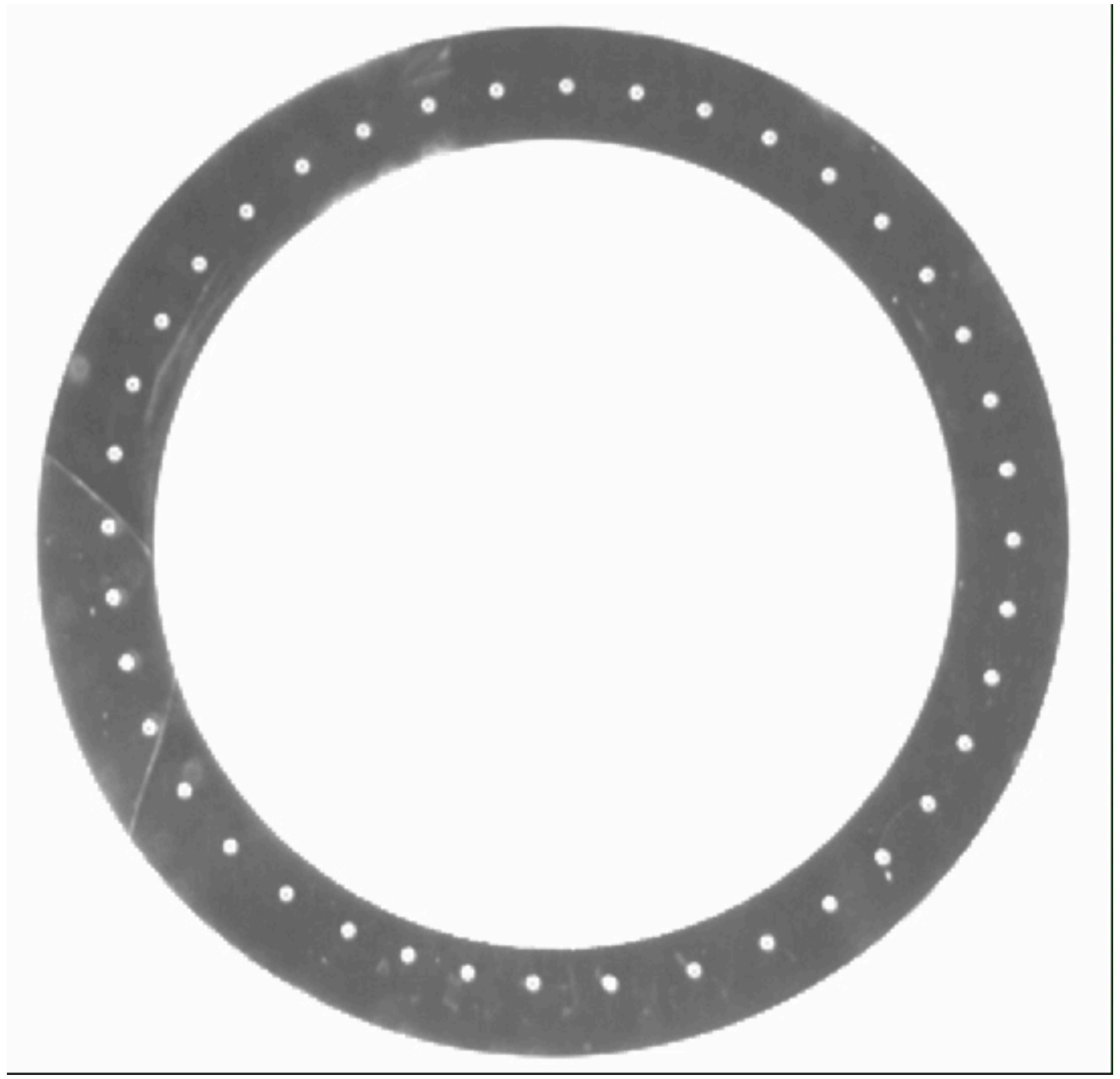
PHYSICAL REVIEW FLUIDS 5, 083601 (2020)

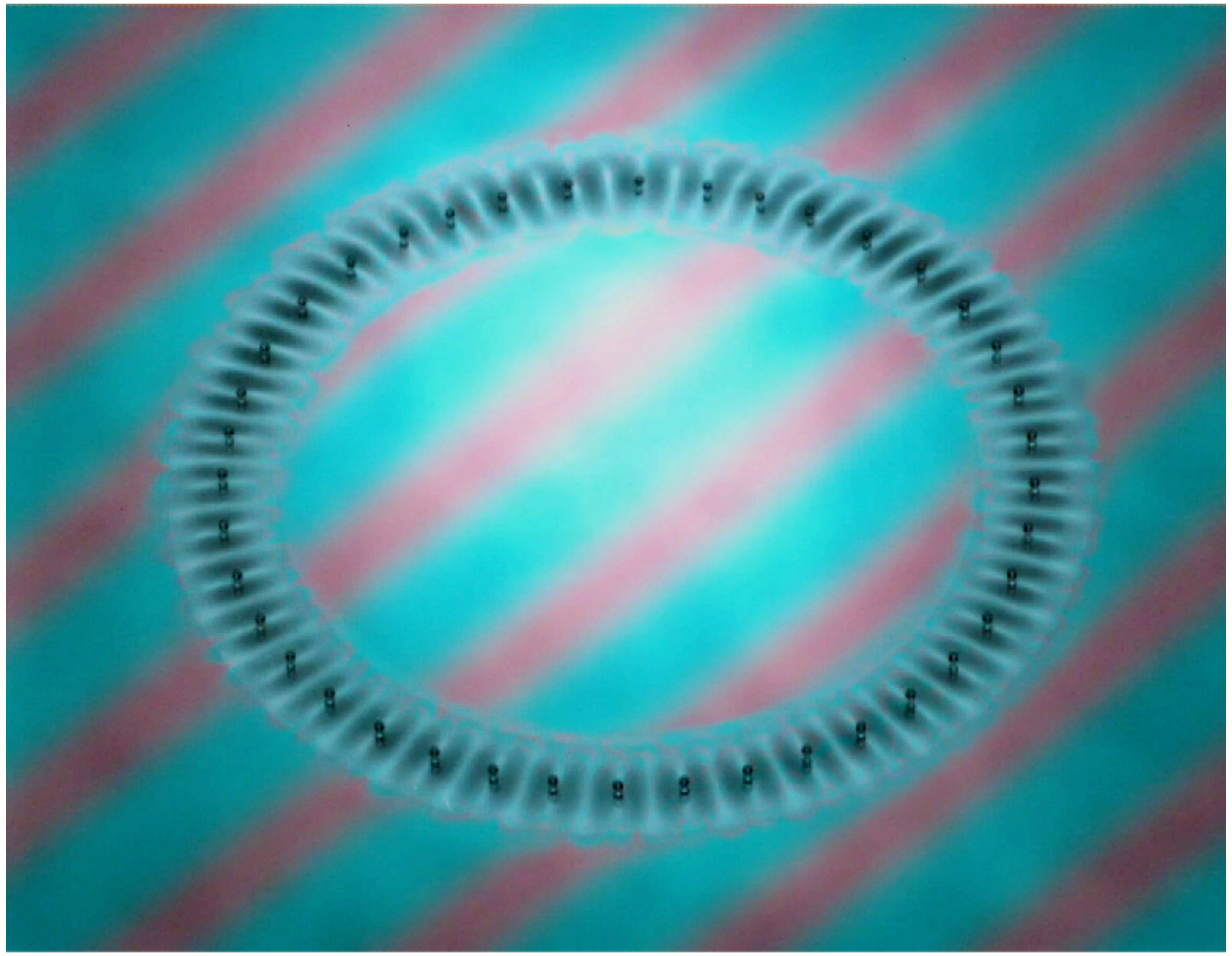
Droplets bound in an annulus

Collective vibrations of confined levitating droplets

S. J. Thomson, M. M. P. Couchman, and J. W. M. Bush*
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

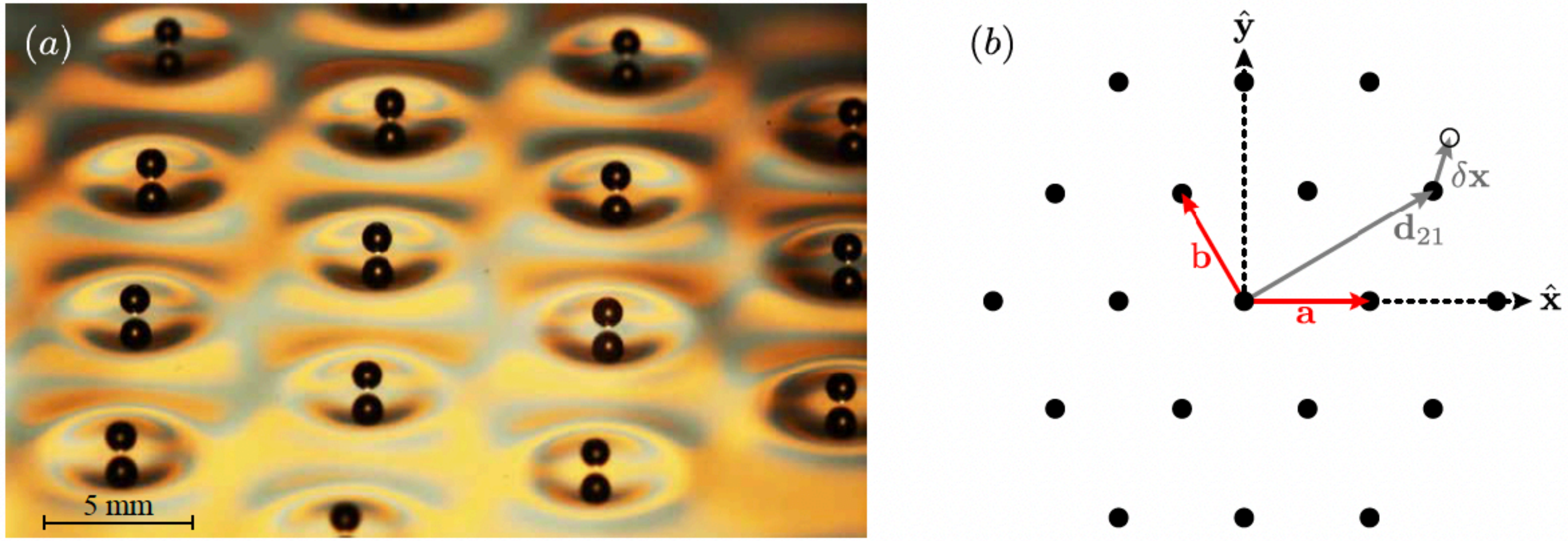
Thomson, Couchman & Bush (2019)





- exhibits binary oscillations, propagating solitary waves
- characteristics of the Toda lattice, a canonical model of crystal vibration
- dynamics greatly enriched through the influence of memory

The stability of droplet lattices



symmetry

Article

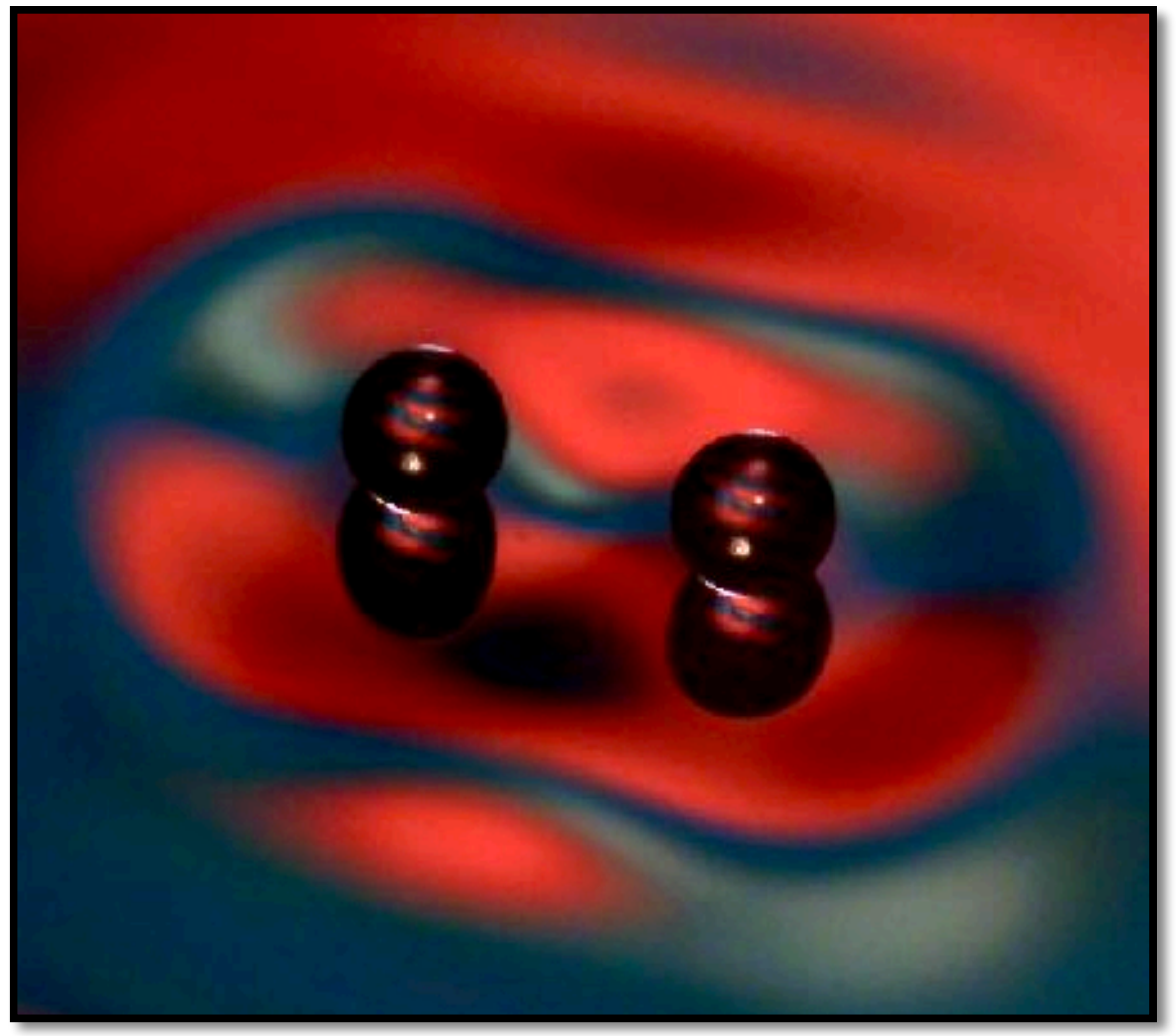
The Stability of a Hydrodynamic Bravais Lattice

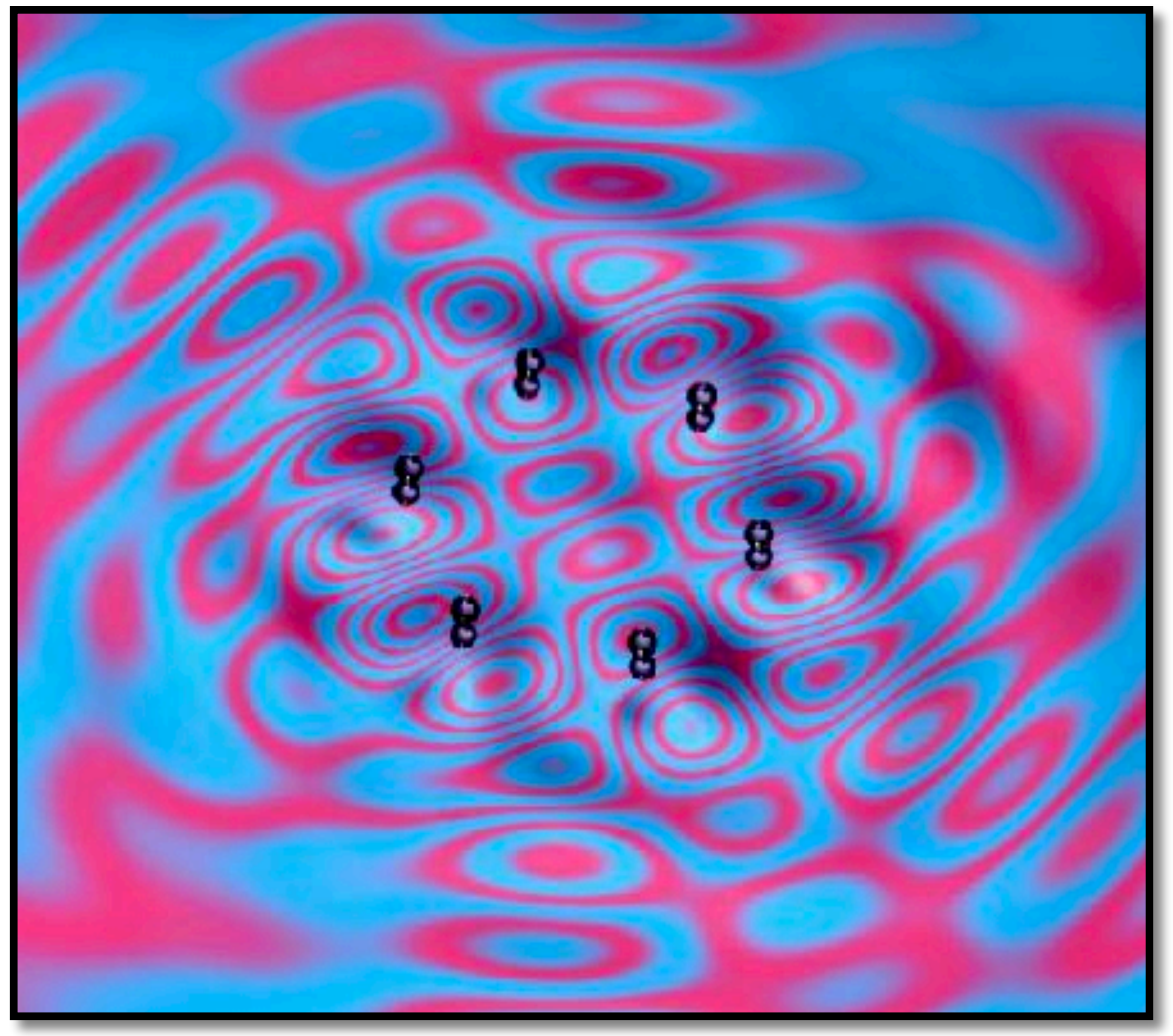
Miles M. P. Couchman [1], Davis J. Evans [2] and John W. M. Bush [2,*]

- the melting of a crystal lattice in response to increased memory

Summary

- variable-impact-phase trajectory equation captures coupling between drop's horizontal and vertical motion
- variations in drop's impact phase influence:
  - amplitude of wave generated at each bounce $(\mathcal{S})$
  - horizontal wave force $(\mathcal{C})$
- vertical dynamics have critical influence on stability of bound states
  - $(2,1)^1$: drop-drop interaction is destabilizing — less stable when bouncing in deeper wave minimum
  - $(2,1)^2$: drop-drop interaction is stabilizing — more stable when bouncing in deeper wave minimum
- previous constant-impact-phase models unable to capture observed stability





Conclusions

- discovered new static and dynamic bound states comprised of droplets bouncing on vibrating bath
  - In-line oscillations of droplet pair
  - Stable droplet rings and associated oscillatory instabilities
  - Propagating solitary waves in periodic droplet chain
- resolved fast bouncing dynamics, developed trajectory equation that captures coupling between horizontal and vertical motion
  - modulations in impact phase critically influence stability of bound states
- provides new degree of freedom in pilot-wave modeling, potentially facilitating discovery of new HQAs