intro to Faraday waves modeling

Introduction to Faraday waves:

The stability of the plane free surface of a liquid in vertical periodic motion

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(*Communicated by Sir Geoffrey Taylor, F.R.S.—Received* 13 *April* 1954)

Benjamin & Ursell (Proc.Royal Soc. '54)

where $F(t)$ is independent of x, y, z, and may be put equal to 0; and (2·1) gives

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0. \qquad (2\cdot3)$$

the walls.) The simplified equations are, from (2·5),

$$\frac{\partial\zeta}{\partial t}=w=\frac{\partial\phi}{\partial z} \quad \text{for } z=0; \qquad (2\cdot8)$$

and from (2·4)
$$\frac{\gamma}{\rho}\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)+\left(\frac{\partial\phi}{\partial t}\right)_{z=0}-(g-f\cos\omega t)\,\zeta=0. \qquad (2\cdot9)$$

The approximation used here for the curvature is familiar in membrane theory (Rayleigh 1894, §194).

There is an important consequence of these boundary conditions. From (2·6) and (2·8) it follows that $\partial^2\zeta/\partial t\partial n=0$ at any point of the curve C bounding the free surface, whence $\partial\zeta/\partial n=$ its initial value $=0$. Thus the angle of contact at the walls is 90°. Also, by applying the operator $\partial/\partial n$ to (2·9), it is found that $\frac{\partial}{\partial n}\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)=0$ on C. These boundary conditions show that ϕ, ζ and $\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)$ can each be expanded in terms of the complete orthogonal set of eigenfunctions $S_m(x,y)$, where

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_m^2\right)S_m(x,y)=0 \qquad (2\cdot10)$$

We will pursue a pilot-wave model along the potential theory framework:

(our notation)

$$f\equiv g\Gamma$$
$$\omega\equiv\Omega$$

$$\zeta(x,y,t) = \sum_0^\infty a_m(t)\, S_m(x,y),$$

$$\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2} = -\sum_0^\infty k_m^2\, a_m(t)\, S_m(x,y),$$

$$\phi(x,y,z,t) = -\sum_1^\infty \frac{da_m(t)}{dt}\, \frac{\cosh k_m(h-z)}{k_m \sinh k_m h}\, S_m(x,y) + G(t), \qquad (2\cdot11)$$

If the parameters p_m and q_m are defined by the equations

$$p_m = \frac{4k_m \tanh k_m h}{\omega^2}\left(g + \frac{k_m^2\gamma}{\rho}\right), \quad q_m = \frac{2k_m f \tanh k_m h}{\omega^2}, \qquad (2\cdot13)$$

Stab. Anal. => Mathieu eqn.

and if $T = \frac{1}{2}\omega t$, then (2·12) takes the form

$$\frac{d^2 a_m}{dT^2} + (p_m - 2q_m \cos 2T)\, a_m = 0, \qquad (2\cdot14)$$

which is the standard form of Mathieu's equation adopted by McLachlan (1947).

relating to *free* vibrations of the liquid. The frequency (= 1/period) of these vibrations is

$$\frac{\omega_m}{2\pi} = \frac{1}{2\pi}\left[\tanh k_m h\left(\frac{k_m^3\gamma}{\rho} + k_m g\right)\right]^{\frac{1}{2}}, \qquad (2\cdot15)$$

and $p_m = \omega_m^2/\omega^2$. Note also that $q_m = 2k_m \tanh k_m h \times$ (amplitude of vibration).

(Benjamin & Ursell, '54)

Most unstable mode is p=1:

$$\omega_F = \frac{\Omega}{2}$$

forcing frequency

(sub-harmonic)

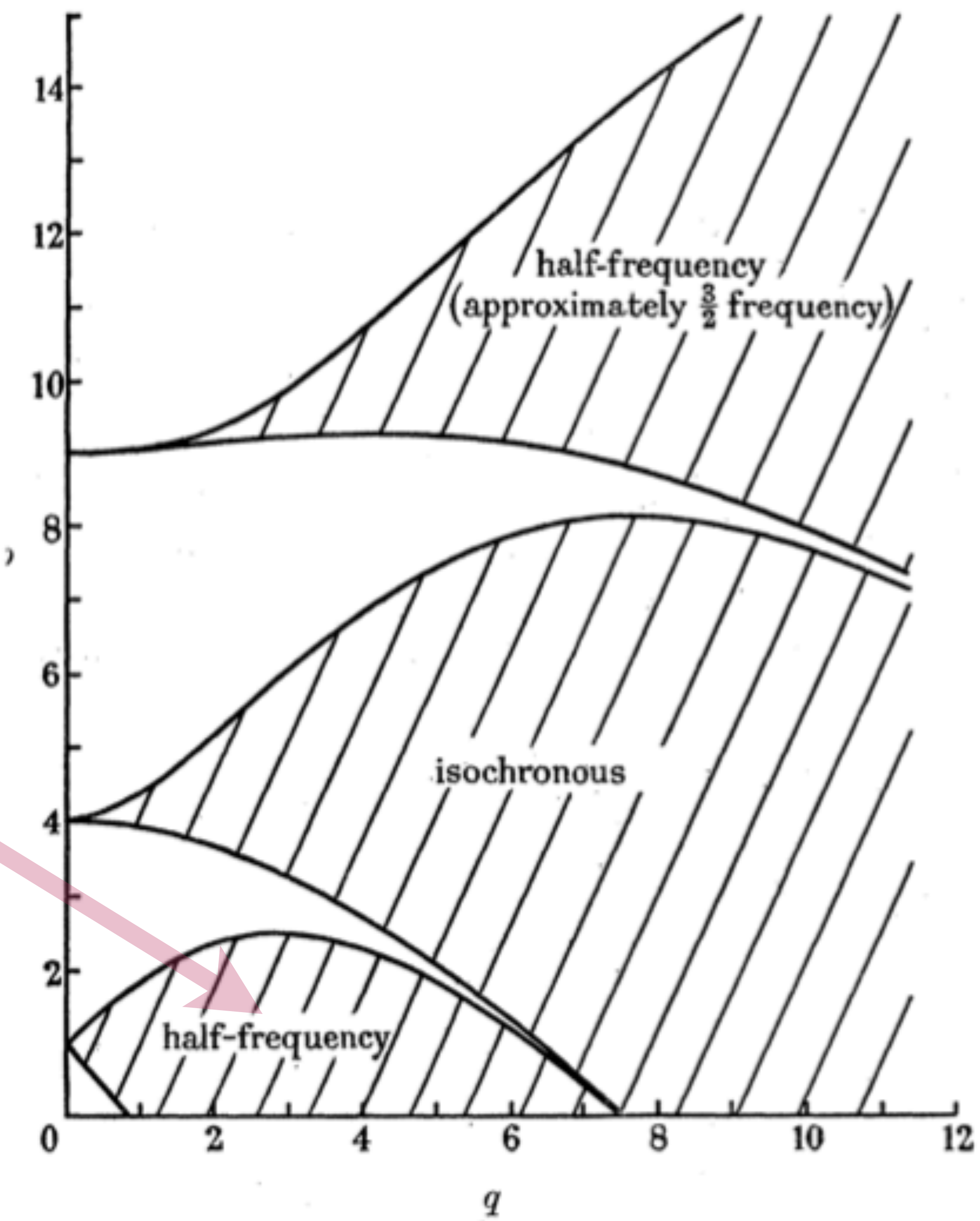


FIGURE 2. Stability chart for the solutions of Mathieu's equation

$$\frac{d^2a}{dT^2} + (p - 2q \cos 2T)\, a = 0.$$

fluid/wave

2D1D

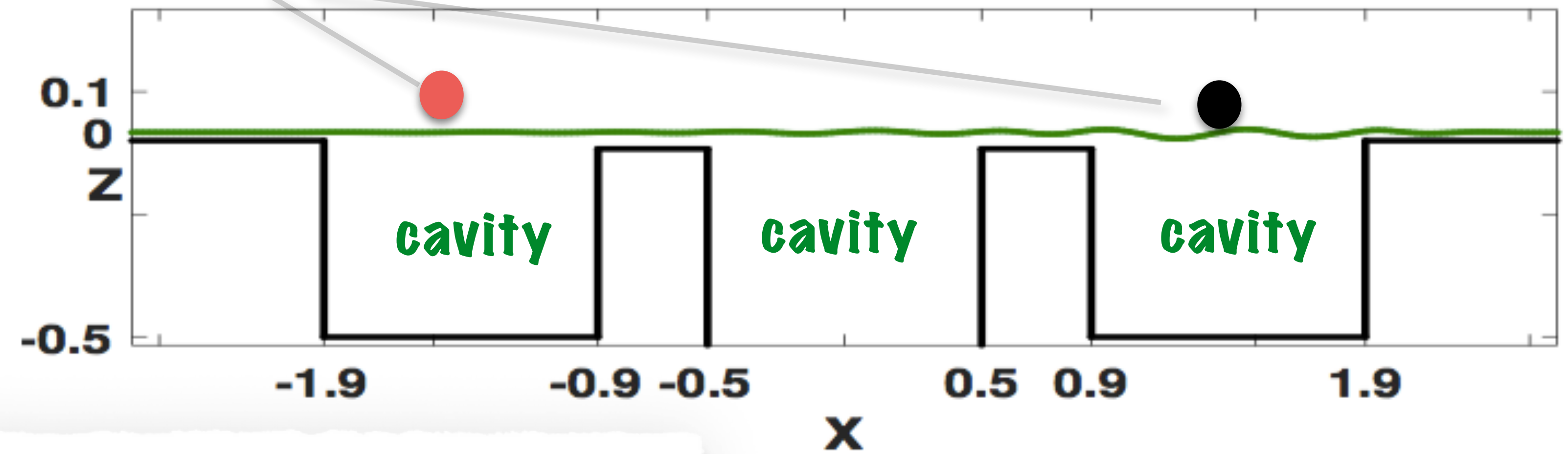
modeling

Will focus on: cavity structure & conformal mapping; Dirichlet-to-Neumann operator

1D dynamics of 2 droplets placed at a distance

$\eta(x,t) =$ wave elevation

$\phi(x,y,t) =$ velocity potential



$$m\ddot{X}_1 + c\,F(t)\dot{X}_1 = -F(t)\,\frac{\partial \eta}{\partial x}(X_1(t),t).$$

2 droplet-dynamics: Newton's Law

$$m\ddot{X}_2 + c\,F(t)\dot{X}_2 = -F(t)\,\frac{\partial \eta}{\partial x}(X_2(t),t).$$

$$F(t) \equiv \mathbb{1}_{T_c=T_F/4}G(t)$$

(Dias, Dyachenko & Zakharov '08)

viscosity

$g(t) = g(1 - \Gamma\sin(\omega_0 t))$

surface tension

droplet's pressure term

DtN: Fourier integral op.

DIRICHLET-to-NEUMANN OPERATOR

wave system

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} + 2\nu\frac{\partial^2 \eta}{\partial x^2},$$

DtN

$$\frac{\partial \phi}{\partial t} = -g(t)\eta + \frac{\sigma}{\rho}\frac{\partial^2 \eta}{\partial x^2} + 2\nu\frac{\partial^2 \phi}{\partial x^2}$$

shaking

$$-\frac{1}{\rho}P_d(x - X_1(t)) - \frac{1}{\rho}P_d(x - X_2(t)),$$

2 wave makers

contact time $T_c \equiv T_F/4$

Modeling ideas

simplified 2D. $(\sigma = \Gamma = P_d = 0)$

Helmholtz decomposition: *velocity = (curl free + div free) parts*

(Dias, Dyachenko & Zakharov, 2008; Lamb, 1932)

$$u \equiv \phi_x - \psi_z, \qquad w \equiv \phi_z + \psi_x$$

$$\phi, \psi \to 0 \text{ as } z \to -\infty$$

Problems that arise from LINEAR Navier-Stokes eq. at $z \leq 0$

Potential component of velocity field: $\Delta\phi = 0$

Vortical component of velocity field: $\psi_t = \nu \, \Delta\psi$

Bernoulli equation $\phi_t = -\dfrac{p - p_0}{\rho} - gz$

LINEARIZED STRESS RELATIONS:

$$0 = \frac{p}{\rho} - 2\nu w_z$$ **normal stress**

$$0 = \nu(u_z + w_x) = \nu \left[2\phi_{xz} - \psi_{zz} + \psi_{xx}\right]$$ **tangential stress**

1D dynamics of 2 droplets placed at a distance

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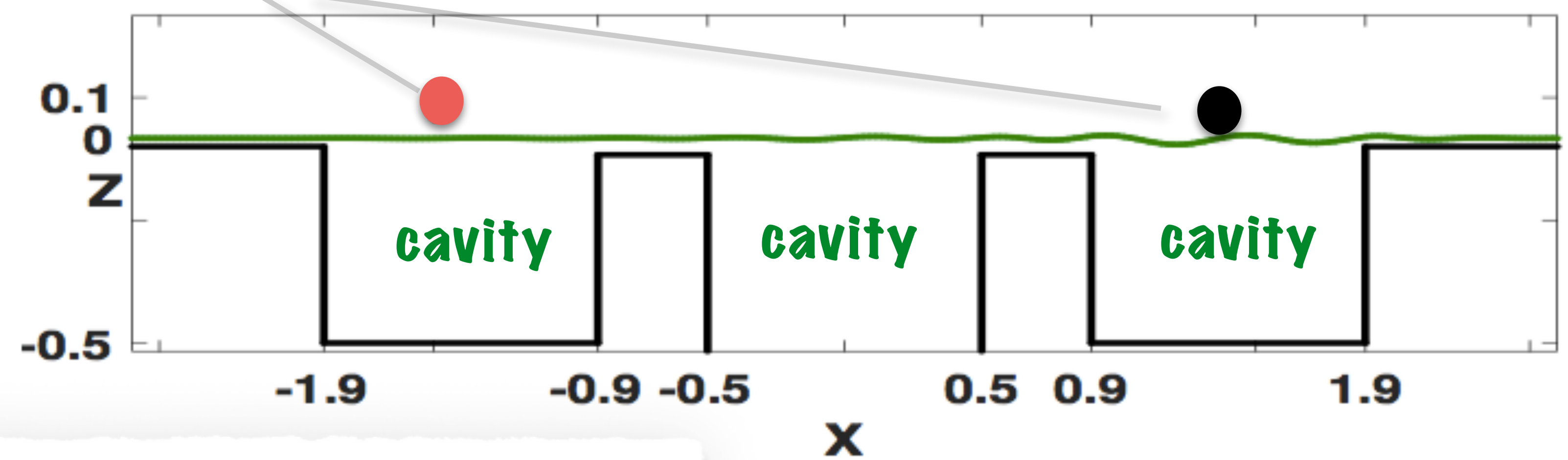

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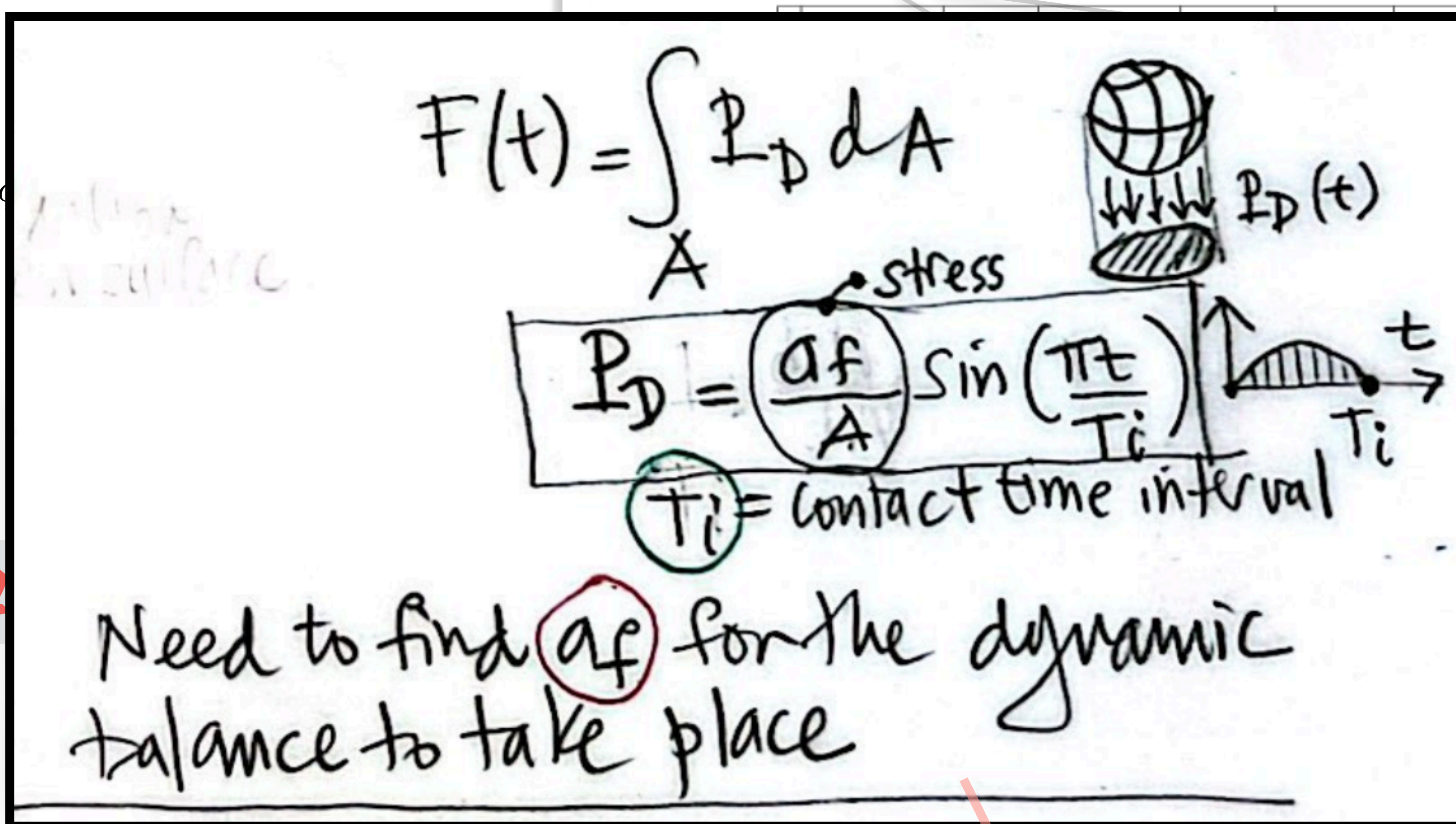
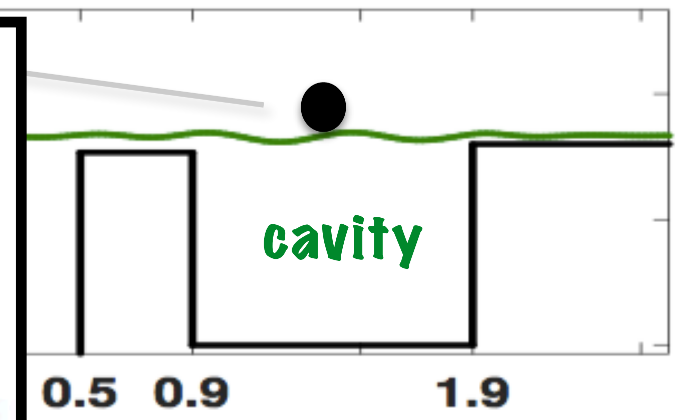
contact time $T_c \equiv T_F/4$

1D dynamics of 2 droplets placed at a distance





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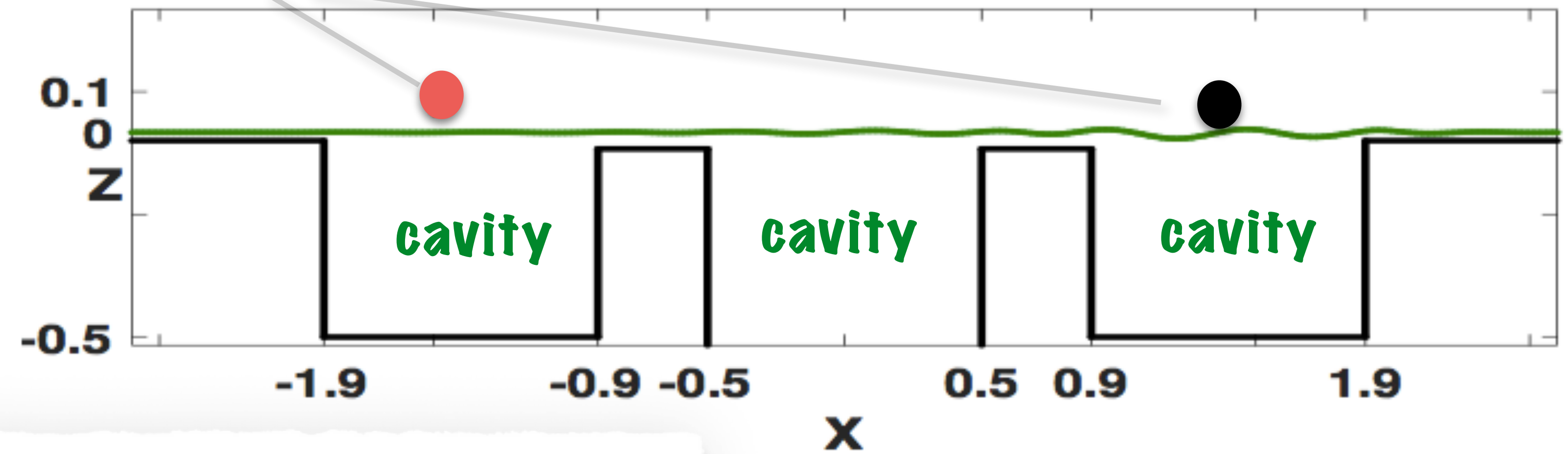

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2 droplet-dynamics: Newton's Law

$$m\ddot{X}_2 + c\,F(t)\dot{X}_2 = -F(t)\,\frac{\partial \eta}{\partial x}(X_2(t),t).$$

conformal mapping and the

DtN: Fourier integral op.

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$\eta(x, t$

$\phi(x, y, t$

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contact time $T_c \equiv T_F/4$

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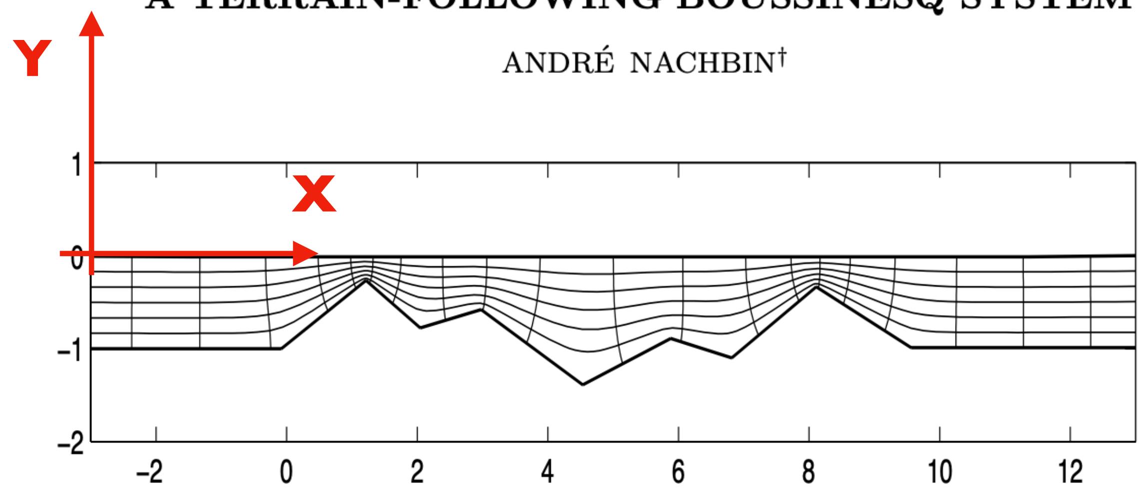
A TERRAIN-FOLLOWING BOUSSINESQ SYSTEM*

ANDRÉ NACHBIN†



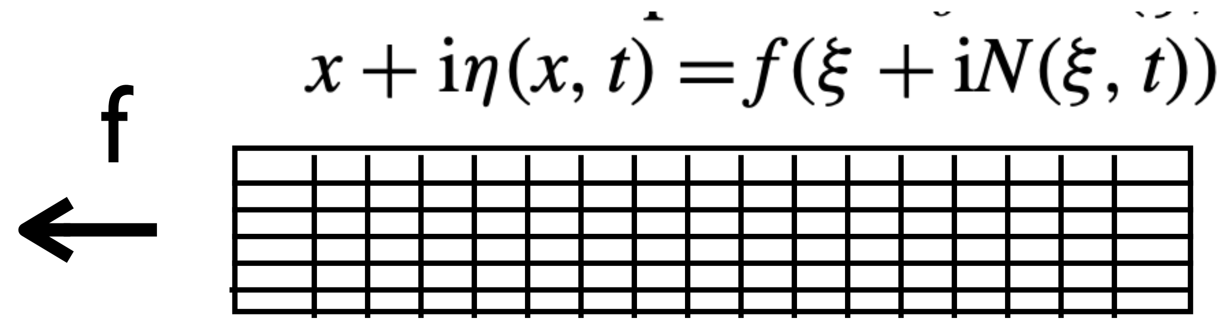


FIG. 4.1. *A schematic figure showing a slowly varying topography in the xy coordinate system together with the ξ and $\tilde{\zeta}$ level-curves. This figure was generated using SC-Toolbox* [4].

4. Nonlinear potential theory equations in terrain-following coordinates. The scaled water wave equations in the fixed orthogonal curvilinear coordinates $(\xi, \tilde{\zeta})$ (cf. Figure 4.1) are

$$\phi_{\xi\xi} + \phi_{\tilde{\zeta}\tilde{\zeta}} = 0, \qquad -\sqrt{\beta} < \tilde{\zeta} < \alpha\sqrt{\beta}N(\xi,t), \tag{4.1}$$

Laplace eq.

with free surface conditions

$$|J|N_t + \alpha\phi_\xi N_\xi - \frac{1}{\sqrt{\beta}}\phi_{\tilde{\zeta}} = 0 \tag{4.2}$$

NL kinematic cond.

and

$$\phi_t + \eta + \frac{\alpha}{2|J|}\left(\phi_\xi^2 + \phi_{\tilde{\zeta}}^2\right) = 0 \tag{4.3}$$

NL Bernoulli cond.

at $\tilde{\zeta} = \alpha\sqrt{\beta}N(\xi,t)$. The bottom condition is

$$\phi_{\tilde{\zeta}} = 0 \quad \text{at} \quad \tilde{\zeta} = -\sqrt{\beta}. \tag{4.4}$$

Neumann cond.

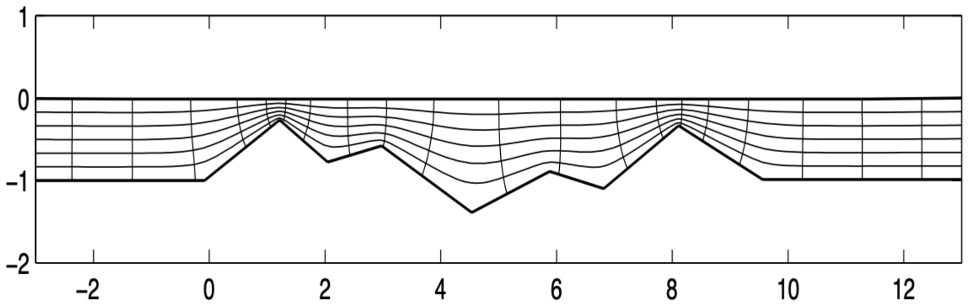


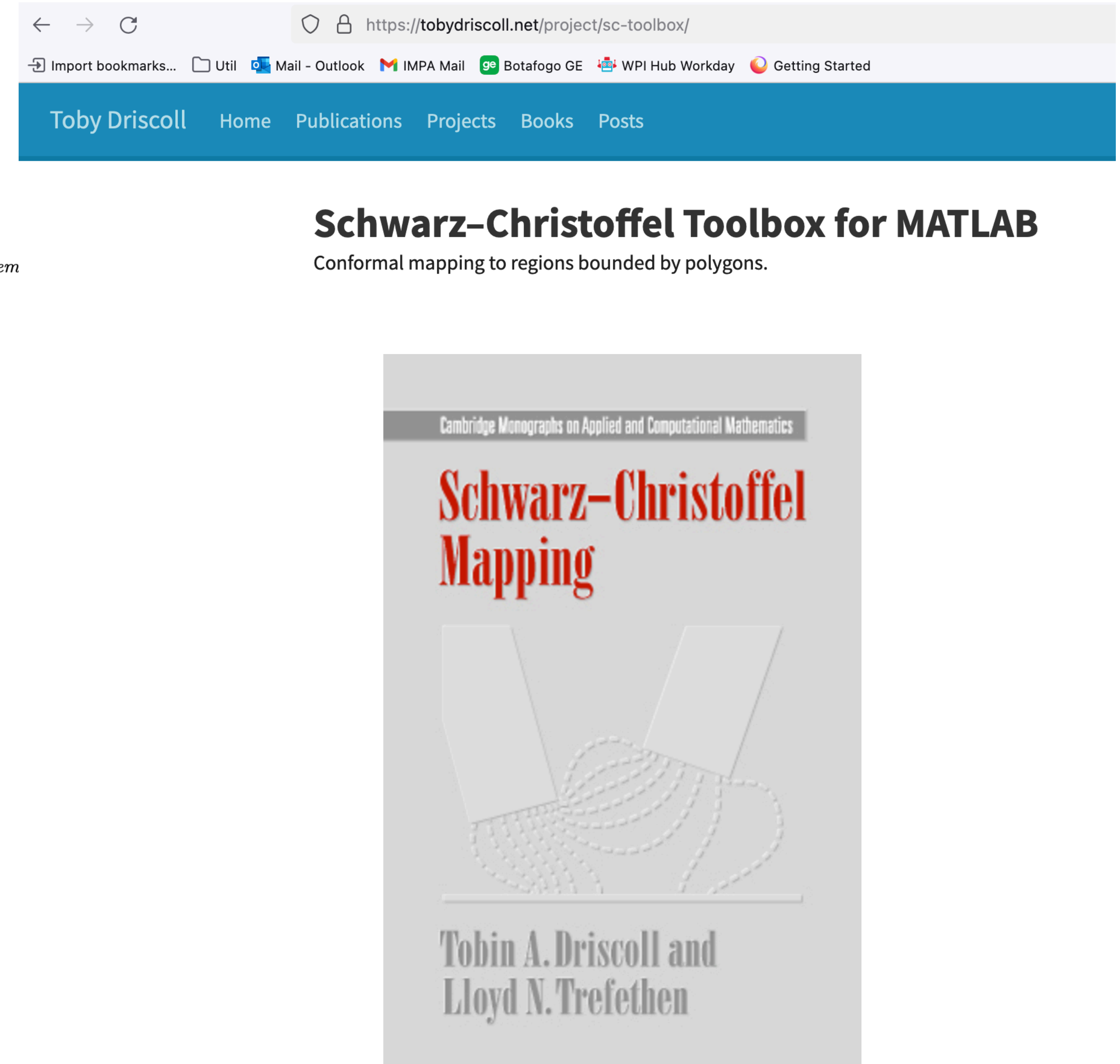
FIG. 4.1. *A schematic figure showing a slowly varying topography in the xy coordinate system together with the ξ and $\tilde{\zeta}$ level-curves. This figure was generated using SC-Toolbox* [4].

$$\phi_x = \frac{1}{|J|}\left[\tilde{y}_{\tilde{\zeta}}\phi_\xi - \tilde{y}_\xi\phi_{\tilde{\zeta}}\right]$$

$$\phi_{\tilde{y}} = \frac{1}{|J|}\left[-x_{\tilde{\zeta}}\phi_\xi + x_\xi\phi_{\tilde{\zeta}}\right],$$

$$|J| = x_\xi\tilde{y}_{\tilde{\zeta}} - \tilde{y}_\xi x_{\tilde{\zeta}} = \tilde{y}_{\tilde{\zeta}}^2 + \tilde{y}_\xi^2.$$

$$\phi_x^2 + \phi_{\tilde{y}}^2 = \frac{1}{|J|}\left(\phi_\xi^2 + \phi_{\tilde{\zeta}}^2\right),$$



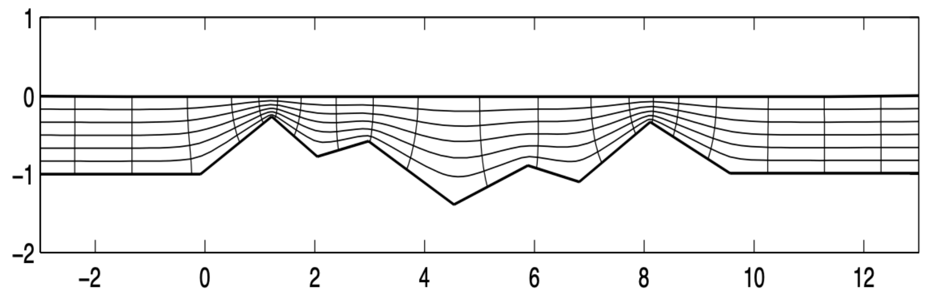


FIG. 4.1. *A schematic figure showing a slowly varying topography in the xy coordinate system together with the ξ and $\tilde{\zeta}$ level-curves. This figure was generated using SC-Toolbox* [4].

In particular, at the undisturbed free surface or for linear problems,

$$\phi_\xi(\xi, 0) = M(\xi)\phi_x$$

and

$$\phi_{\tilde{\zeta}}(\xi, 0) = M(\xi)\phi_{\tilde{y}}.$$

At the undisturbed level we define the *variable free surface coefficient*

$$M(\xi) \equiv \tilde{y}_{\tilde{\zeta}}(\xi, 0) = 1 + m(\xi),$$

$$\phi_x = \frac{1}{|J|}\left[\tilde{y}_{\tilde{\zeta}}\phi_\xi - \tilde{y}_\xi\phi_{\tilde{\zeta}}\right]$$

$$\phi_{\tilde{y}} = \frac{1}{|J|}\left[-x_{\tilde{\zeta}}\phi_\xi + x_\xi\phi_{\tilde{\zeta}}\right],$$

$$|J| = x_\xi\tilde{y}_{\tilde{\zeta}} - \tilde{y}_\xi x_{\tilde{\zeta}} = \tilde{y}^2_{\tilde{\zeta}} + \tilde{y}^2_\xi.$$

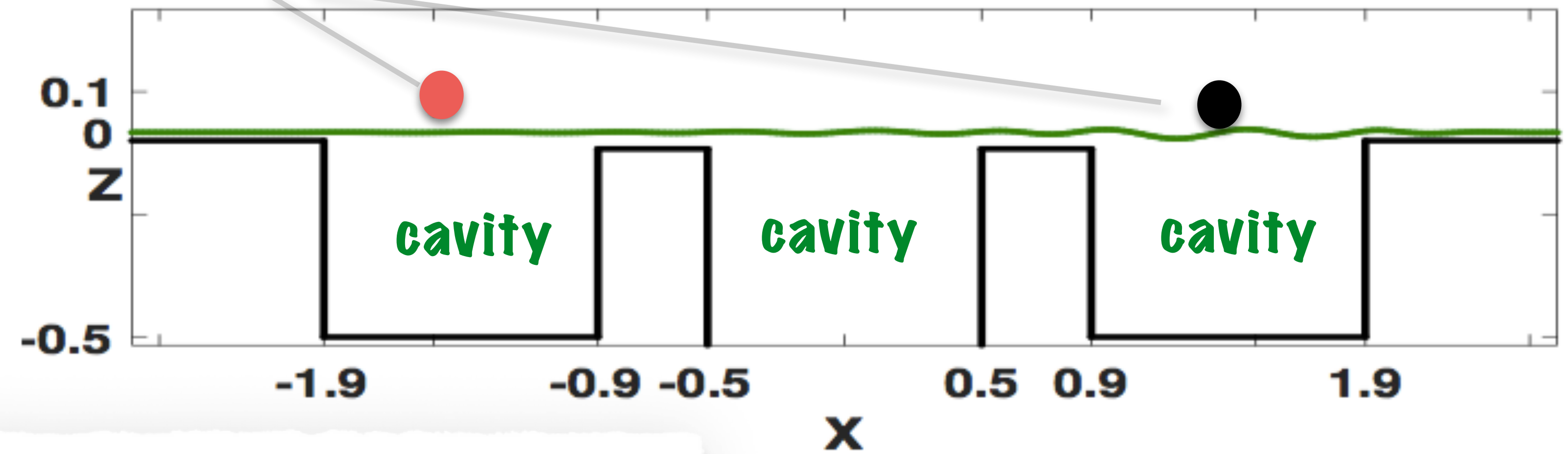
$$\phi^2_x + \phi^2_{\tilde{y}} = \frac{1}{|J|}\left(\phi^2_\xi + \phi^2_{\tilde{\zeta}}\right),$$

Laplace operator

1D dynamics of 2 droplets placed at a distance

$\eta(x,t) = \text{wave elevation}$

$\phi(x,y,t) = \text{velocity potential}$



● $$m\ddot{X}_1 + c\,F(t)\dot{X}_1 = -F(t)\,\frac{\partial\eta}{\partial x}(X_1(t),t).$$

2 droplet-dynamics: Newton's Law

● $$m\ddot{X}_2 + c\,F(t)\dot{X}_2 = -F(t)\,\frac{\partial\eta}{\partial x}(X_2(t),t).$$

DtN: Fourier integral op.

DIRICHLET-to-NEUMANN OPERATOR

wave system

$$\frac{\partial\eta}{\partial t} = \underbrace{\frac{\partial\phi}{\partial z}}_{\text{DtN}} + 2\nu\frac{\partial^2\eta}{\partial x^2},$$

$$\frac{\partial\phi}{\partial t} = -\underbrace{g(t)}_{\text{shaking}}\eta + \frac{\sigma}{\rho}\frac{\partial^2\eta}{\partial x^2} + 2\nu\frac{\partial^2\phi}{\partial x^2} - \frac{1}{\rho}P_d(x-X_1(t)) - \frac{1}{\rho}P_d(x-X_2(t)),$$

contact time

In preparation for the DtN operator

$$K(P,Q) = K(|P-Q|) = \ln(r)$$

— Green's identity with $u \in C^2(\bar{\Omega})$, $\Delta u = 0$ and $v = K(x,\xi)$, $\Delta v = \delta_\xi$ yields (Green's third identity)

$$u(\xi) = -\oint_{\partial\Omega} \left(K(x,\xi) \frac{du}{dn}(x) - u(x) \frac{dK(x,\xi)}{dn} \right) dS_x , \quad x \in \partial\Omega, \; \xi \in \Omega$$

$\frac{du}{dn}(x)$ → Neumann data; $u(x)$ → Dirichlet data

Note: if we know both Neumann and Dirichlet we are DONE!

- $K(x,\xi)$ → single layer potential
- Neumann → source density
- Dirichlet → dipole density (moment density)
- $\frac{dK(x,\xi)}{dn}$ → double layer potential

D D N

$$\theta\, u(\bar{x}) = \oint_{\partial\Omega} \left(u(x) \frac{dE}{dn}(x,\bar{x}) - \frac{du}{dn}(x)\, E(x,\bar{x}) \right) dS$$

Functional relation Dirichlet–Neumann

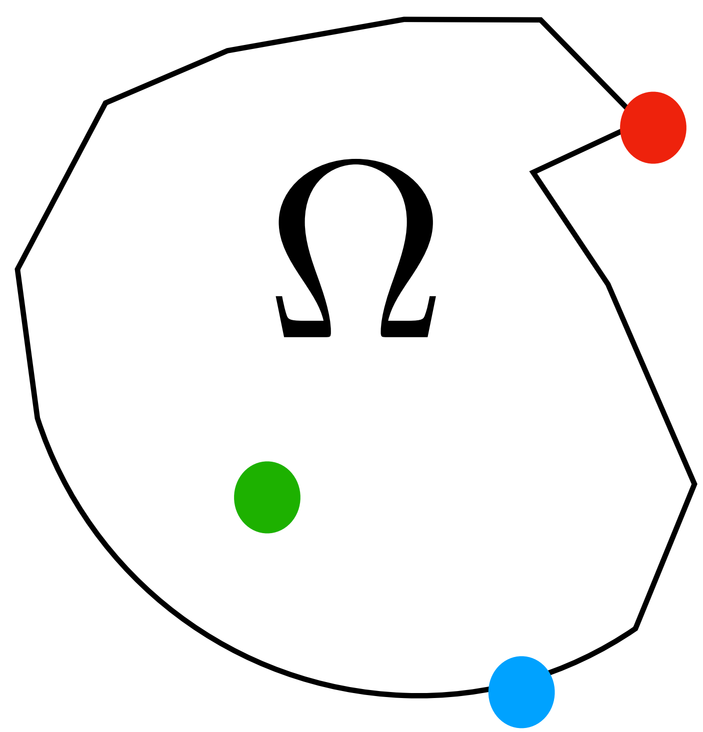
D N

$x \in \partial\Omega$

$\theta = \pi$, $\bar{x} \in \partial\Omega$ smooth

$\theta =$ internal angle $\partial\Omega$–corner

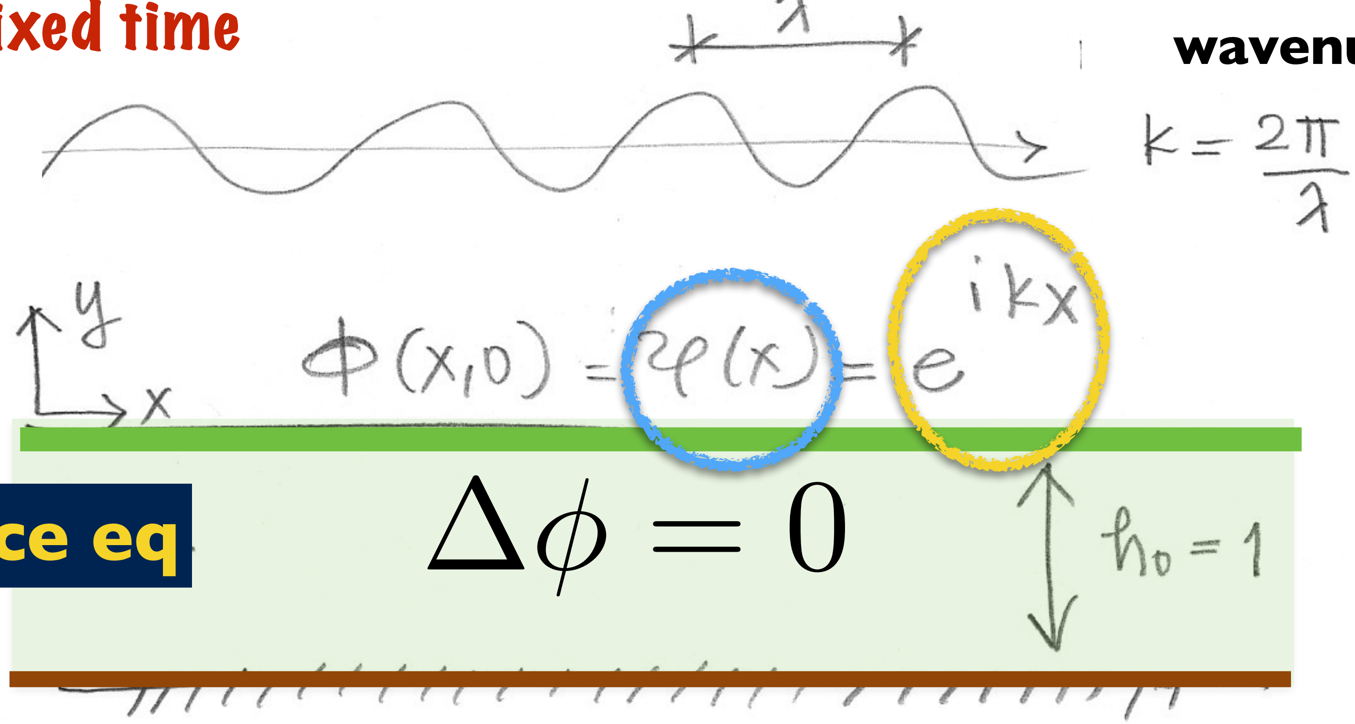
$\theta = 2\pi$ *interior*



An exercise in Separation of Variables

At some fixed time

wavenumber



Dirichlet data:
single Fourier mode

Laplace eq

$$d\phi/dn = 0$$

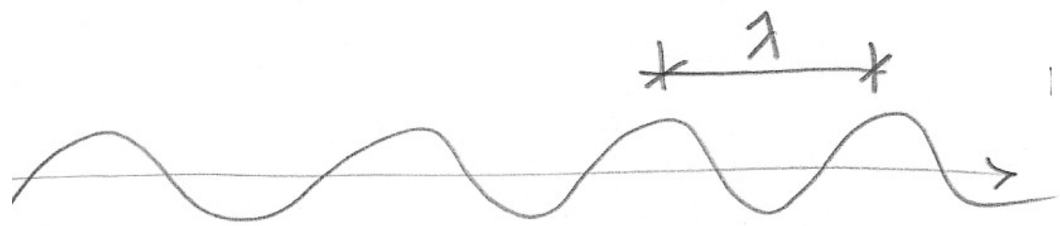
Neumann
Flat bottom

Velocity potential

$\Phi = \Phi(x,y)$ $\quad (u,v) = \nabla\phi$

incomprss. $\quad u_x + v_y = (\phi_x)_x + (\phi_y)_y = 0$

irrotacional $\quad v_x - u_y = (\phi_y)_x - (\phi_x)_y = 0$



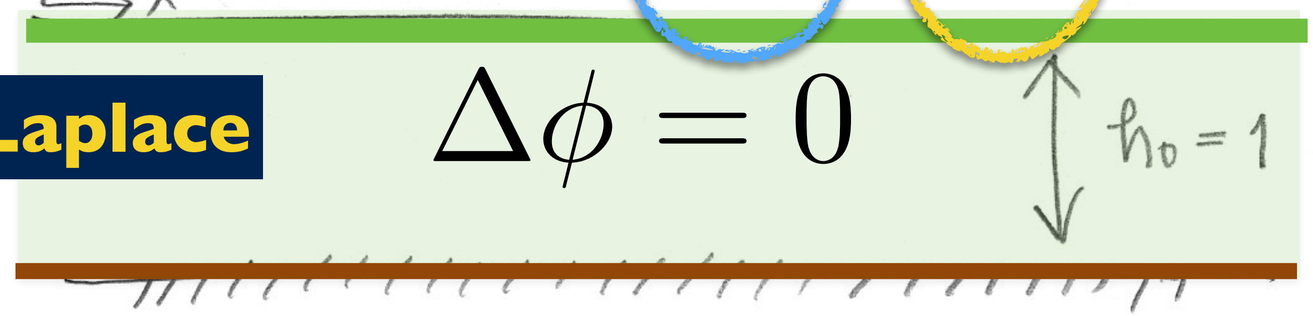
wavenumber

$k = \frac{2\pi}{\lambda}$

$\Phi(x,0) = \varphi(x) = e^{ikx}$

Dirichlet data:
single Fourier mode

eq. de Laplace



$d\phi/dn = 0$

Velocity potential

$\Phi = \Phi(x,y) \qquad (u,v) = \nabla\phi$

incomprss. $u_x + v_y = (\phi_x)_x + (\phi_y)_y = 0$

irrotacional $v_x - u_y = (\phi_y)_x - (\phi_x)_y = 0$

Harmonic extension of Fourier mode; satisfies that Neumann at bottom.

$$\Phi(x,y) = \left(\frac{\cosh(k(h_0+y))}{\cosh(kh_0)}\right) e^{ikx}$$

Harmonic function : superimpose all Fourier modes

$$\Phi(x,y) = \int_{-\infty}^{\infty} \hat{\varphi}(k) \left[\frac{\cosh(k(h_0+y))}{\cosh(kh_0)}\right] e^{ikx}\, dk$$

The harmonic function representation

$$\phi(x,y) = \int_{-\infty}^{\infty} \hat{\varphi}(k) \left[\frac{\cosh(k(h_0+y))}{\cosh(kh_0)} \right] e^{ikx}\, dk$$

To get Neumann data at the top:

$$\frac{\partial \phi}{\partial y}(x,0) = \int_{-\infty}^{\infty} \left[k \tanh(kh_0) \right] \hat{\varphi}(k) e^{ikx}\, dk$$

S(k)= symbol of the integral Fourier operator

Dirichlet-to-Neumann operator (DtN):

$$\underbrace{\frac{\partial \phi}{\partial y}(x,0)}_{\text{NEUMANN}} = \mathrm{DtN}[\underset{\uparrow \text{DIRICHLET}}{\varphi}](x)$$

DtN = pseudo-differential operator

$k = 2\pi/\lambda, \quad k \ll 1$

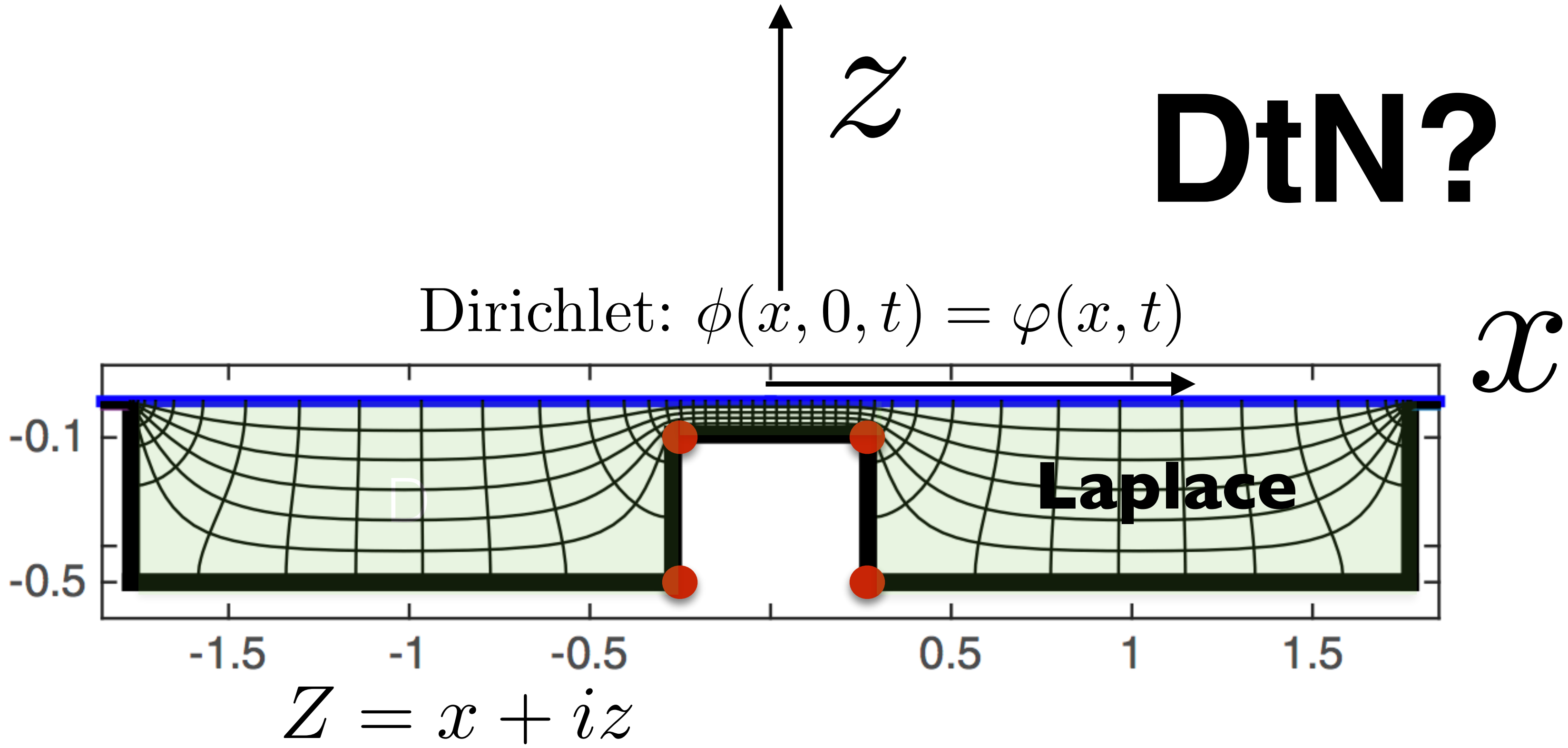
long waves

$$\tanh(kh_0) \approx kh_0 - \frac{(kh_0)^3}{3}$$

Example:
differencial operator

$$\frac{d}{dx}\phi(x,0) = \int_{-\infty}^{\infty} \underset{\text{symbol}}{(ik)}\, \hat{\varphi}(k) e^{ikx}\, dk$$

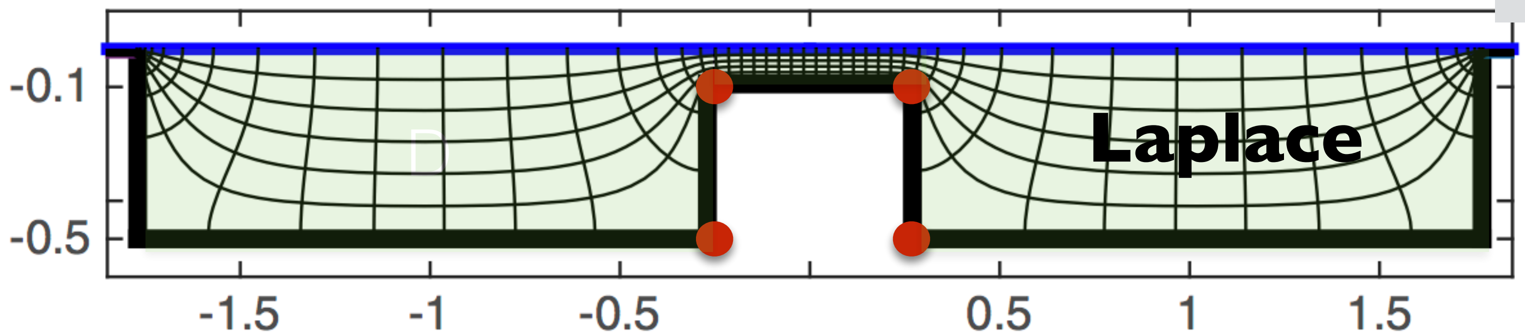
DtN?


z
Dirichlet: $\phi(x,0,t) = \varphi(x,t)$
x
Laplace
-0.1
-0.5
-1.5
-1
-0.5
0.5
1
1.5
$Z = x + iz$

Z-plane **w-plane:** uniform strip

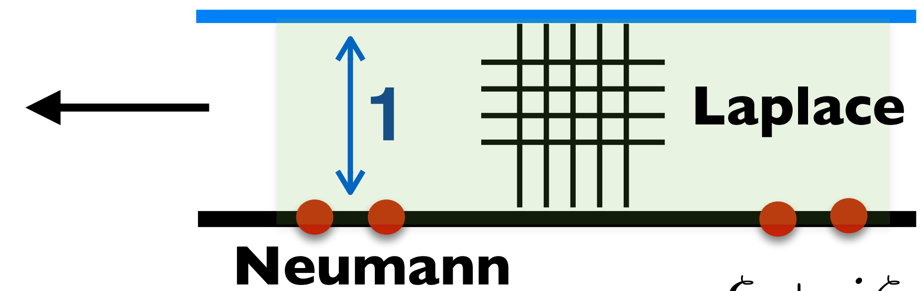
$$DtN_Z[\varphi](x,t) = \phi_z(x,0,t) = \frac{1}{M(\xi(x,0))} DtN_w[\varphi](\xi(x,0)).$$

Dirichlet: $\phi(x,0,t) = \varphi(x,t)$



$Z = x + iz$

Schwarz-Christoffel Toolbox, by Driscoll
a **numerical conformal mapping**



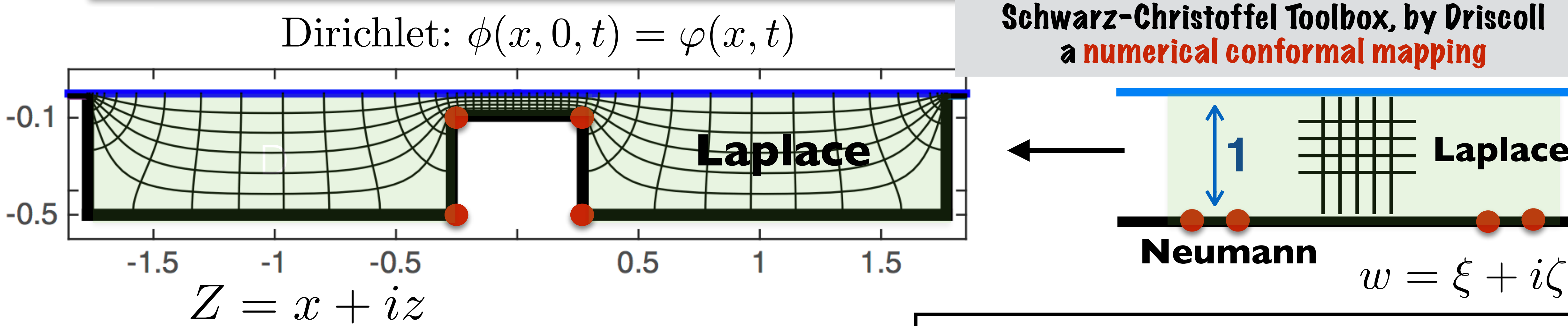
$w = \xi + i\zeta$

$$DtN_w[\varphi](\xi,t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k \tanh(k)\hat{\varphi}(k,t)e^{ik\xi}\,dk.$$

Z-plane **w-plane: uniform strip**

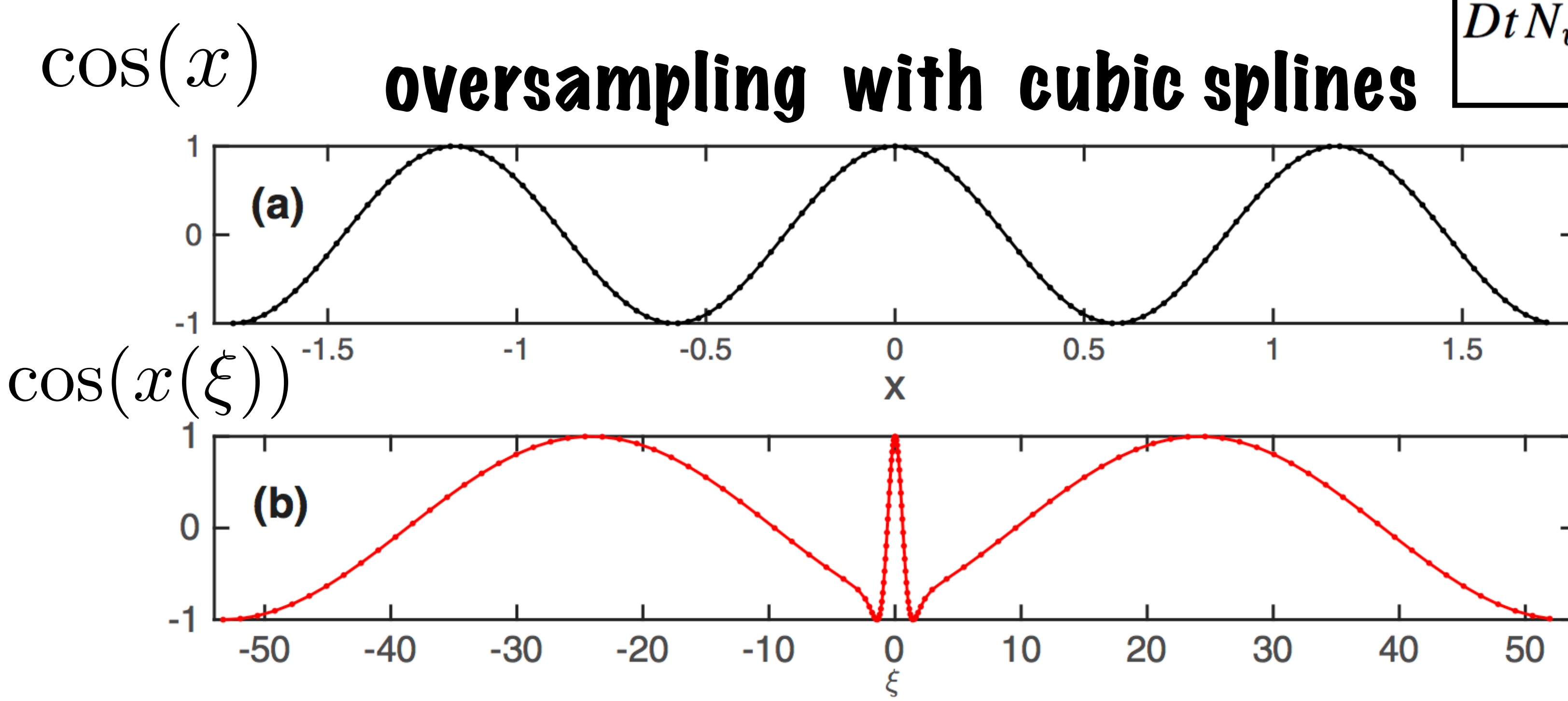
$$DtN_Z[\varphi](x,t) = \phi_z(x,0,t) = \frac{1}{M(\xi(x,0))} DtN_w[\varphi](\xi(x,0)).$$

Schwarz-Christoffel Toolbox, by Driscoll
a numerical conformal mapping



$$DtN_w[\varphi](\xi,t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k \tanh(k)\hat{\varphi}(k,t)e^{ik\xi}\,dk.$$

oversampling with cubic splines



$$\sqrt{|J|(\xi,0)} \equiv M(\xi)$$

$$|J| = \text{Jacobian}$$

PHYSICAL REVIEW FLUIDS 2, 034801 (2017)

Tunneling with a hydrodynamic pilot-wave model

André Nachbin,[1,3] Paul A. Milewski,[2] and John W. M. Bush[3]

J. Fluid Mech. (2012), *vol.* 695, *pp.* 288–309. © Cambridge University Press 2012 288
doi:10.1017/jfm.2012.19

Water waves over a variable bottom: a non-local formulation and conformal mappings

A. S. Fokas[1,2] and A. Nachbin[3]†

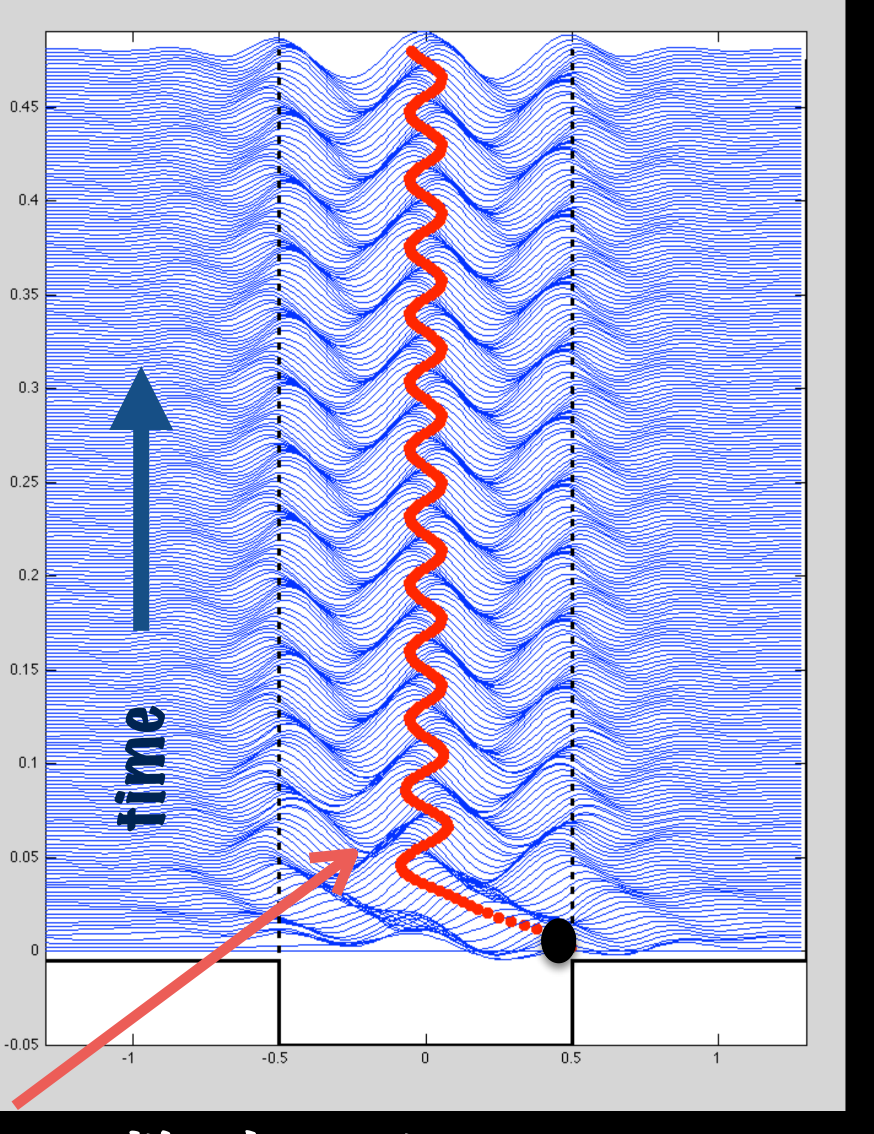
particles on a vibrating background

a particle on a potential of its own making

ONE CAVITY
a particle on a potential of its own making

Newton's Law:
droplet/particle dynamics

$$m\ddot{X}_1 + c\,F(t)\dot{X}_1 = -F(t)\,\frac{\partial \eta}{\partial x}(X_1(t), t).$$



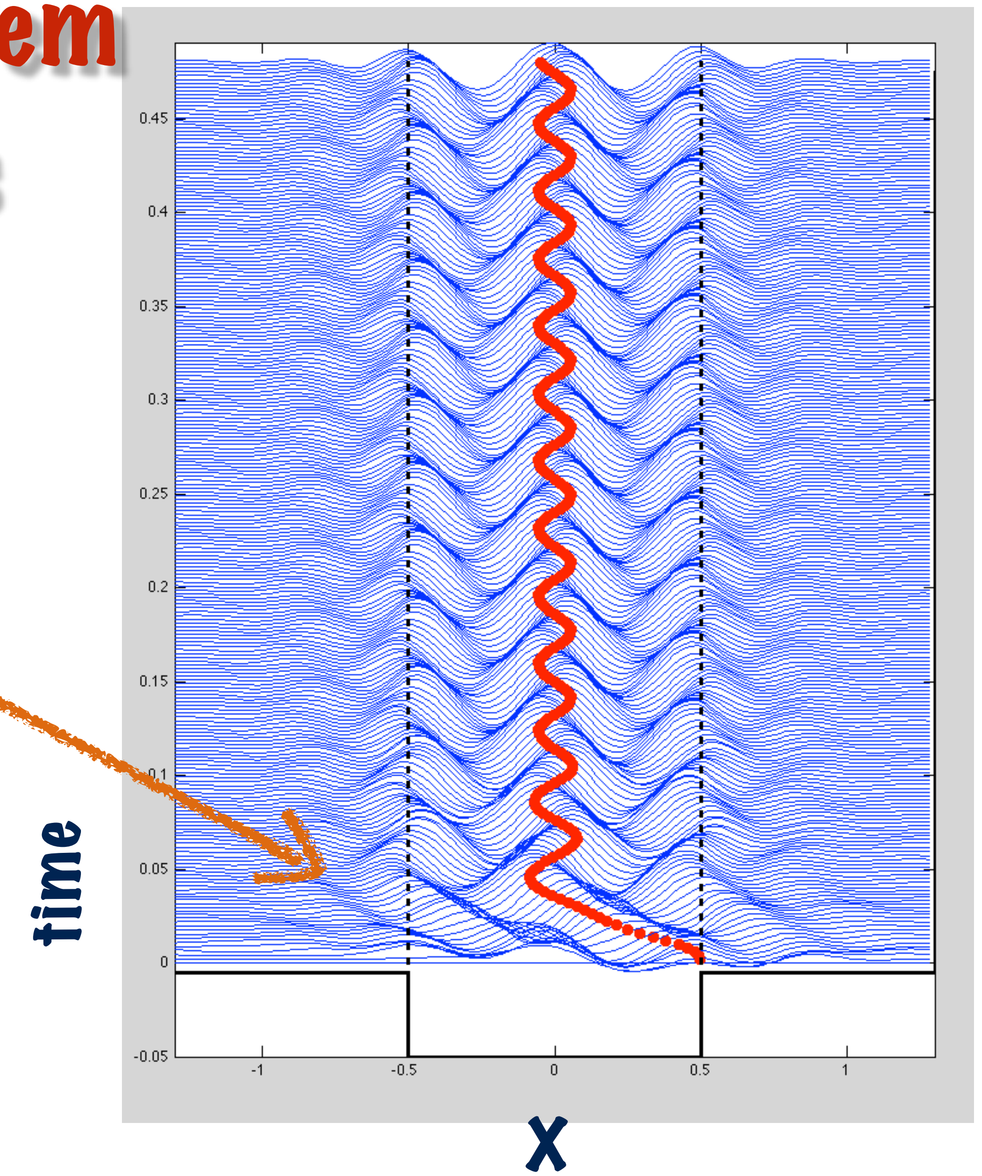
wave acting as a potential ; harmonic oscillator-like dynamics

To understand the underlying dynamical system lets take a pause with simpler models

Take a single droplet in a single cavity

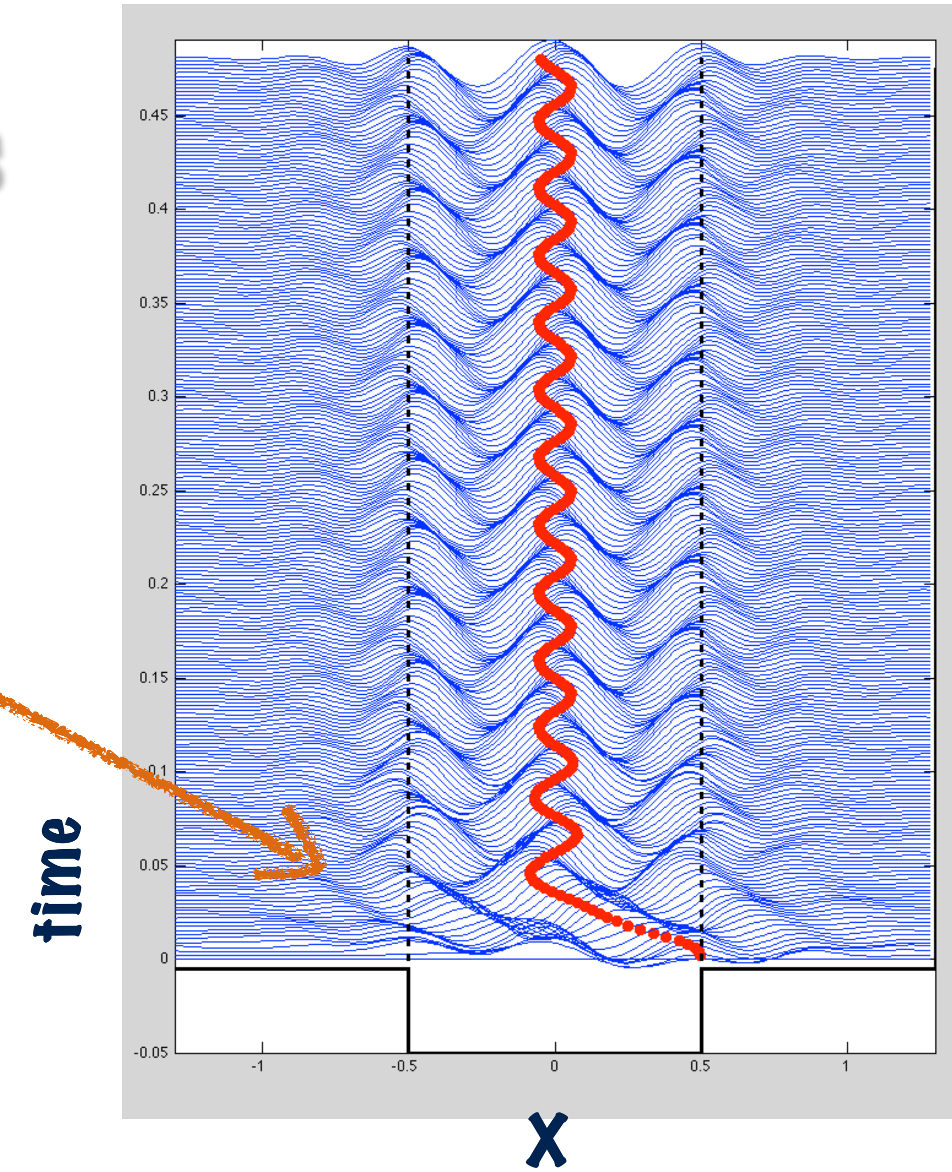
Looks like a harmonic oscillator

We observe a SLOSHING WAVE



To understand the underlying dynamical system lets take a pause with simpler models

Take a single droplet in a single cavity

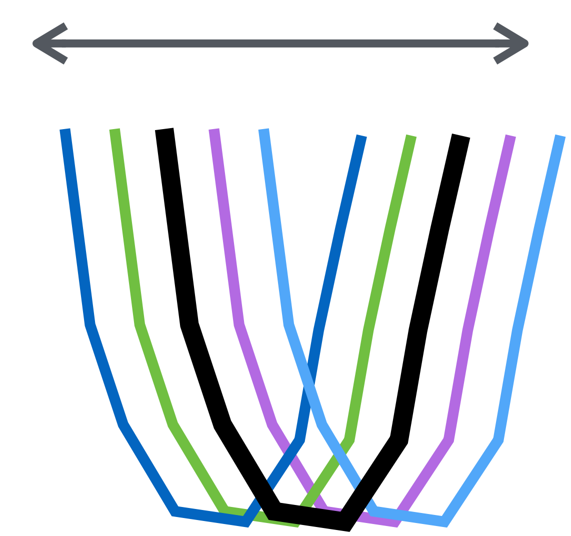


Take the simple damped and forced oscillator

$$m\frac{d^2\mathbf{X}}{dt^2} + d\,\frac{d\mathbf{X}}{dt} + \frac{d\mathbf{V}(\mathbf{X},t)}{d\mathbf{X}} = 0$$

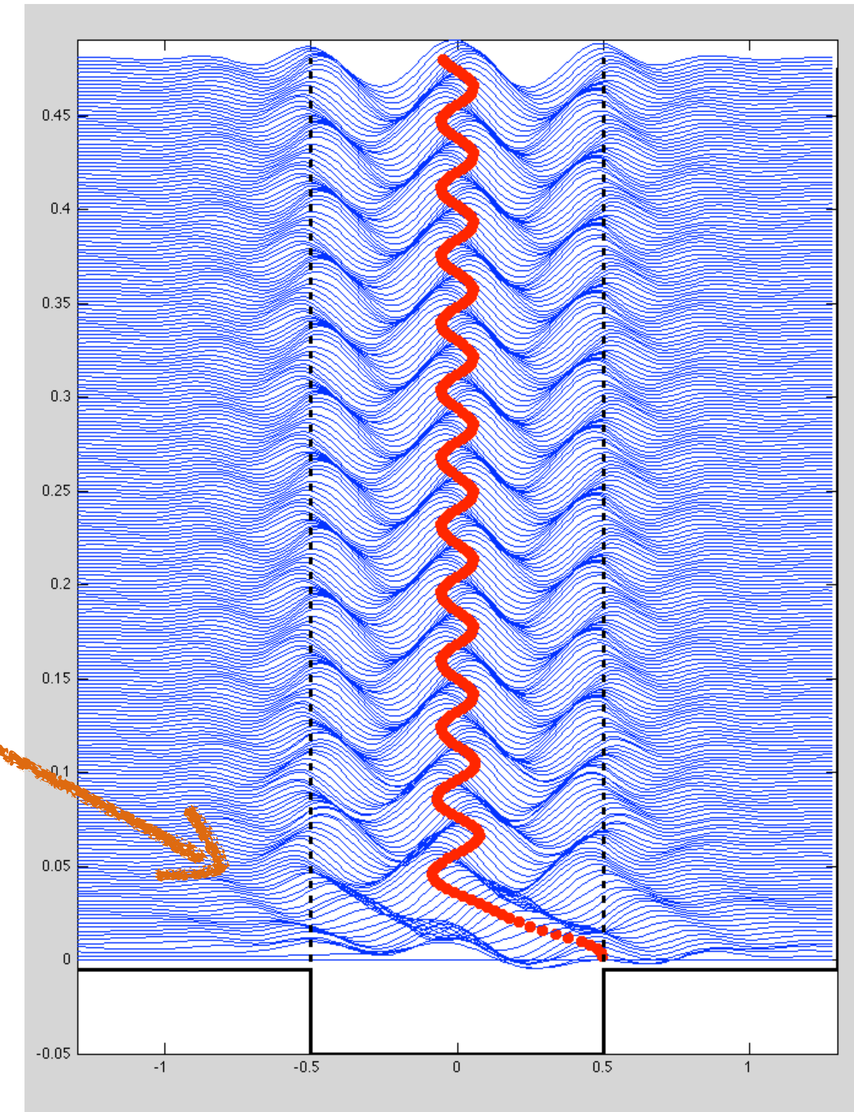
using the "ROCKING" Potential

$$\mathbf{V}(\mathbf{X},t) = \frac{K_0}{2}\left(\mathbf{X} - \frac{\varepsilon}{K_0}\sin(\omega t)\right)^2$$



To understand the underlying dynamical system lets take a pause with simpler models

Take a single droplet in a single cavity



Take the simple damped and forced oscillator

$$m\frac{d^2\mathbf{X}}{dt^2} + d\ \frac{d\mathbf{X}}{dt} + \frac{d\mathbf{V}(\mathbf{X},t)}{d\mathbf{X}} = 0$$

using the "ROCKING" Potential

$$\mathbf{V}(\mathbf{X},t) = \frac{K_0}{2}(\mathbf{X} - \frac{\varepsilon}{K_0}\sin(\omega t))^2$$

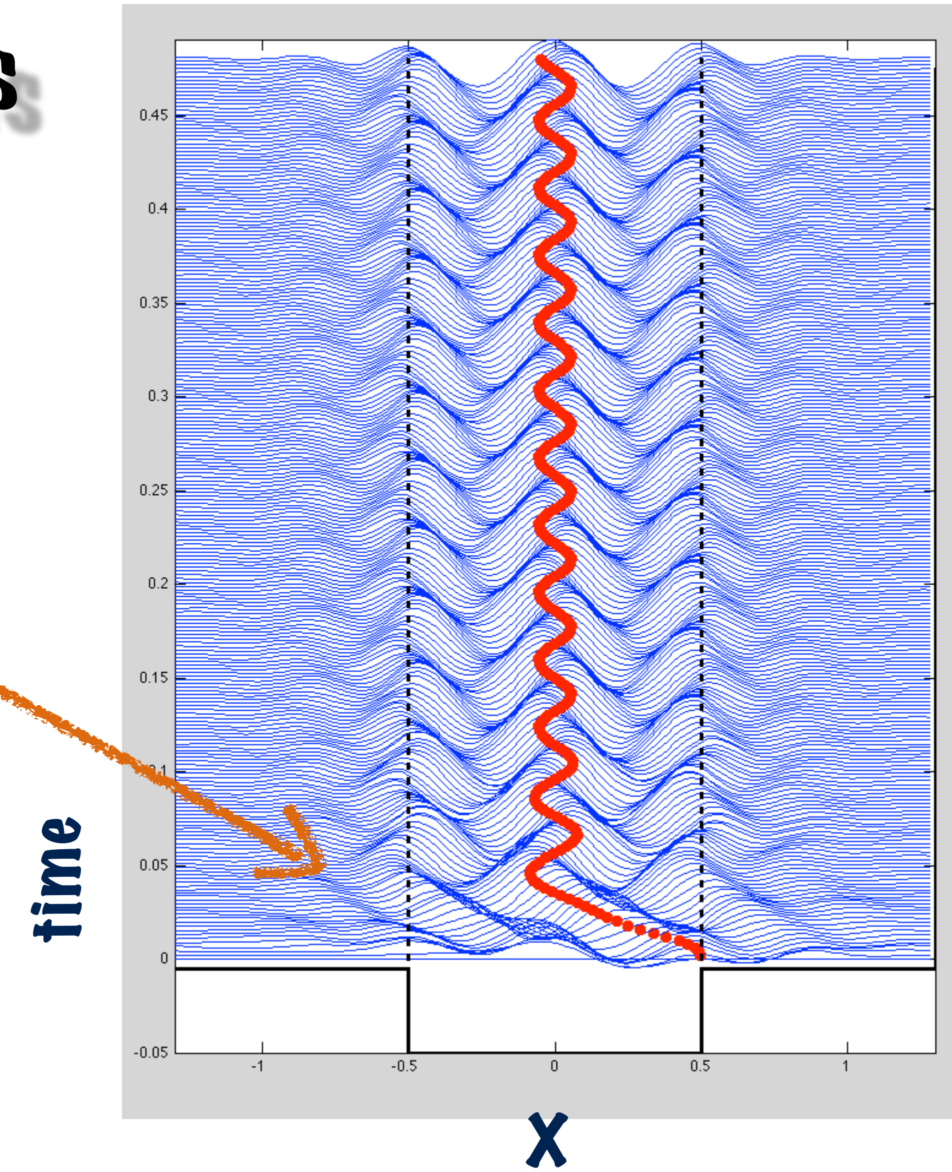
mass-spring

$$m\frac{d^2\mathbf{X}}{dt^2} + d\frac{\mathbf{dX}}{dt} + K_o \cdot \mathbf{X} = \varepsilon\sin(\omega t)$$

Lets take a pause with simpler models

Lets take a pause with simpler models

Take a single droplet in a single cavity



Take the simple damped and forced oscillator

$$m\frac{d^2\mathbf{X}}{dt^2} + d\,\frac{d\mathbf{X}}{dt} + \frac{d\mathbf{V}(\mathbf{X},t)}{d\mathbf{X}} = 0$$

using the "ROCKING" Potential

$$\mathbf{V}(\mathbf{X},t) = \frac{K_0}{2}\left(\mathbf{X} - \frac{\varepsilon}{K_0}\sin(\omega t)\right)^2$$

("rocking" parabola)

mass-spring

$$m\frac{d^2\mathbf{X}}{dt^2} + d\frac{\mathbf{dX}}{dt} + K_o \cdot \mathbf{X} = \varepsilon \sin(\omega t)$$

sloshing wave

$$m\frac{d^2\mathbf{X}}{dt^2} + c\,F(t)\,\frac{\mathbf{dX}}{dt} = -\mathbb{1}_{t_c=\frac{1}{5}T_F}\,G(t)\,\frac{d\eta}{dx}(\mathbf{X},t)$$

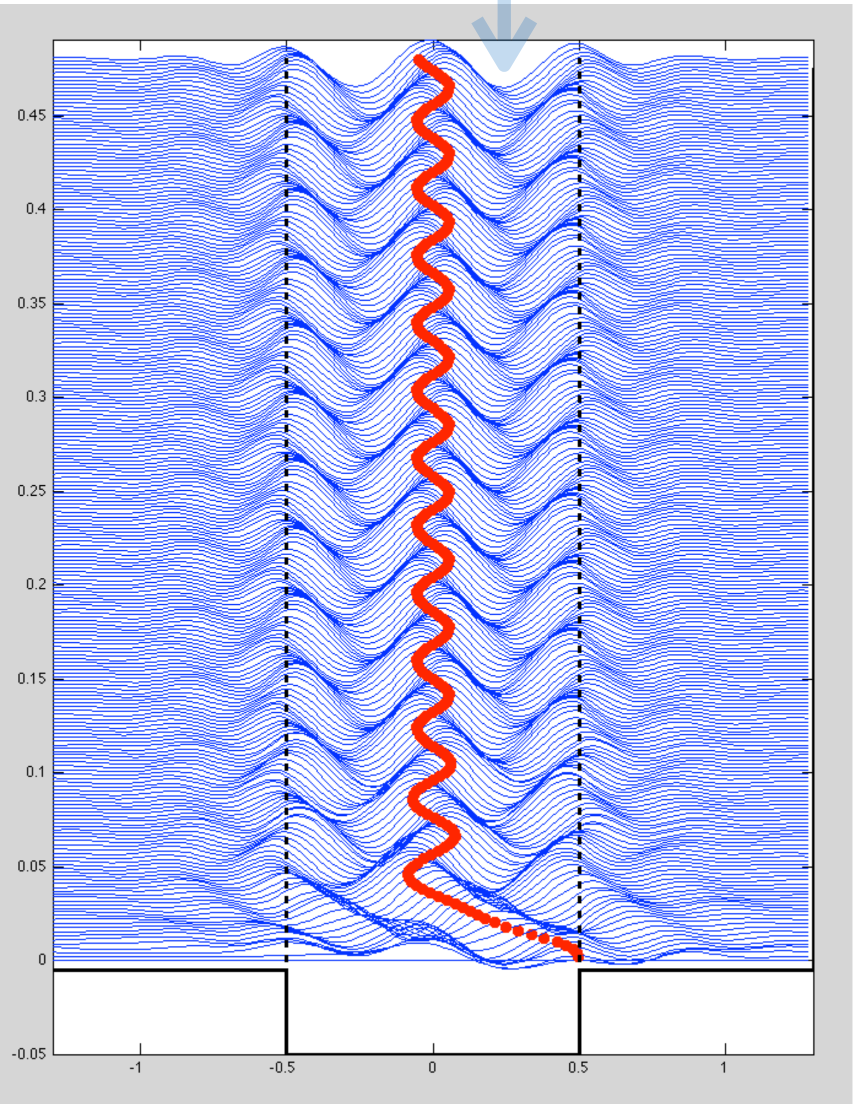
due to flight $F(t) \equiv \mathbb{1}_{t_c=\frac{1}{5}T_F}\,G(t)$

("rocking" cosine) $$\mathbf{V}(\mathbf{x}, t) = \mathbf{K}_o(1 - \cos(\mathbf{x} - \varepsilon \sin(\omega t))$$

$$\frac{d\mathbf{V}}{d\mathbf{x}} = \mathbf{K}_o \sin(\mathbf{x} - \varepsilon \sin(\omega t))$$

"ROCKING" POTENTIAL

SLOSHING WAVE



$$m\frac{d^2\mathbf{x}}{dt^2} + d\,\frac{d\mathbf{x}}{dt} + \frac{d\mathbf{V}}{d\mathbf{x}} = 0$$

$$\mathbf{V}(\mathbf{x},t) = \mathbf{K}_o(1-\cos(\mathbf{x}-\varepsilon\sin(\omega t)))$$

$$\frac{d\mathbf{V}}{d\mathbf{x}} = \mathbf{K}_o\sin(\mathbf{x}-\varepsilon\sin(\omega t))$$

forced **DUFFING's eq.**

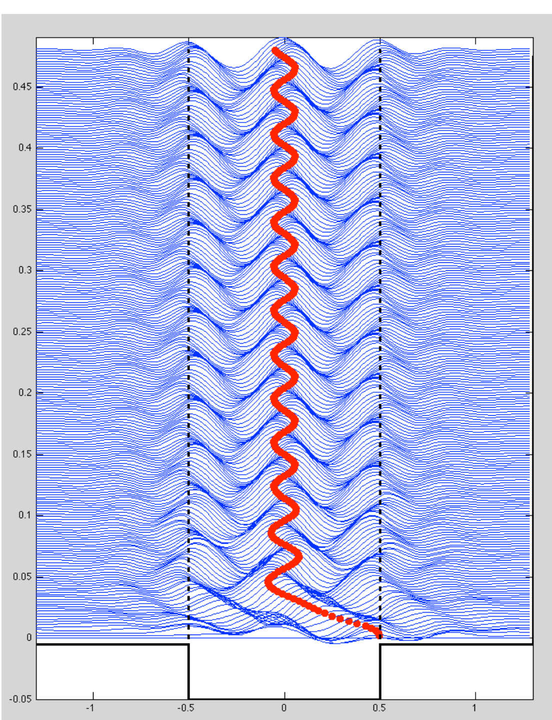
SCHOLARPEDIA
the peer-reviewed
open-access encyclopedia

small deflections: expand...

Duffing oscillator is an example of a periodically forced oscillator with a nonlinear elasticity, written as

$$\ddot{x} + \delta\dot{x} + \beta x + \alpha x^3 = \gamma\cos\omega t\,,$$

where the damping constant obeys $\delta \geq 0$, and it is also known as a simple model which yields chaos, as well as van der Pol oscillator.



for a fixed and small sinusoidal **WAVE SLOSHING** this resembles a **DUFFING** eq.

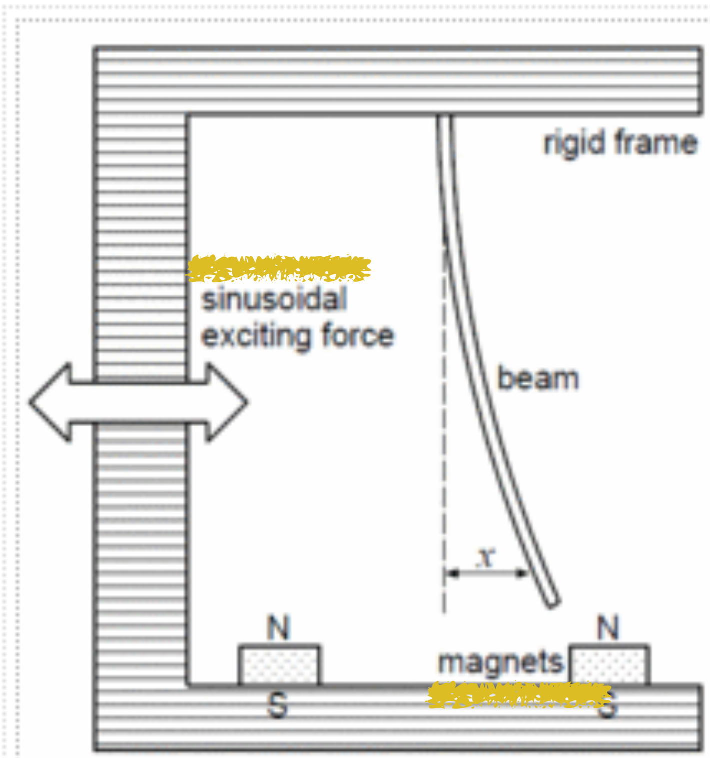


Figure 3: For $\beta < 0$, the Duffing oscillator can be regarded as a model of a periodically forced steel beam which is deflected toward the two magnets.

Scientific Computing

PRL 102, 240401 (2009) PHYSICAL REVIEW LETTERS week ending 19 JUNE 2009

Unpredictable Tunneling of a Classical Wave-Particle Association

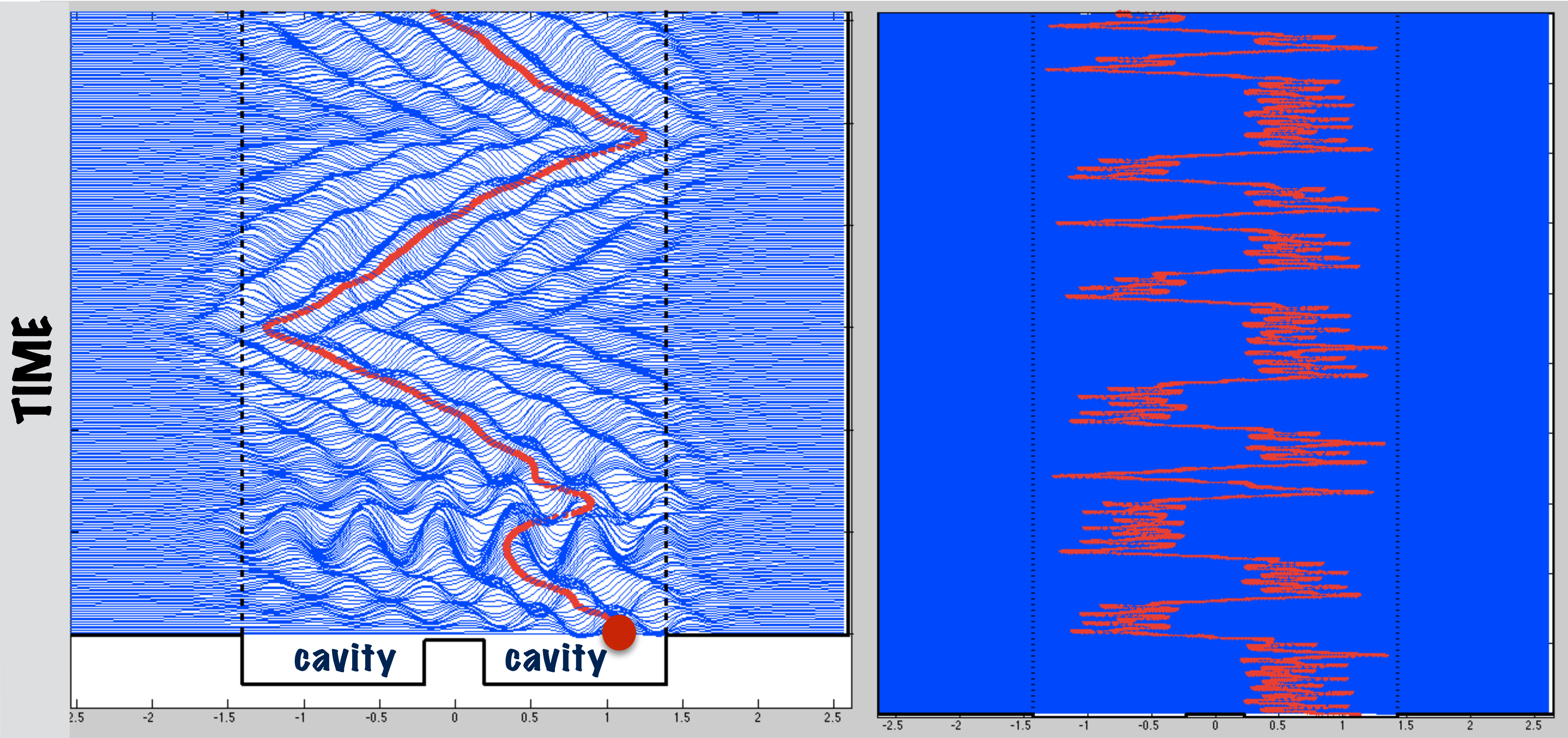
A. Eddi,[1] E. Fort,[2] F. Moisy,[3] and Y. Couder[1]

PHYSICAL REVIEW FLUIDS 2, 034801 (2017)

Tunneling with a hydrodynamic pilot-wave model

André Nachbin,[1,3] Paul A. Milewski,[2] and John W. M. Bush[3]

Effects at a distance



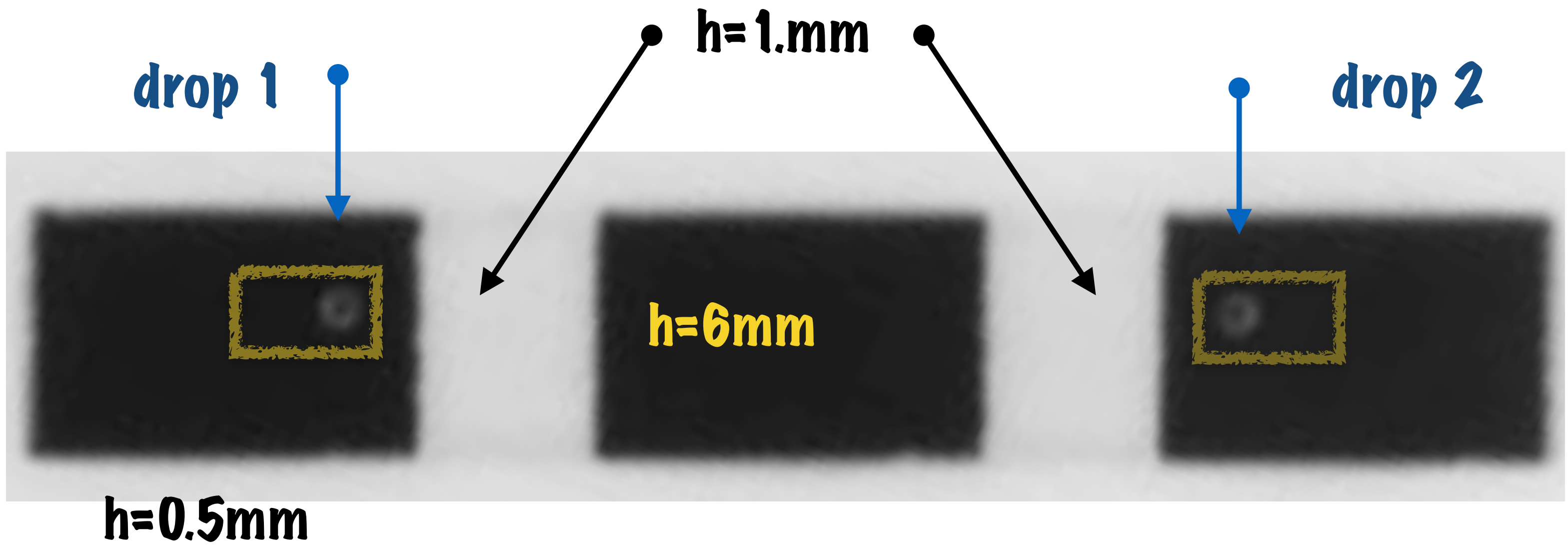
2D fluid/1D waves with BOUNDARIES

only HORIZONTAL DYNAMICs

CONFIRMING nonlocal effect

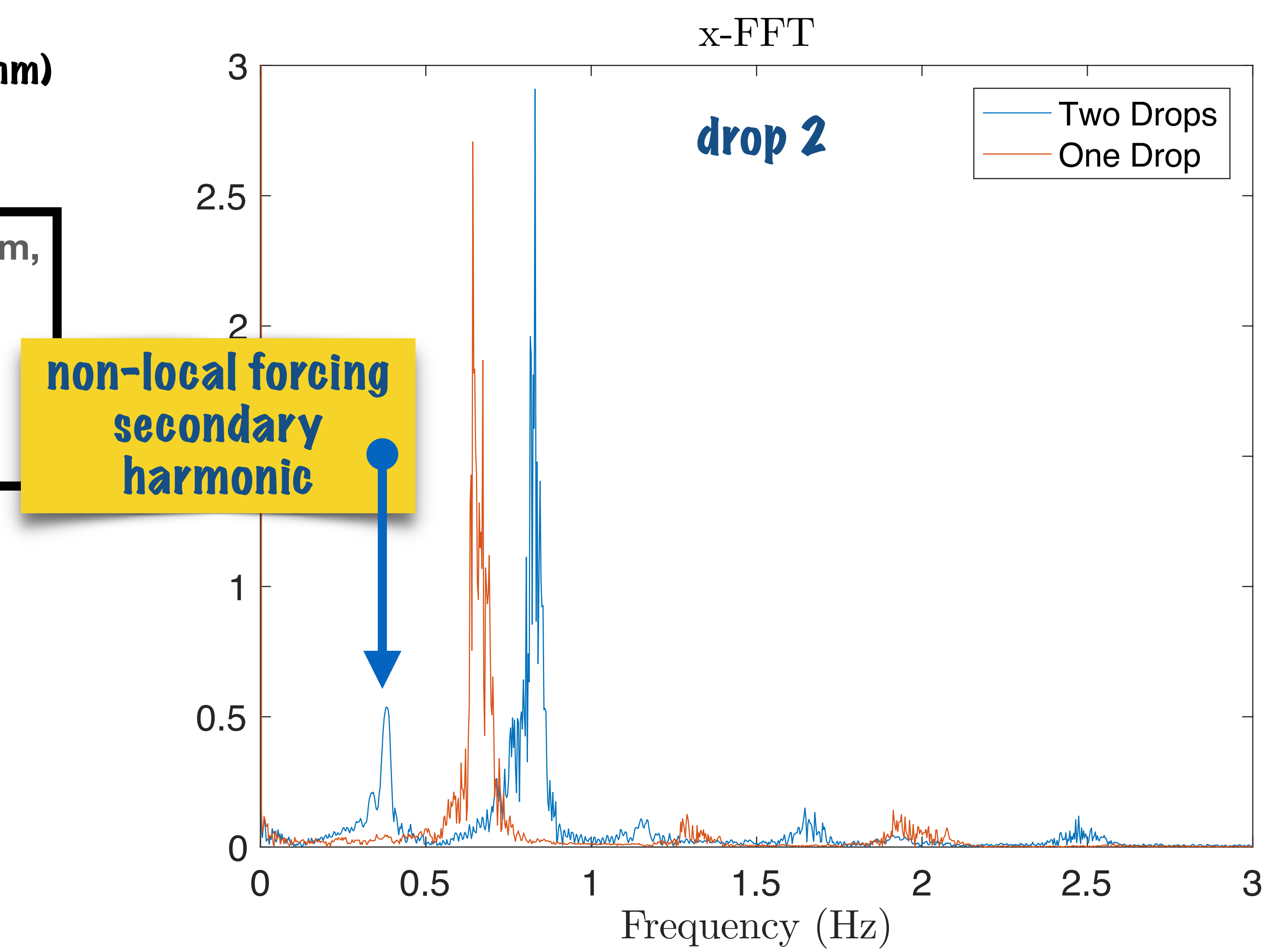
laboratory experiment by Miles Couchman (MIT, 2016)

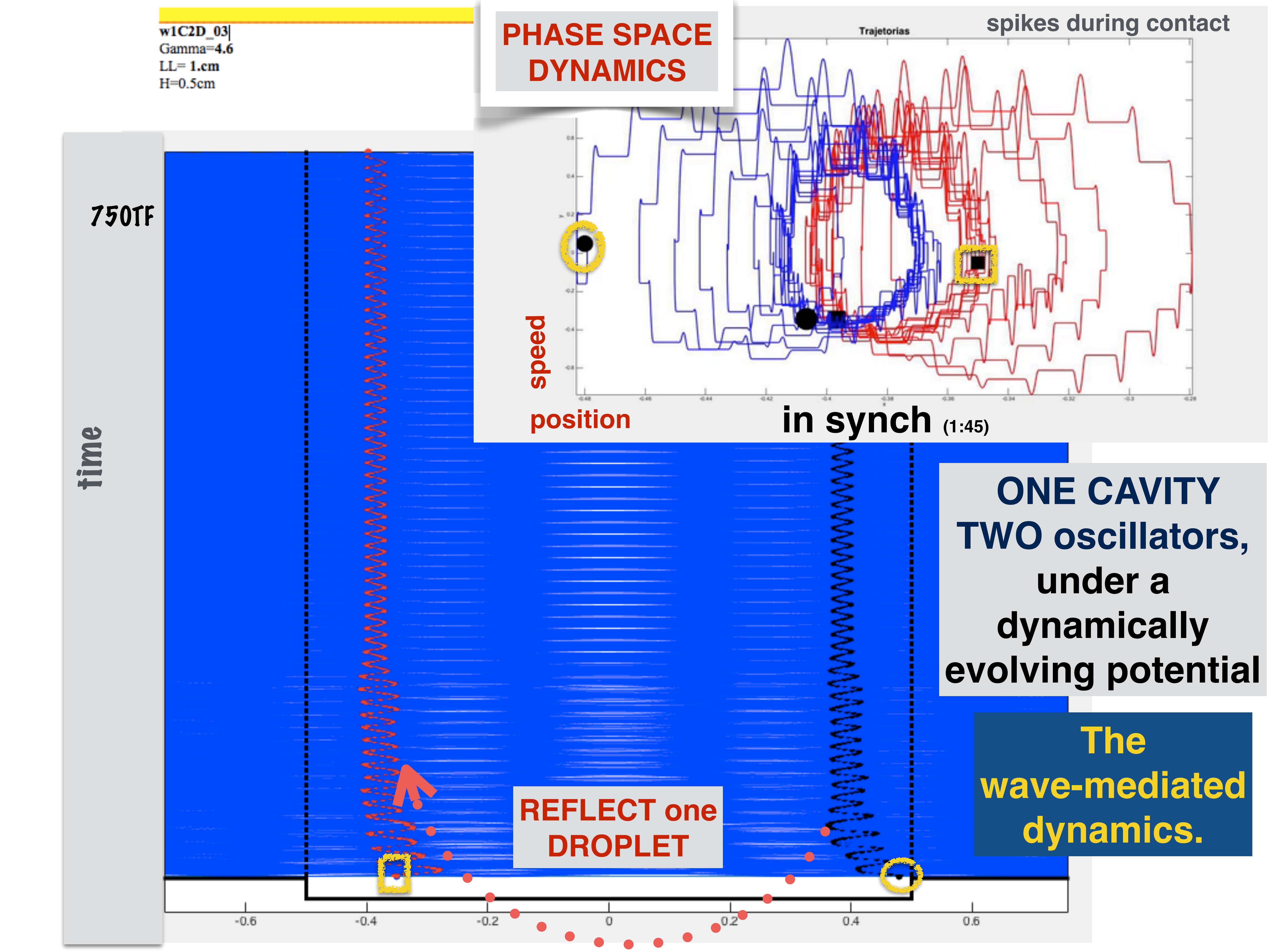
3 Cavities



2 x 1.5 Faraday wavelengths (lambda_F = 4.75mm)

Non-local features of a hydrodynamic pilot-wave system,
Nachbin, Couchman & Bush
APS Division of Fluid Dynamics (Fall) 2016
Bibcode: 2016APS..DFDL16005N





w1C2D_03
Gamma=4.6
LL= 1.cm
H=0.5cm
PHASE SPACE DYNAMICS
spikes during contact
Trajetorias
750TF
time
speed
position
in synch (1:45)
ONE CAVITY TWO oscillators, under a dynamically evolving potential
The wave-mediated dynamics.
REFLECT one DROPLET
-0.6
-0.4
-0.2
0
0.2
0.4
0.6

Coupled oscillators that can spontaneously synchonize

Kuramoto model: Winfree '67; Kuramoto '75

(phenomenological model for phase transition from **incoherence** to a **coherent** state)

- limit-cycle -
coupled phase oscillators

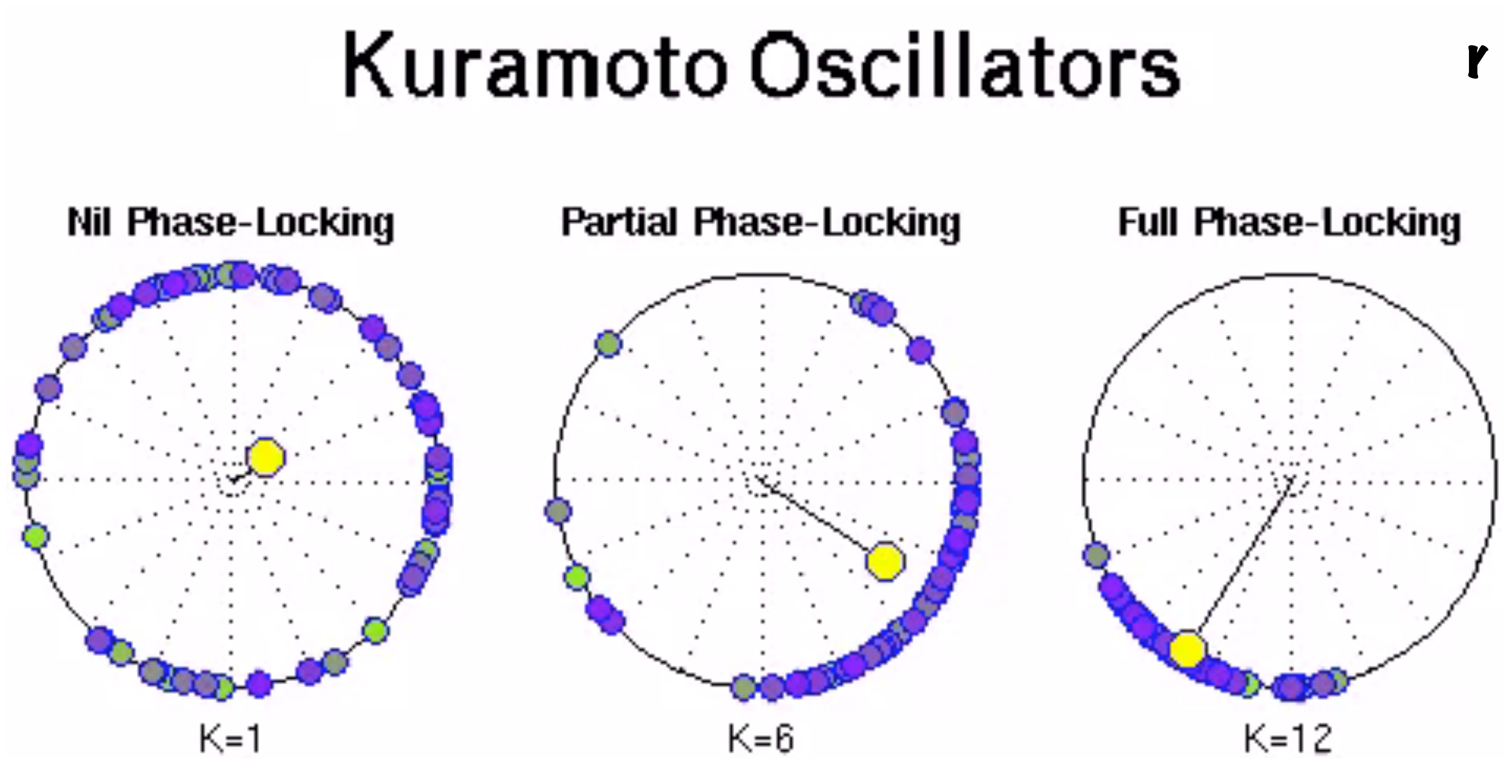
$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} K_{ij} \sin(\theta_j - \theta_i), \quad i = 1, ..., N,$$

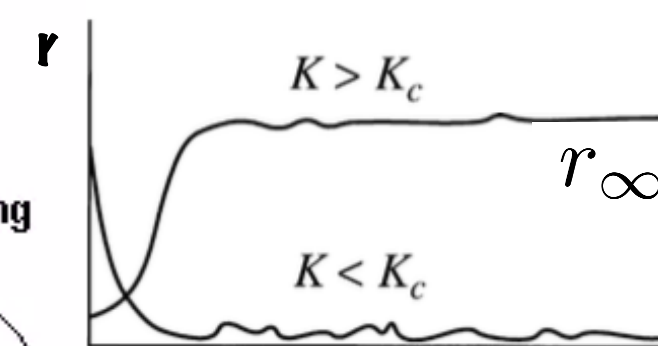
phase change | nat. freq. (random) | coupling matrix | nonlinear coupling

order parameter
r

$$r\, e^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

centroid

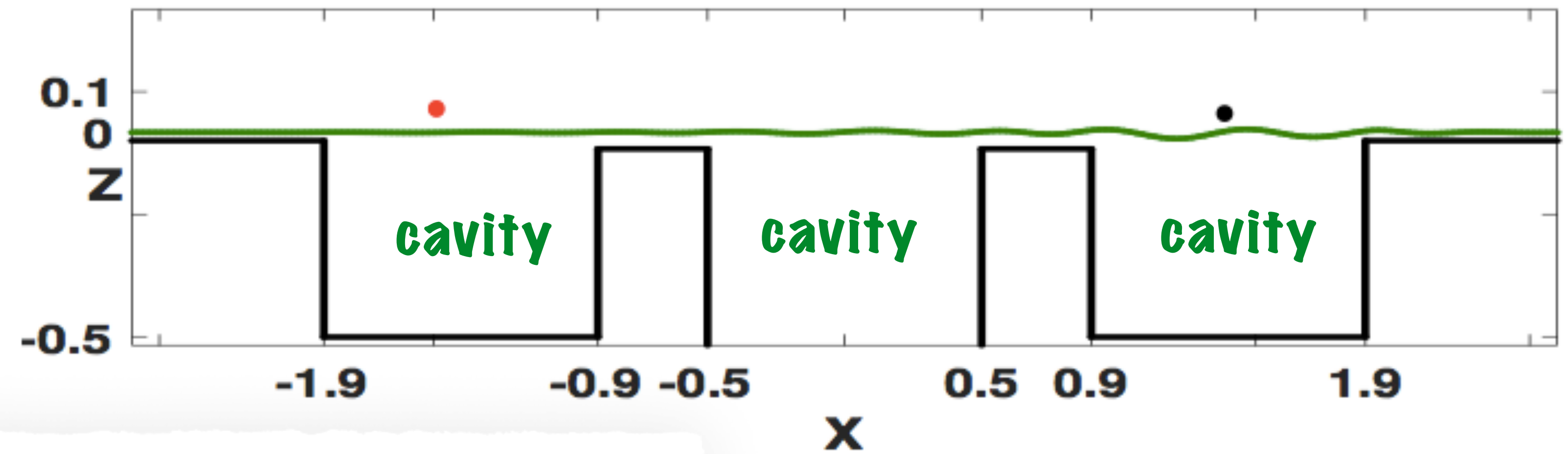




critical Kc

https://en.wikipedia.org/wiki/Kuramoto_model

OSCILLATING DROPLETs that can SPONTANEOUSLY SYNC



$$m\ddot{X}_1 + c\,F(t)\dot{X}_1 = -F(t)\,\frac{\partial \eta}{\partial x}(X_1(t), t).$$

$$m\ddot{X}_2 + c\,F(t)\dot{X}_2 = -F(t)\,\frac{\partial \eta}{\partial x}(X_2(t), t).$$

droplets on a **POTENTIAL** of their own **MAKING**: the **WAVE**

IMPLICIT

WAVE-MEDIATED COUPLING THROUGH

a PDE w/ FEEDBACK

contact time $T_c \equiv T_F/4$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} + 2\nu\frac{\partial^2 \eta}{\partial x^2},$$

$$\frac{\partial \phi}{\partial t} = -g(t)\eta + \frac{\sigma}{\rho}\frac{\partial^2 \eta}{\partial x^2} + 2\nu\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{\rho}P_d(x - X_1(t)) - \frac{1}{\rho}P_d(x - X_2(t)),$$

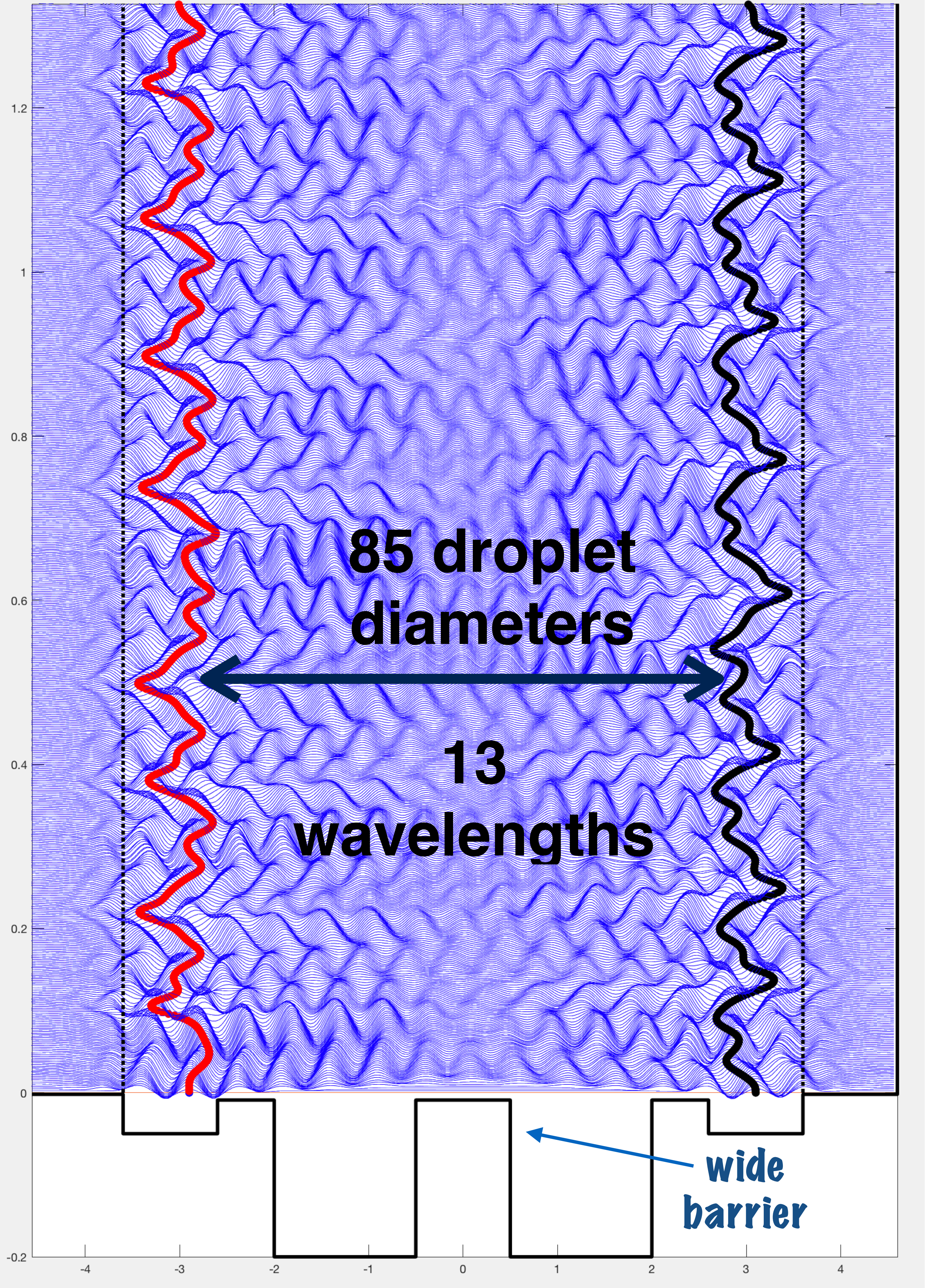
2 wave makers

TWO
DROPLETS
FAR APART

Walking droplets correlated at a distance

André Nachbin

Citation: Chaos **28**, 096110 (2018); doi: 10.1063/1.5050805



droplet's disordered oscillation in time

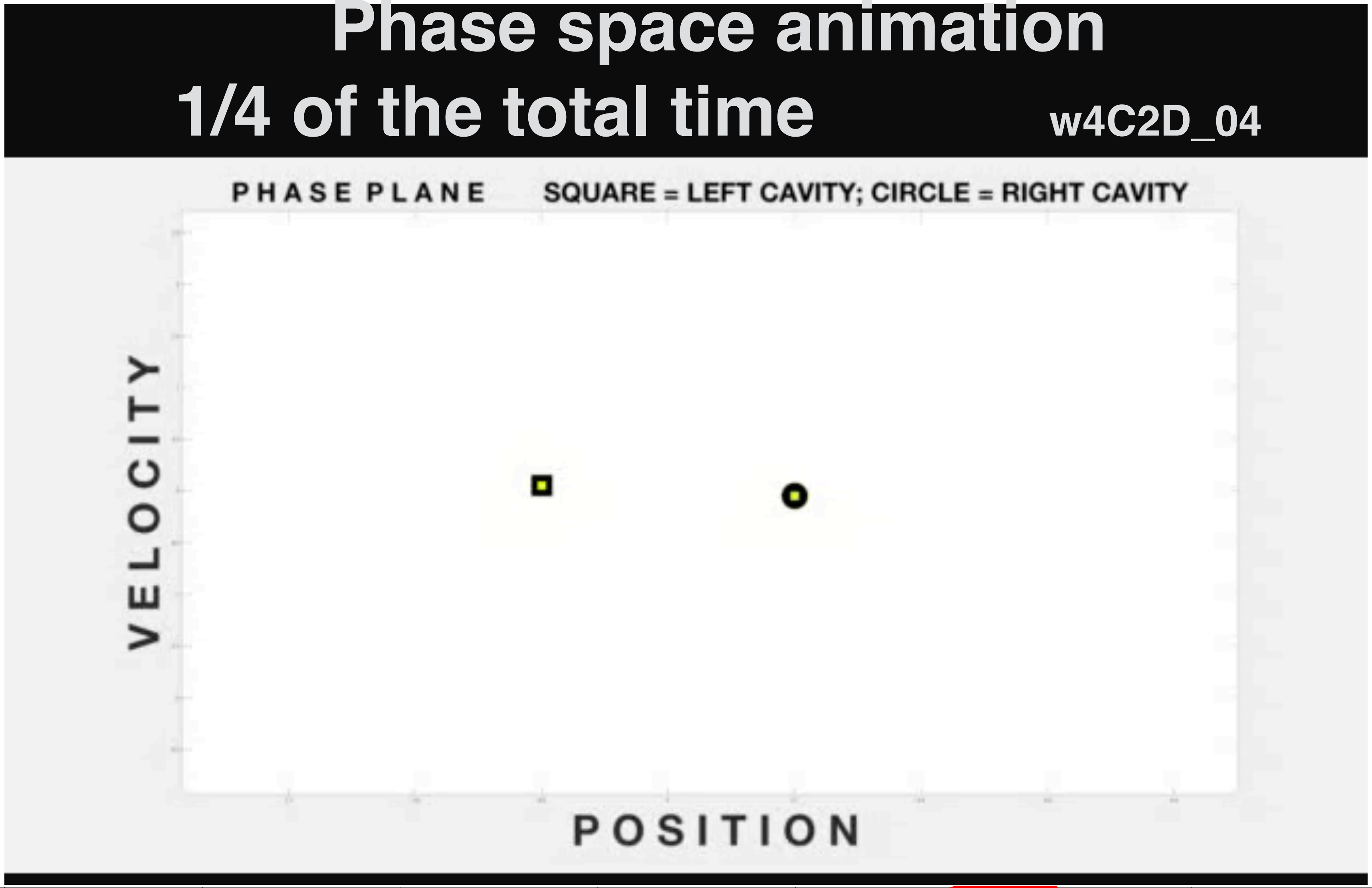
Not in SYNC!

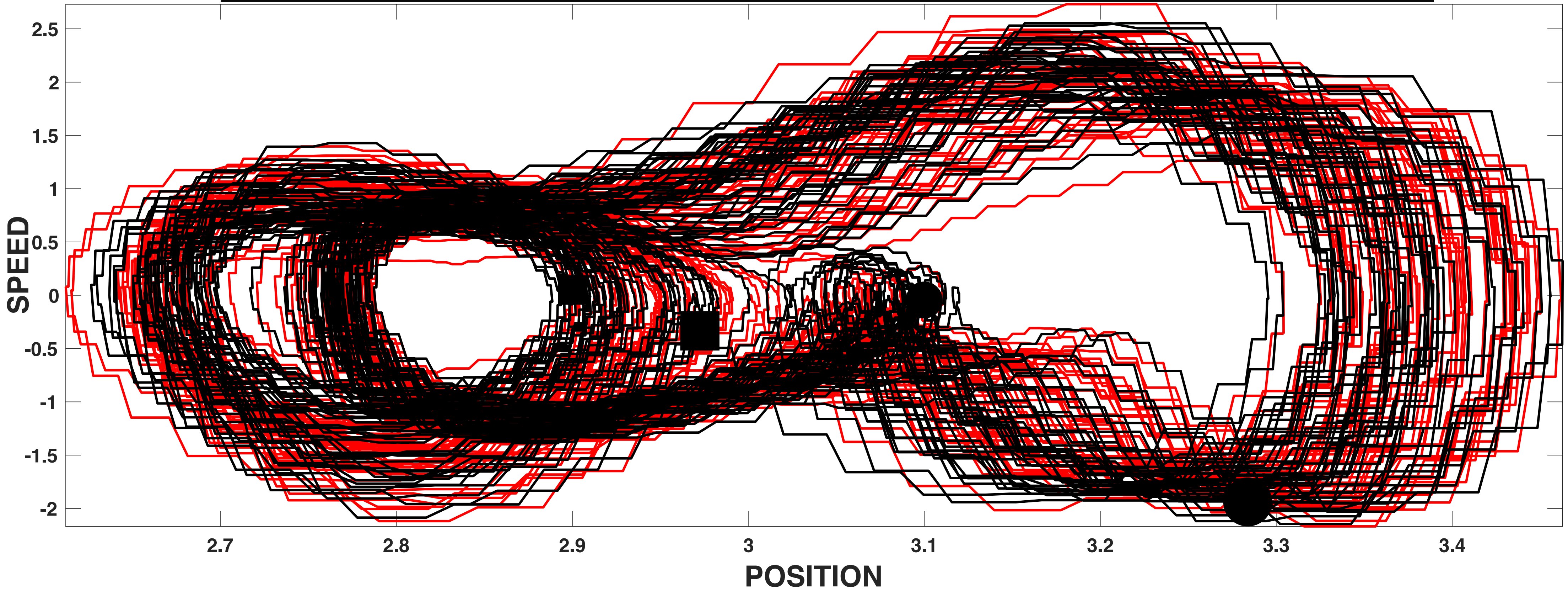
Is the dynamics SEPARABLE?

↓

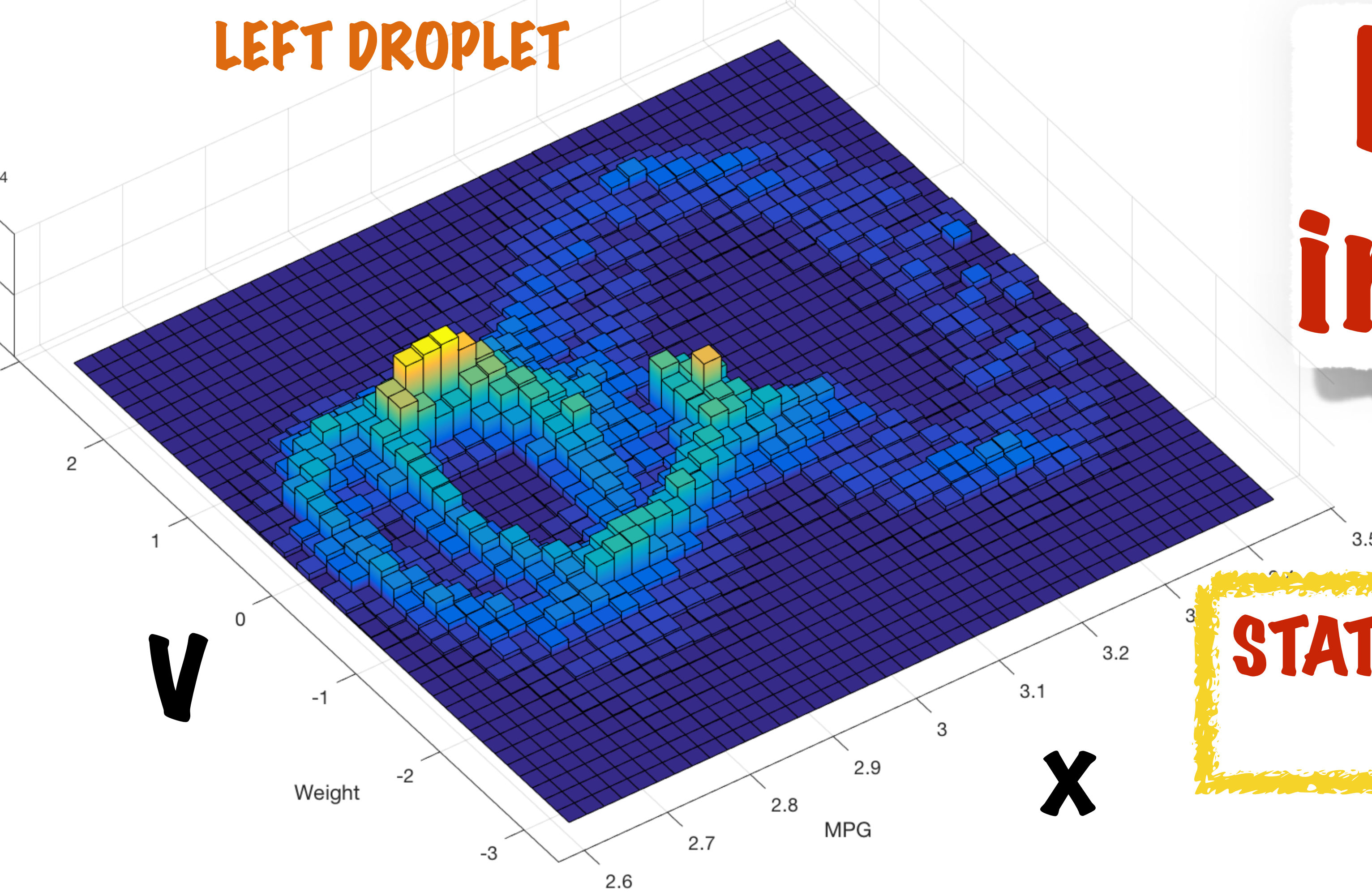
in the sense of being 2 independent sub-systems?

FAILs to SYNC



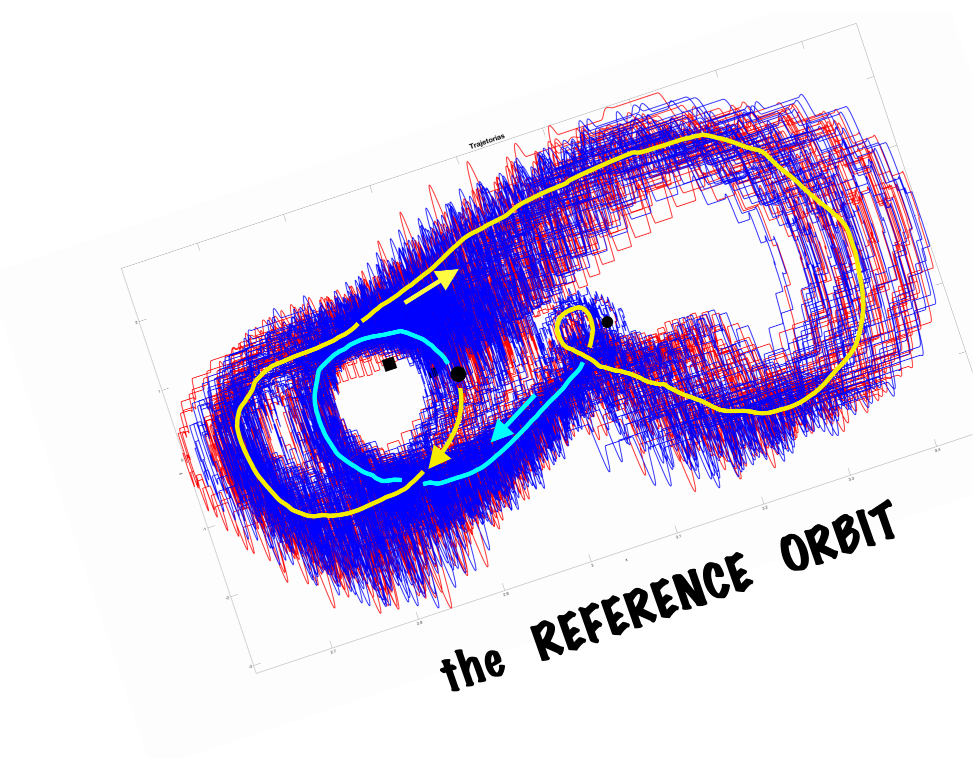


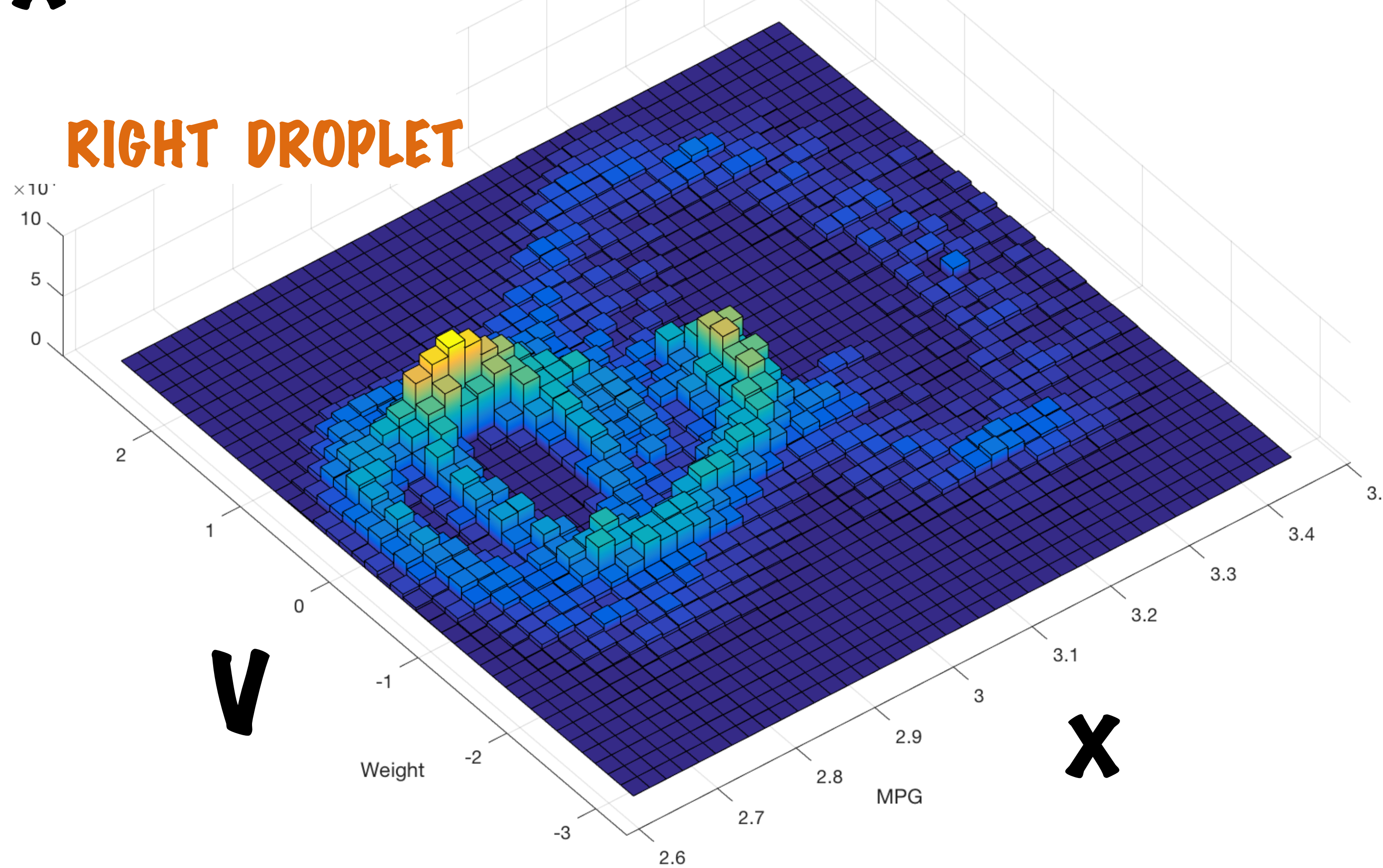
HISTOGRAMS in PHASE SPACE



STATISTICALLY INDISTINGUISHABLE

STATISTICAL COHERENCE



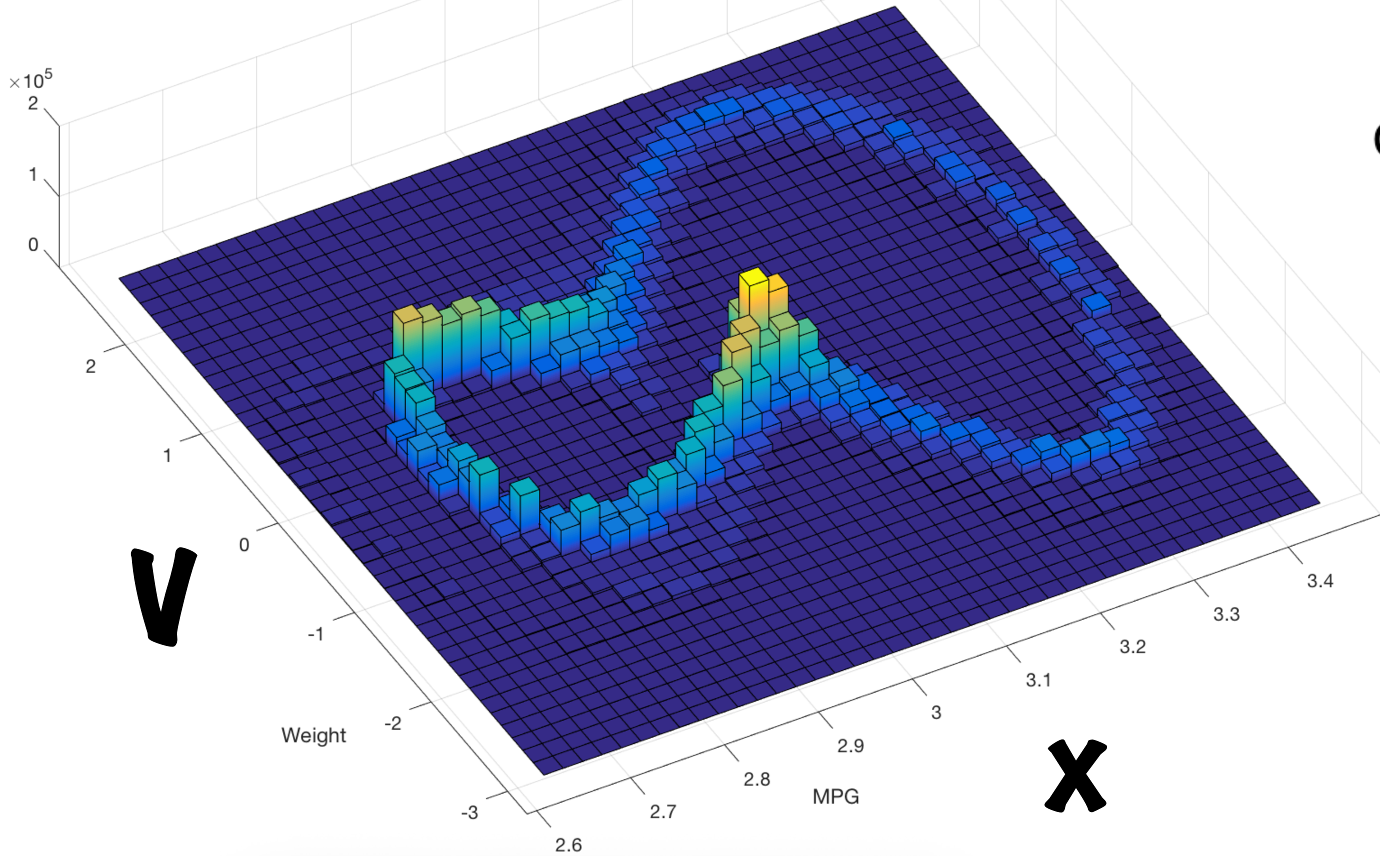


SINGLE DROPLET DYNAMICS
RIGHT DROPLET

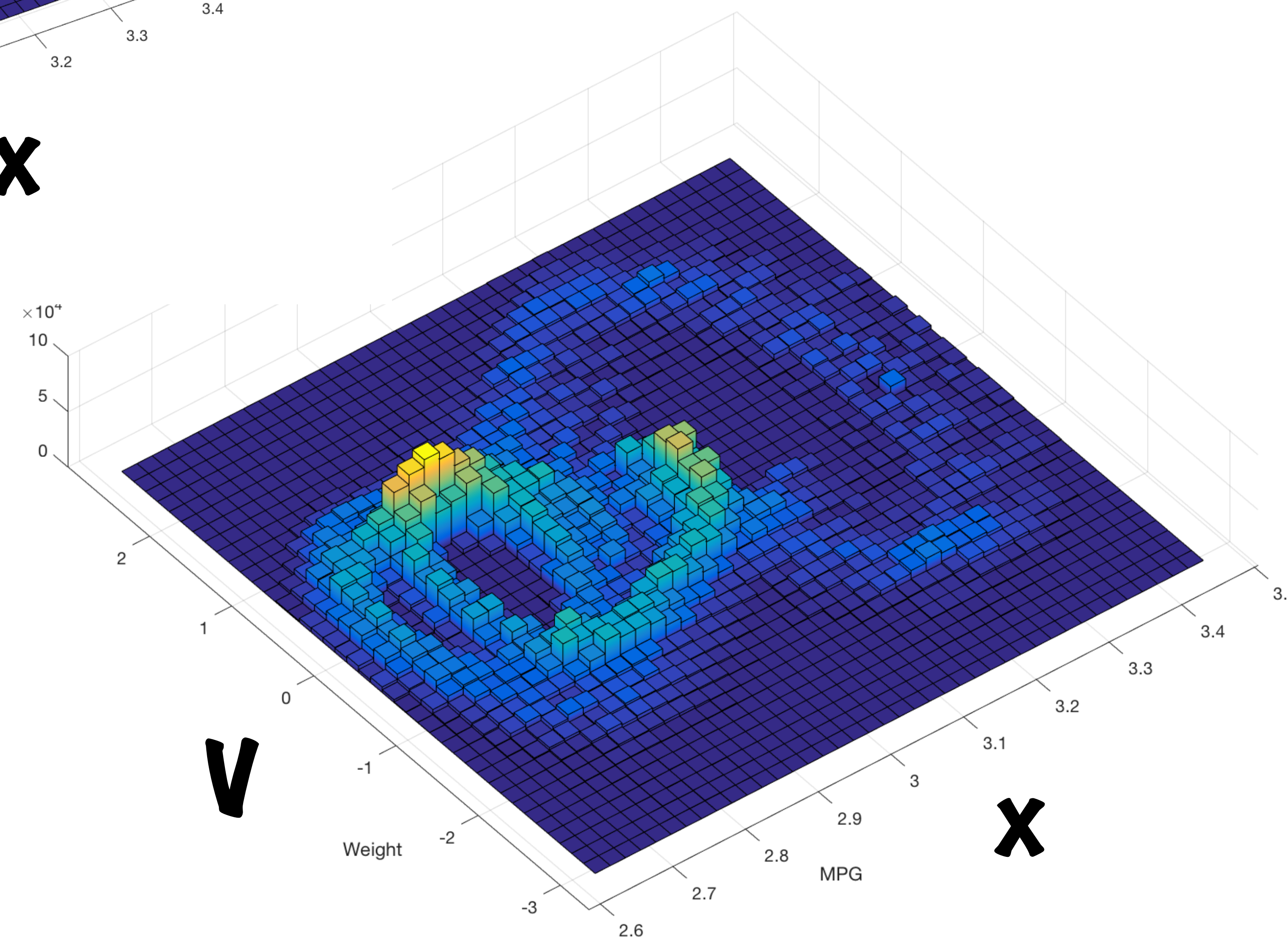
Walking droplets correlated at a distance

André Nachbin

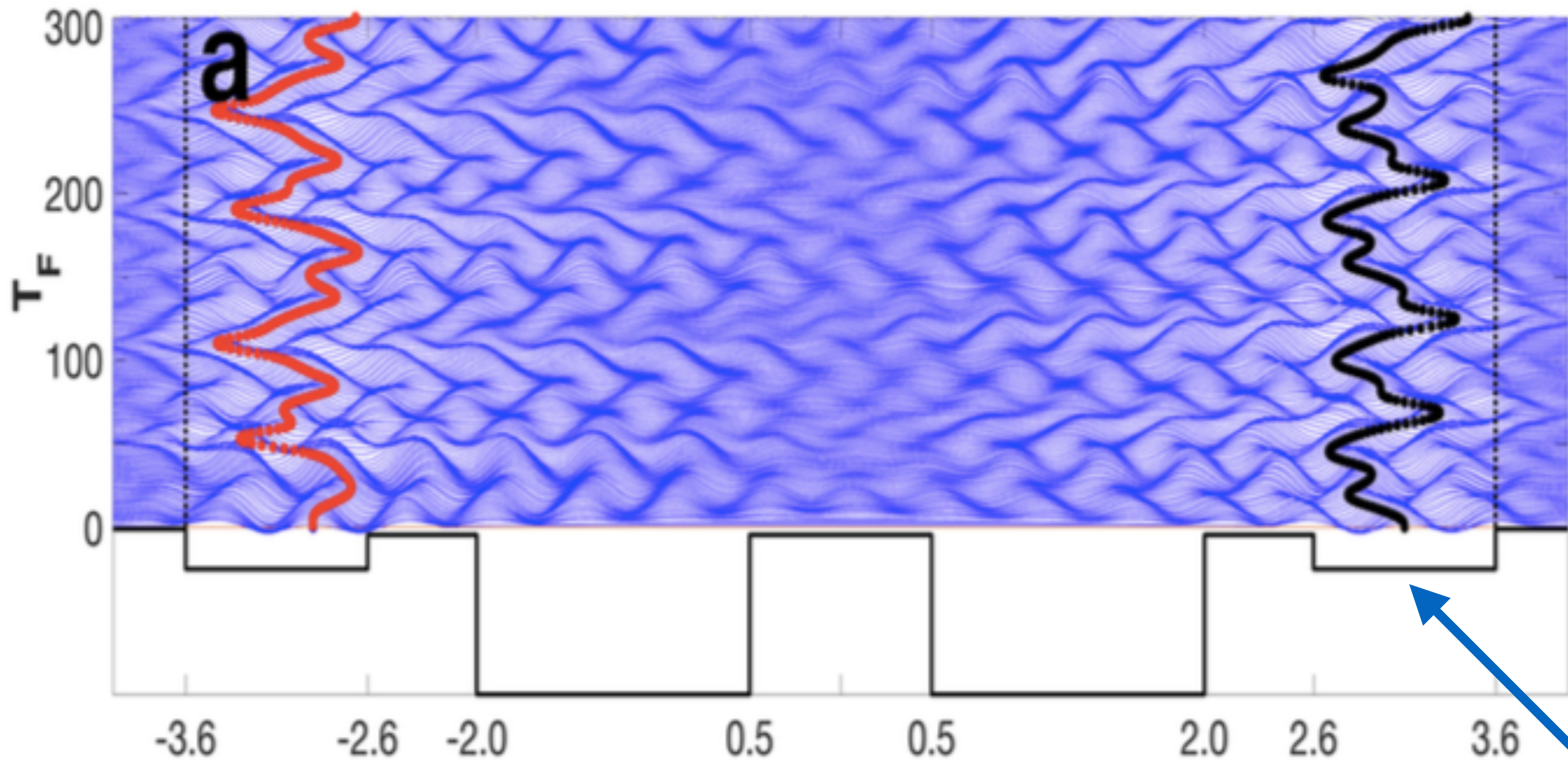
Citation: Chaos **28**, 096110 (2018); doi: 10.1063/1.5050805



Phase space dynamics is described by the **SYSTEM as a WHOLE** and NOT by each subsystem independently

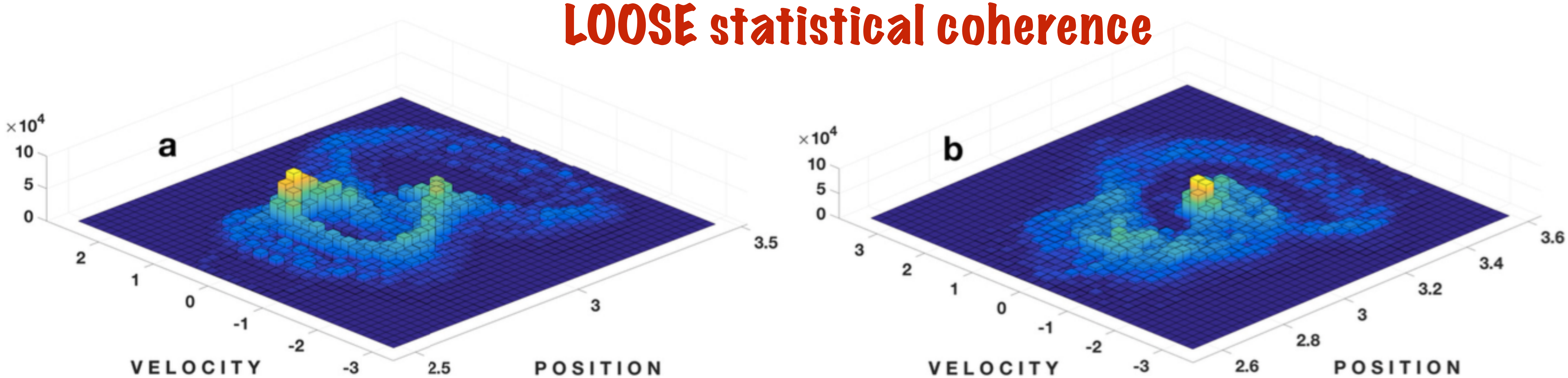


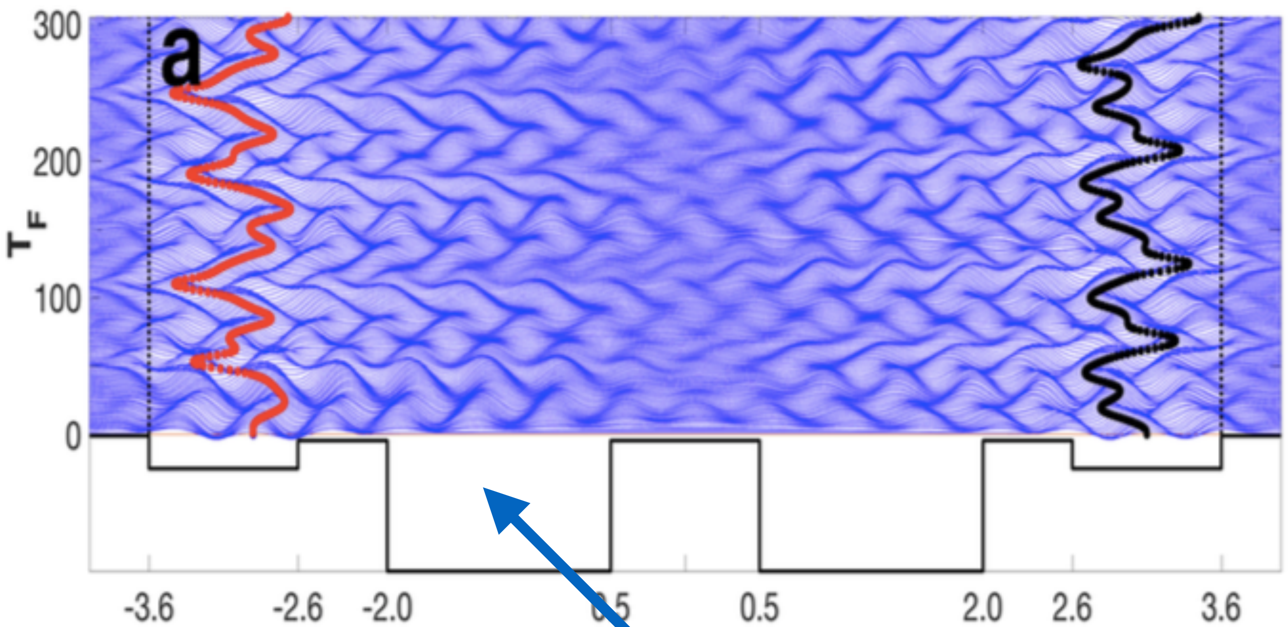
dynamics not separable
statistics not factorizable



perturbed to 1.2cm
"different particle"

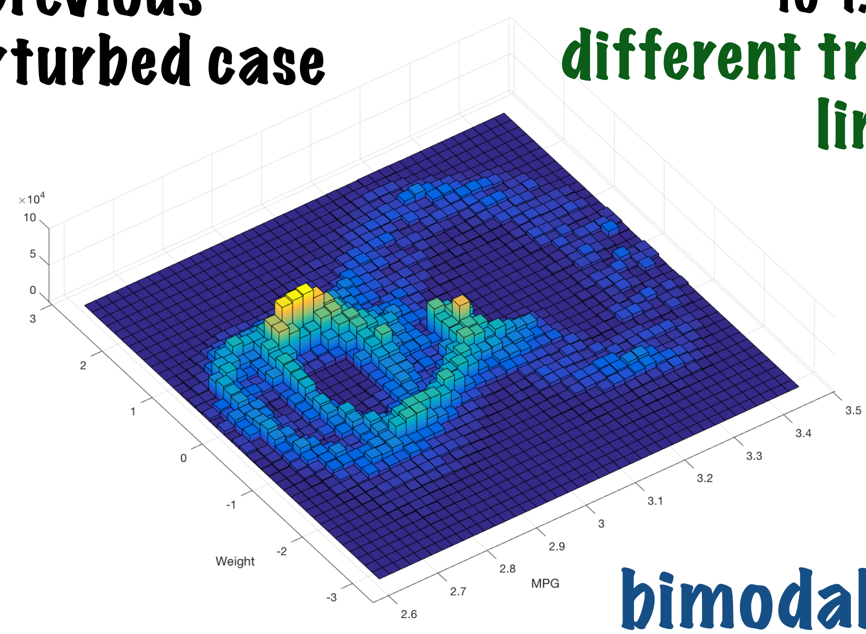
LOOSE statistical coherence



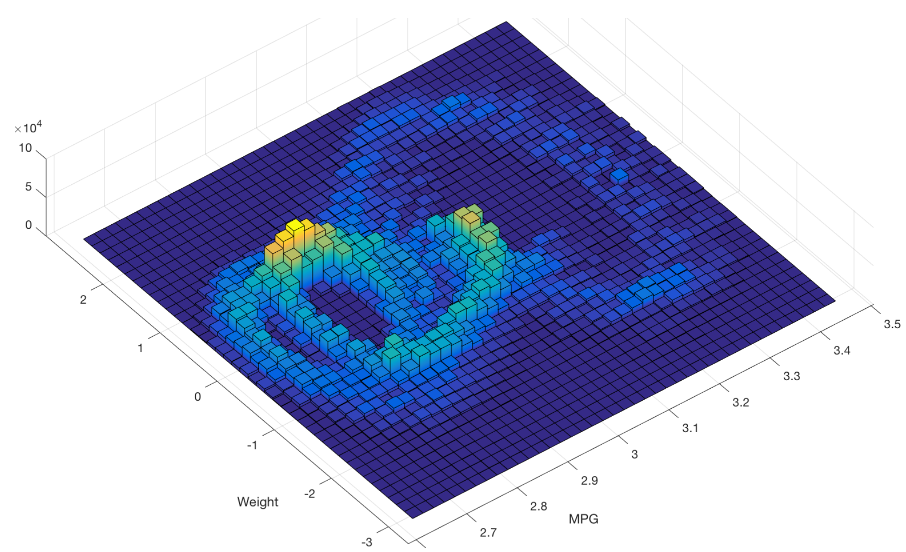


perturbed cavity
to 1.7cm
different transmission line

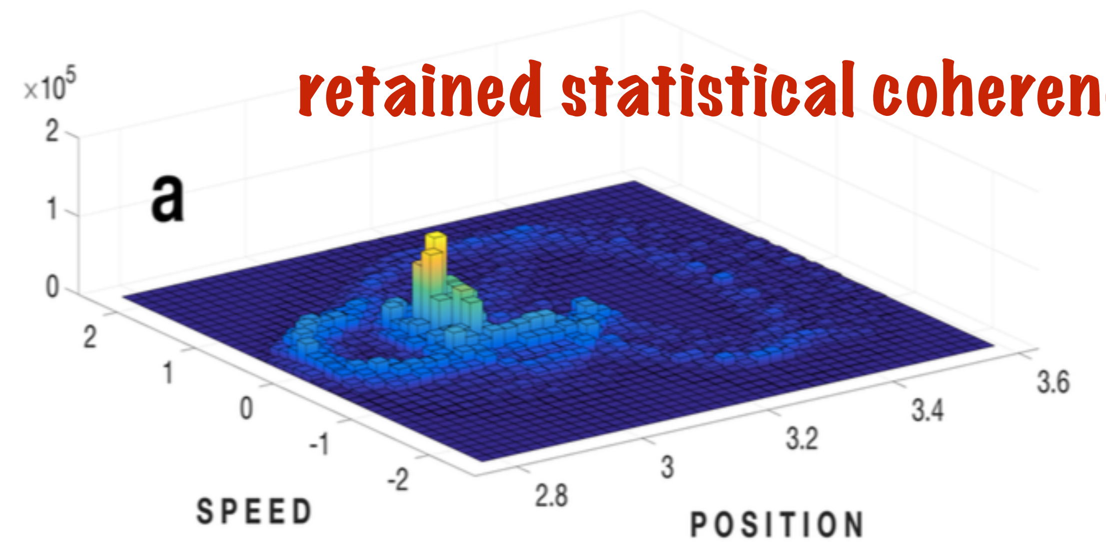
previous
unperturbed case



bimodal



retained statistical coherence



different distribution