Problem Set 2: 18.S996 Hydrodynamic quantum analogs Due April 10

Problem 1. Consider the Klein-Gordon equation:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + \frac{m^2c^2}{\hbar^2}\phi = 0 \tag{1}$$

a) Show that if one seeks a solution of the form $\phi(x,t) = e^{i\omega_c t}\Psi(x,t)$, then the wave envelope $\Psi(x,t)$ satisfies the linear Schrodinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi \tag{2}$$

provided $\Psi(x,t)$ evolves slowly relative to the Compton frequency, $\omega_c = mc^2/\hbar$.

b) Calculate the phase velocity and group velocity of the Klein-Gordon equation. Compare your results to those of water waves.

c) If a particle is accompanied by a wave of wavelength λ, specifically the particle speed is equal to the wave's group velocity, show that the particle momentum satisfies the de Broglie relation $\mathbf{p} = \hbar\mathbf{k}$.

d) Discuss the relation between strobing the hydrodynamic pilot-wave system at the Faraday frequency, and strobing the Klein-Gordon wave at the Compton frequency.

Problem 2. The Madelung transformation

The non-linear Schrodinger (Gross-Pitaevksii) equation governs the evolution of dilute Bose-Einstein condensates, in which the interaction potential between particles takes a particularly simple form:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi + g_0|\Psi|^2\Psi \tag{3}$$

a) Execute the Madelung transformation on this nonlinear Schrodinger equation. Specifically, express the wavefunction $\Psi(\mathbf{x},t) = \sqrt{\rho(\mathbf{x},t)}\, e^{iS(\mathbf{x},t)/\hbar}$ in terms the probability density $\rho(\mathbf{x},t)$ and action $S(\mathbf{x},t)$, and so deduce equations governing the evolution of these real-valued functions.

b) For particle motion in a plane, relate the equations deduced above to those governing small-amplitude waves in shallow-water hydrodynamics in a term by term basis. Identify the probability density with the fluid depth, and the quantum velocity of probability with the depth-averaged flow velocity in the fluid layer.

c) Discuss the relation between surface tension σ in hydrodynamics and $\hbar$ in quantum mechanics.

d) Interpret the resulting systems physically, discussing specifically the quantum potential.

3. *Bohmian mechanics*

a) Describe how Bohmian mechanics follows from Q2. Starting with the Madelung-transformed linear Schrodinger equation (set $g_0 = 0$ in Q2a), follow Bohm's assumption that the particle velocity is equal to the quantum velocity of probability $\nabla S/m$. Thus, deduce Bohm's second-order guidance equation, in which the quantum potential appears explicitly.

b) Solve the linear Schrodinger equation in a circular corral. Show that if a single mode is selected, Bohmian mechanics predicts zero particle velocities.

(OPEN-ENDED, POSSIBLY TRICK) BONUS QUESTIONS: choose 1 only

c) If there are two modes, compute Bohmian trajectories. Is the pdf associated with the resulting trajectories consistent with the wavefunction?

d) Calculate the probability distribution function $\rho(r,t)$ that emerges from a particle executing a random walk over the quantum potential. You choose the form of the random walk.

e) Give a coherent, rational account of Bohmian mechanics for a free particle.