HQA Lecture 16

I. More complete hydrodynamic modeling

II. Interactions with boundaries

The state of affairs

Unbounded single-particle systems: well characterized by strobe model

- has provided the first Paradigm for the emergence of quantum behavior
    - monochromatic wavefield confines drop to a discrete number of states
    - chaotic pilot-wave dynamics yields multimodal statistics
- **note**: strobe model is by construction *nonlocal*

Multiple-particle systems: revealed shortcomings of strobe model

- result from limited description of far-field wave form
- result from neglect of transient wave form generated at impact
- motivates development of more complete, local description of wave generation

"The WORKS"

- a hydrodynamically complete treatment of the problem
- couples droplet dynamics of MB to full treatment of the waves
- drop serves as a local source of viscously damped Faraday waves
- captures both transient and standing wave components

J. Fluid Mech. (2015), *vol.* 778, *pp.* 361–388.
doi:10.1017/jfm.2015.386

Faraday pilot-wave dynamics: modelling and computation

Paul A. Milewski[1,†], Carlos A. Galeano-Rios[2], André Nachbin[2] and John W. M. Bush[3]

[1]Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK

[2]IMPA/National Institute of Pure and Applied Mathematics, Est. D. Castorina, 110, Rio de Janeiro, RJ 22460-320, Brazil

[3]Department of Mathematics, MIT, Cambridge, MA, USA

Long-wave, low viscosity approximation

Quasi-potential theory

- viscous effects confined to thin region adjoining free surface

$$\Delta\phi = 0, \qquad \text{for } -h(\mathbf{x}) \leqslant z \leqslant 0$$

$$\nabla\phi \cdot \mathbf{n} = 0, \qquad \text{for } z = -h(\mathbf{x})$$

$$\phi_t = -g(t)\eta + 2\nu\Delta_\perp\phi + \frac{\sigma}{\rho}\Delta_\perp\eta, \qquad \text{for } z = 0$$

$$\eta_t = \phi_z + 2\nu\Delta_\perp\eta, \qquad \text{for } z = 0$$

↓ Long wave limit

Damped wave equation

$$\phi_t = -g(t)\eta + 2\nu\Delta\phi - \frac{1}{\rho}P_D$$

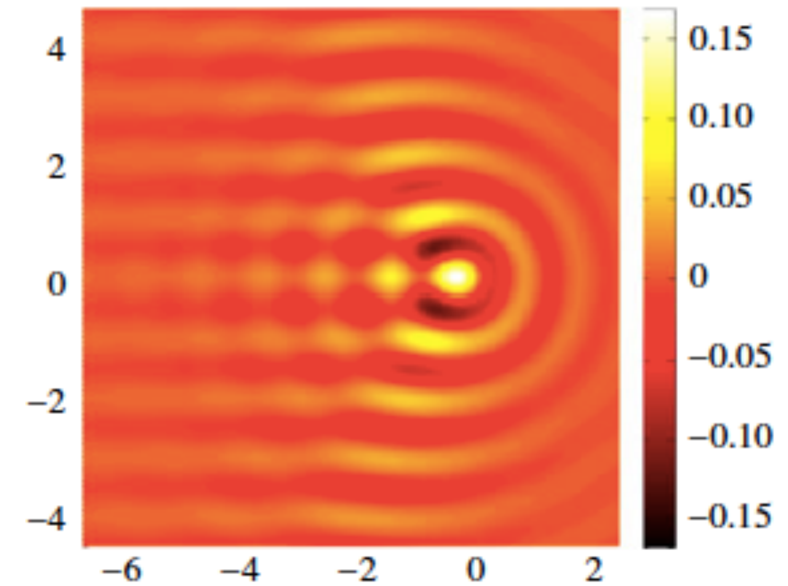
$$\eta_t = \nabla\cdot(h\nabla\phi) + 2\nu\Delta\eta$$

Milewski et al., JFM 2015

Weakly viscous quasi-potential wave generation

(Milewski, Galeano-Rios, Nachbin & Bush, JFM 2015)

- droplet serves as generator of waves for which viscous effects are confined to a thin boundary layer adjoining the upper surface



$$\nabla^2 \phi = 0$$

B.C.s **1.** $\phi \to 0 \quad \text{as} \quad z \to -\infty$

2. $$\phi_t = g(t)\eta + \frac{\sigma}{\rho}\nabla_H^2 \eta + 2\nu \nabla_H^2 \phi - \frac{1}{\rho} P_D(\mathbf{x} - \mathbf{x}_p, t), \quad z = 0$$

WAVE-MAKING DROP

3. $$\eta_t = \phi_z + 2\nu \nabla_H^2 \eta, \quad z = 0$$

- coupled to trajectory equation developed by Molacek & Bush (2013ab)
- drop serves as a localized pressure perturbation to the free surface

Droplet dynamics

Milewski et al. JFM (2015)

Free flight

$$m\frac{\mathrm{d}^2Z}{\mathrm{d}t^2} = -mg(t),$$

$$m\frac{\mathrm{d}^2\boldsymbol{X}}{\mathrm{d}t^2} = -6\pi R_0\mu_{air}\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t}$$

Impact

$$\left(1+\frac{c_3}{\ln^2\left|\frac{c_1R_0}{Z-\bar{\eta}}\right|}\right)m\frac{\mathrm{d}^2Z}{\mathrm{d}t^2}+\frac{4}{3}\frac{\pi\nu\rho R_0c_2}{\ln\left|\frac{c_1R_0}{Z-\bar{\eta}}\right|}\frac{\mathrm{d}}{\mathrm{d}t}(Z-\bar{\eta})+\frac{2\pi\sigma}{\ln\left|\frac{c_1R_0}{Z-\bar{\eta}}\right|}(Z-\bar{\eta}) = -mg(t),$$

$$m\frac{\mathrm{d}^2\boldsymbol{X}}{\mathrm{d}t^2}+\left(c_4\sqrt{\frac{\rho R_0}{\sigma}}F(t)+6\pi R_0\mu_{air}\right)\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = -F(t)\boldsymbol{\nabla}\bar{\eta}|_{\boldsymbol{x}=\boldsymbol{X}}.$$

Wave making

- drop applies pressure $P_D = \frac{F(t)}{\pi R(t)^2}$ during impact, where

$$F(t) = \max\left[m\frac{\mathrm{d}^2}{\mathrm{d}t^2}Z+mg(t),\,0\right] \quad \text{and} \quad \pi R(t)^2 = \pi\min(2|Z-\bar{\eta}|R_0,\,(R_0/3)^2)$$

Comparison of standing wave fields

- Note: the WORKS also captures the transient wave generated at impact

16 *P. A. Milewski, C. Galeano Rios, A. Nachbin and J. W. M. Bush,*

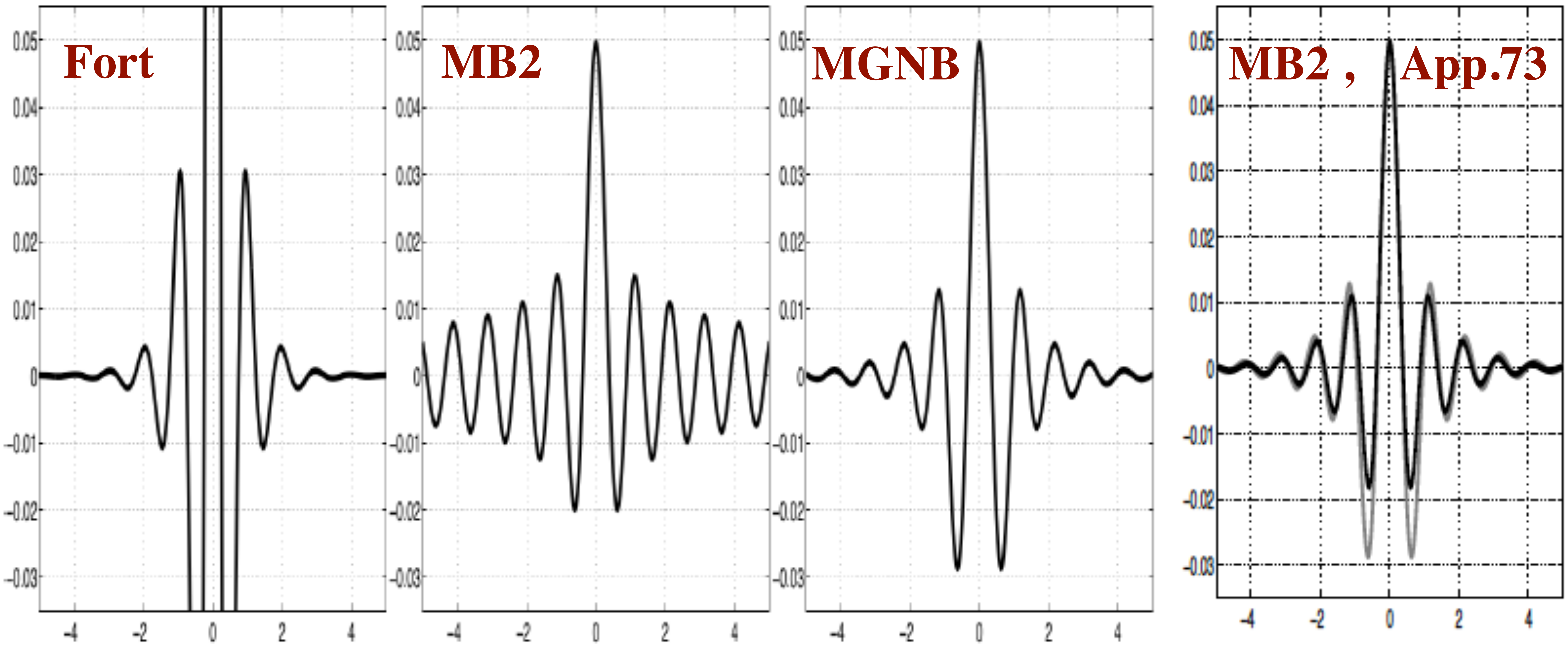


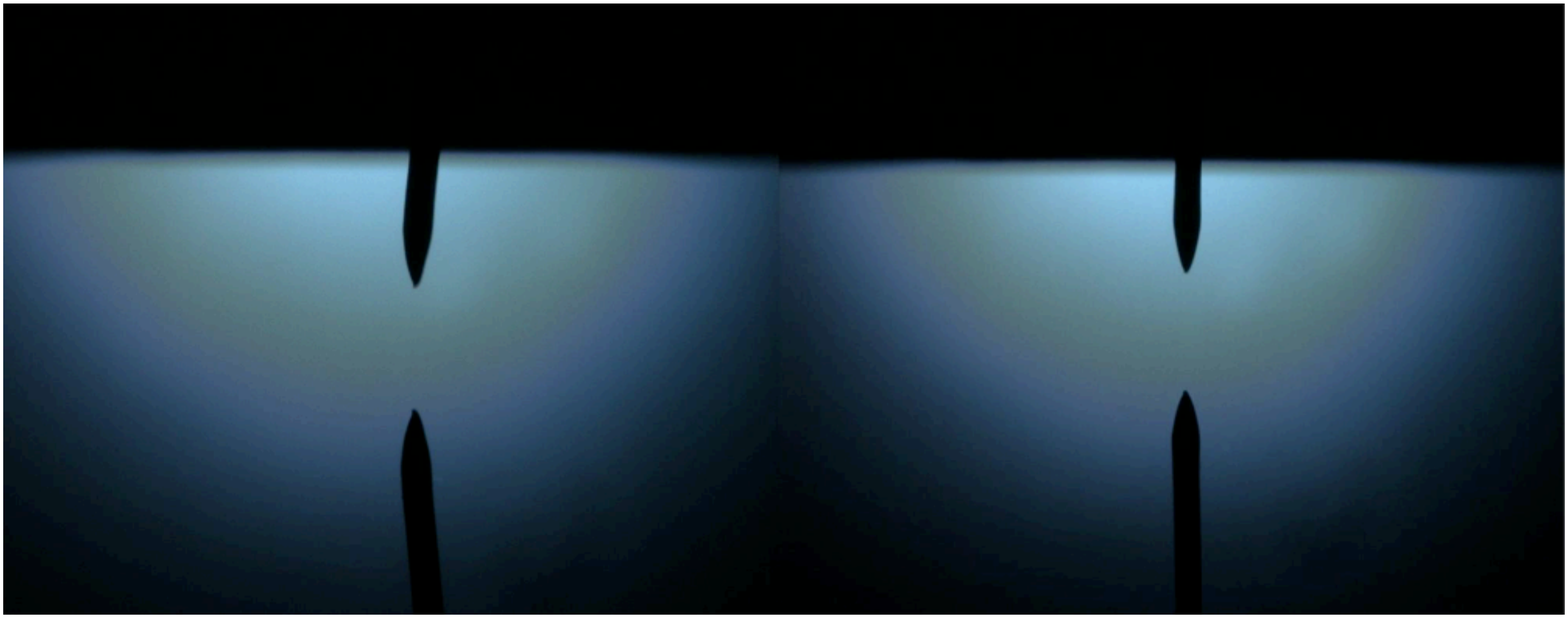
FIGURE 7. Different wave models: bouncer's cross sectional free surface height at the moment before impact with $\Gamma = 3.1$ $(2,1)^1$. From left to right: the wave model (3.9) (singular at $r = 0$), the Bessel model (3.10), and the result of our computation. The fourth graph at the right has our computation repeated in grey superimposed with a higher order correction of the Bessel model (3.10). For the wave model (3.9) we used a spatial damping with decay length 1.6 λ_F as they report by Eddi et al. (2011).

Disturbance of forced and unforced interfaces

- withdraw millimetric needle from interface

No forcing

Vibrational forcing



- waves quickly disperse
- field of Faraday waves persist

- vibration predisposes bath to monochromatic wave field with Faraday wavelength

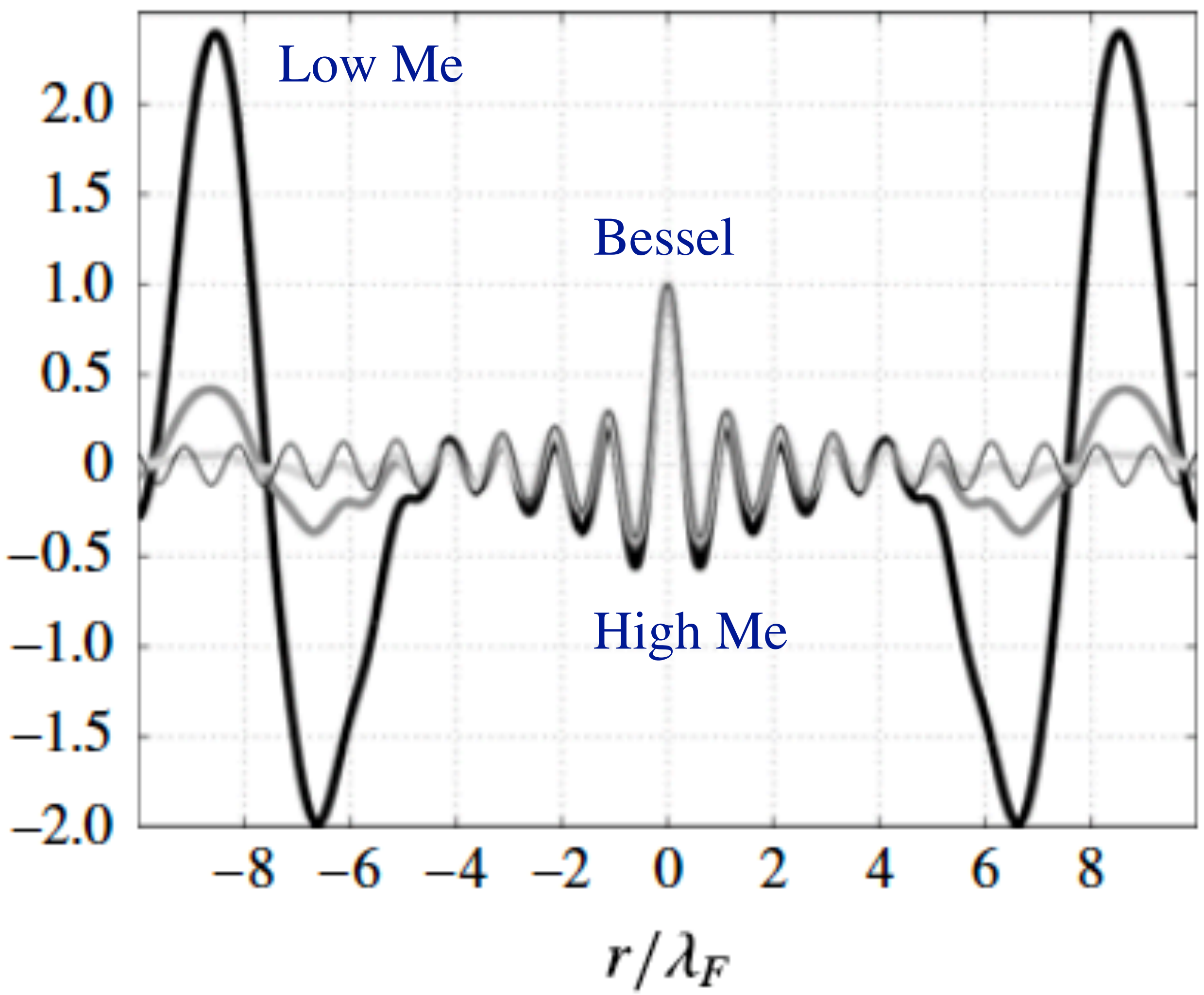
The wave form generated by a single impact

Unforced

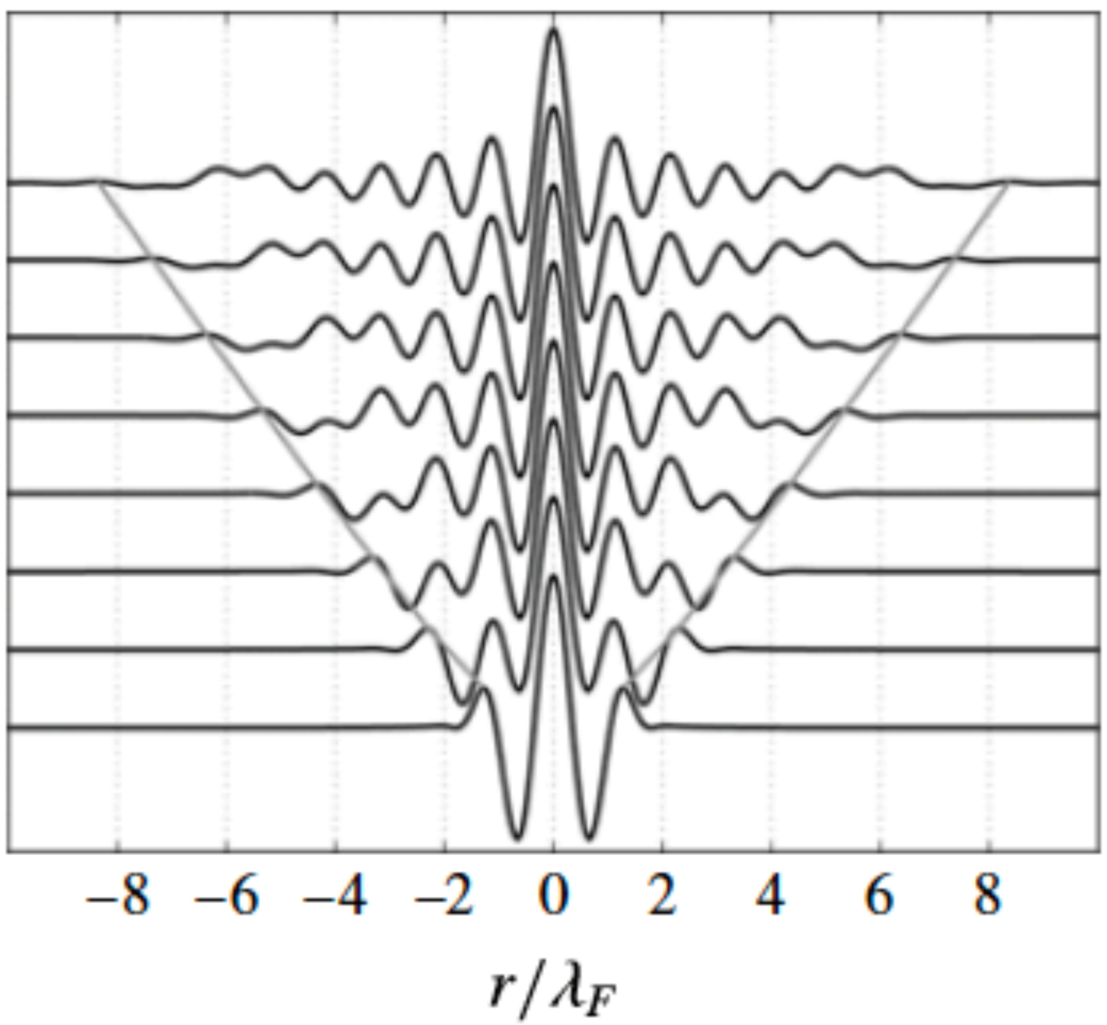
Forced

The wave form generated by a single impact

Dependence on Me



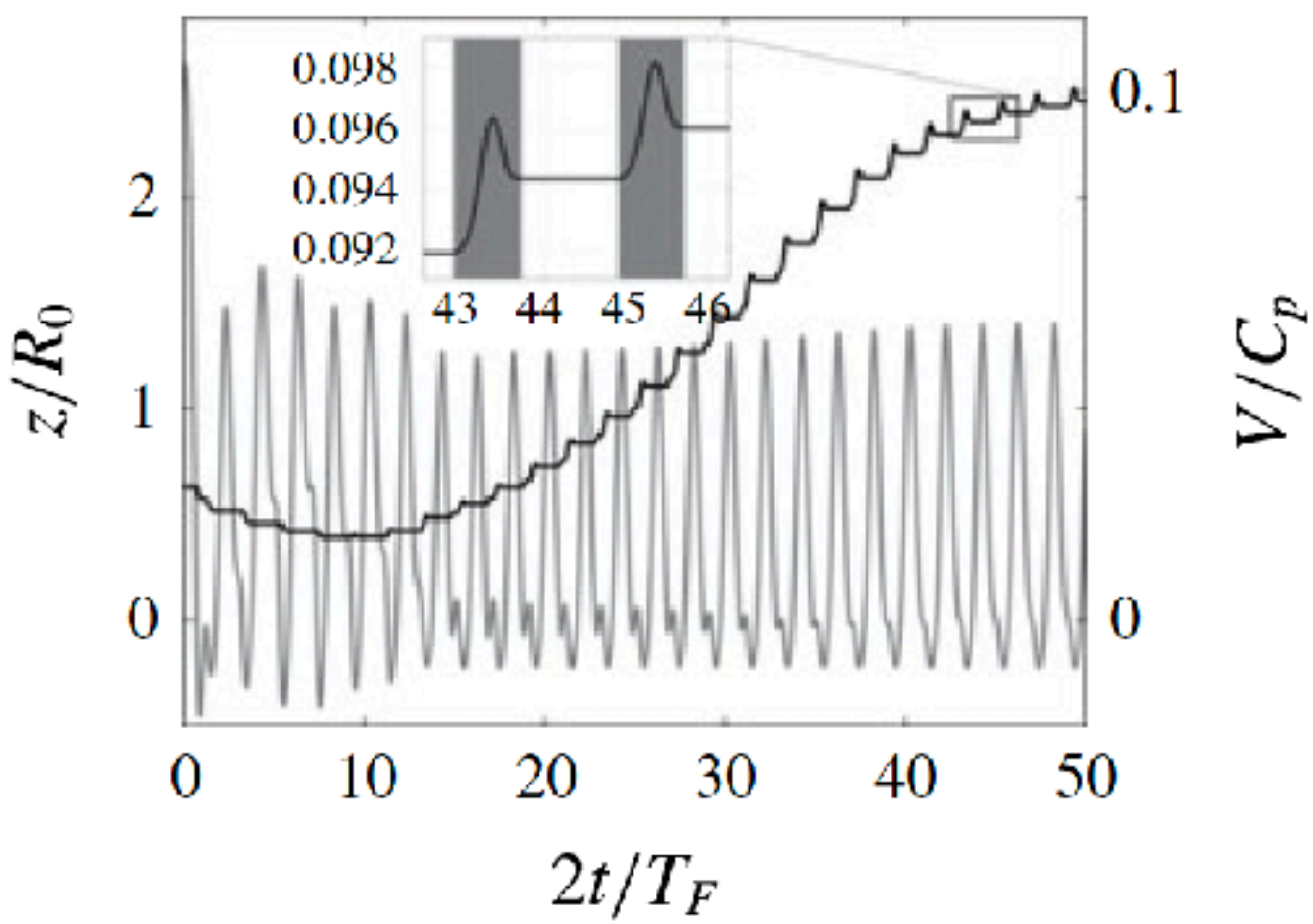
High Me

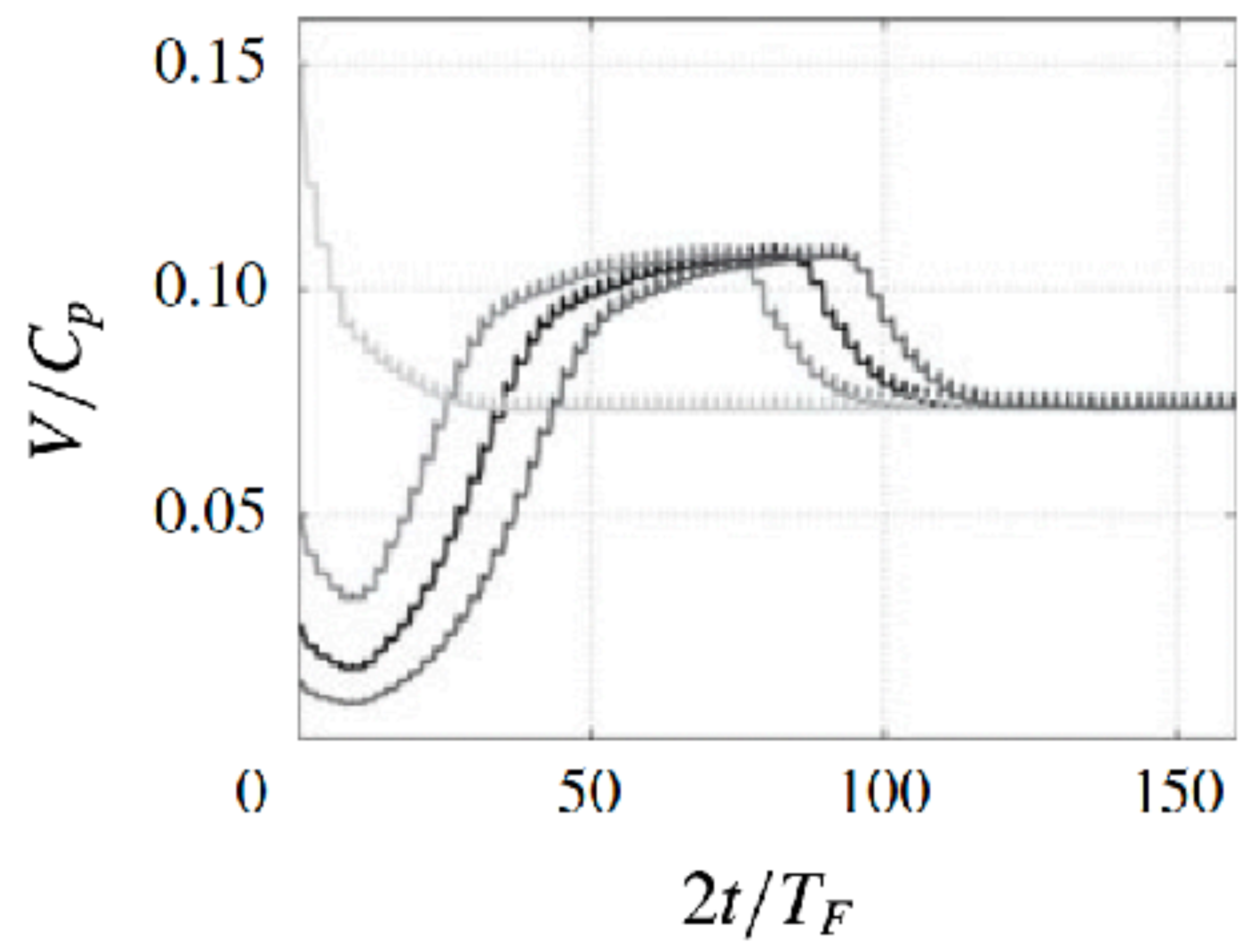


- relative magnitude of standing and transient components increases with Me

Numerical results

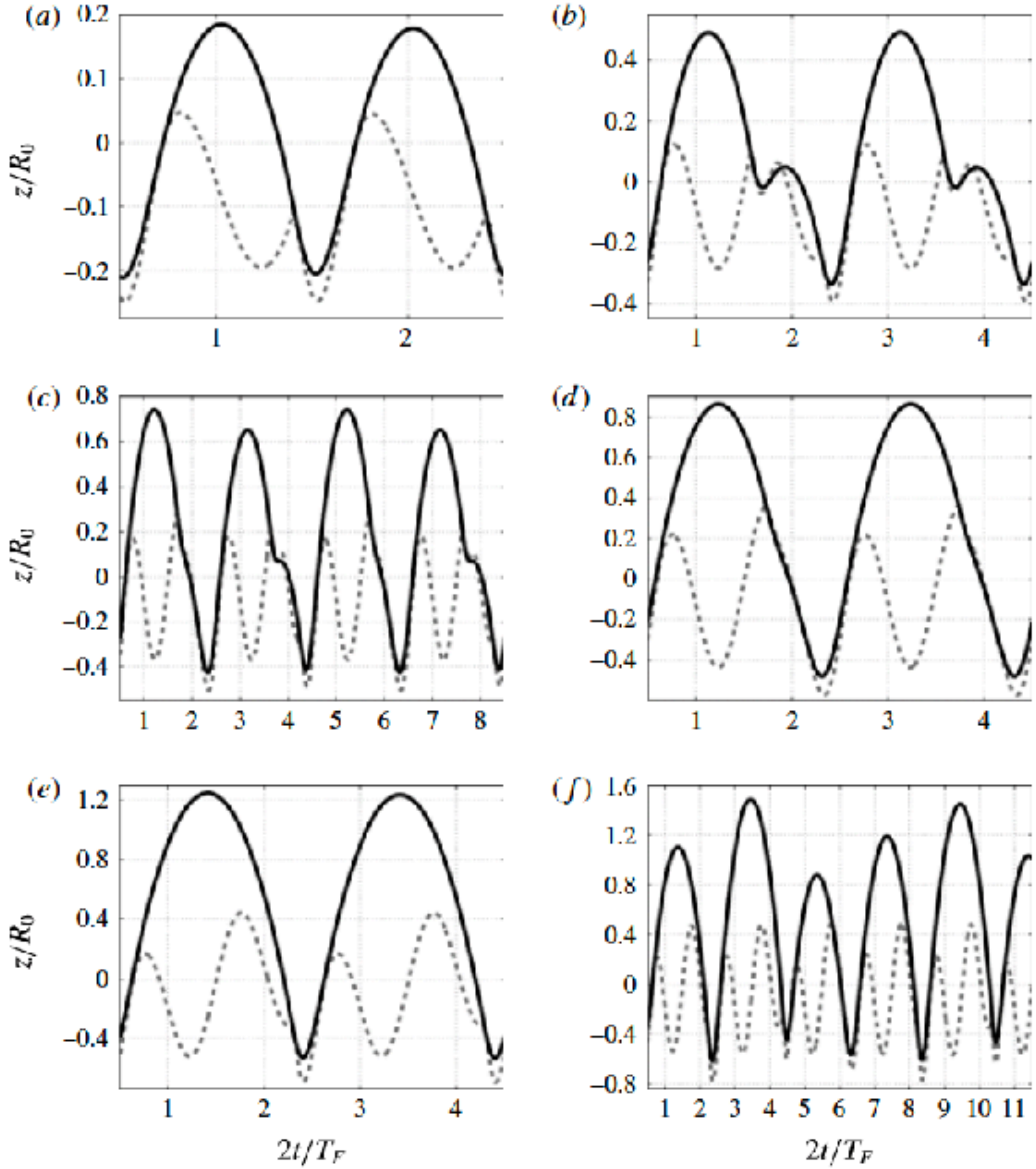
- fast dynamics and transient wave generation fully resolved
- transient walker start-up



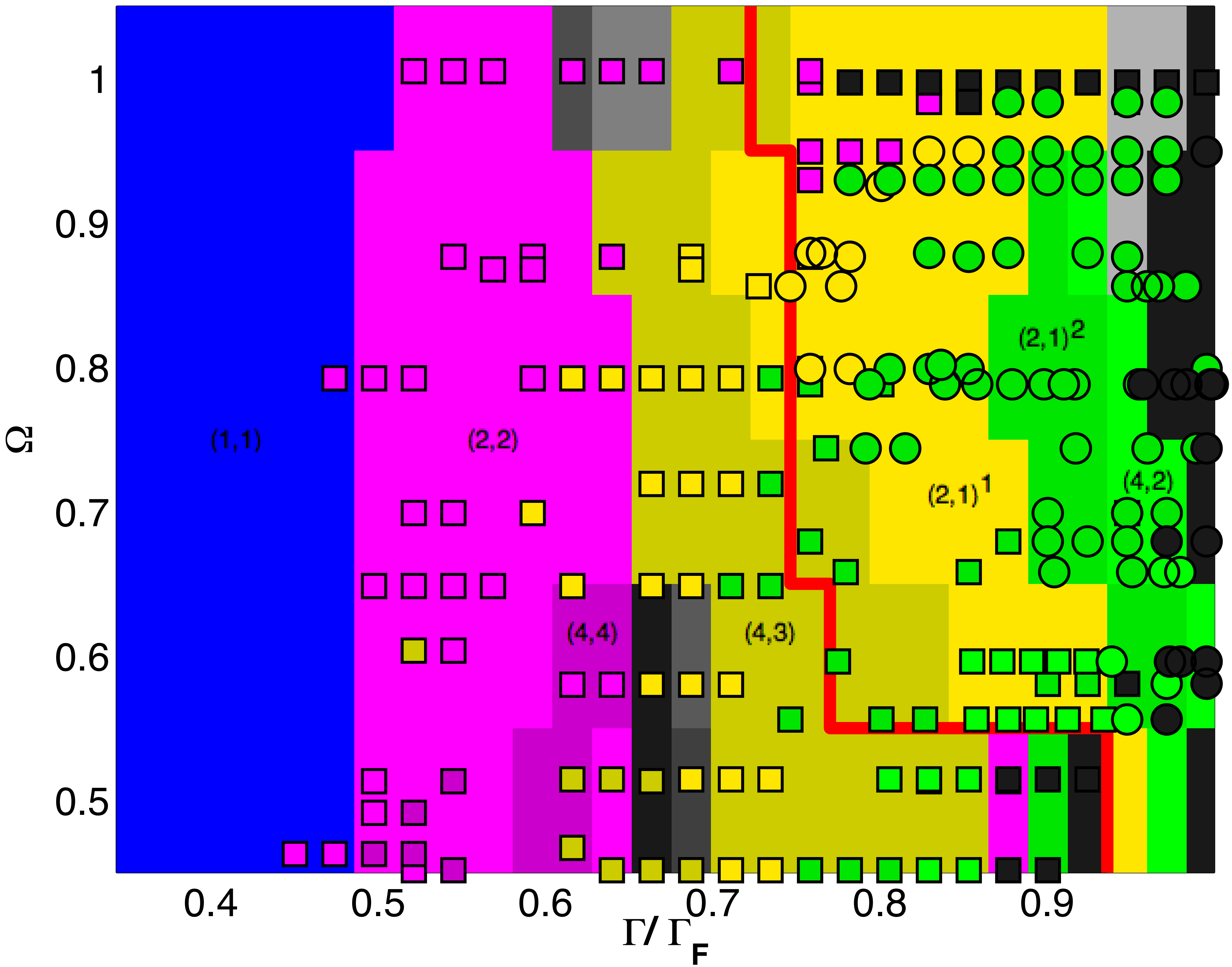


Numerical results

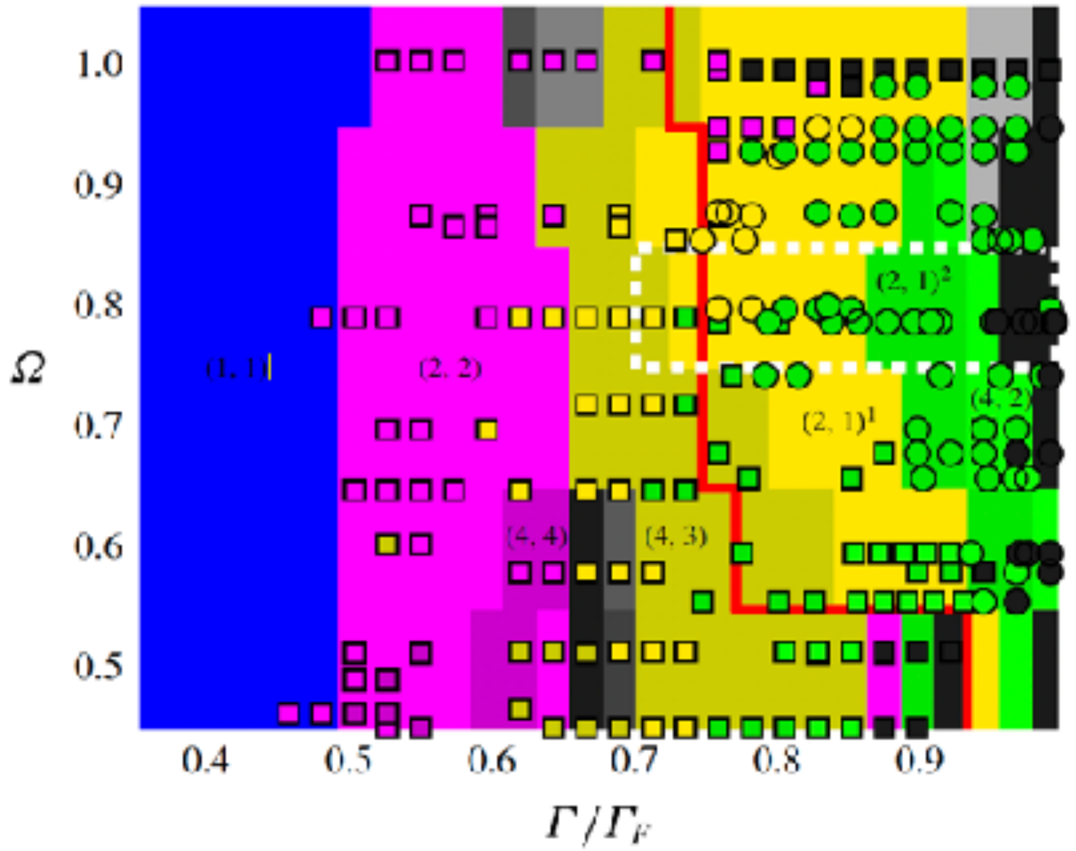
- captures different bouncing modes, regime diagrams



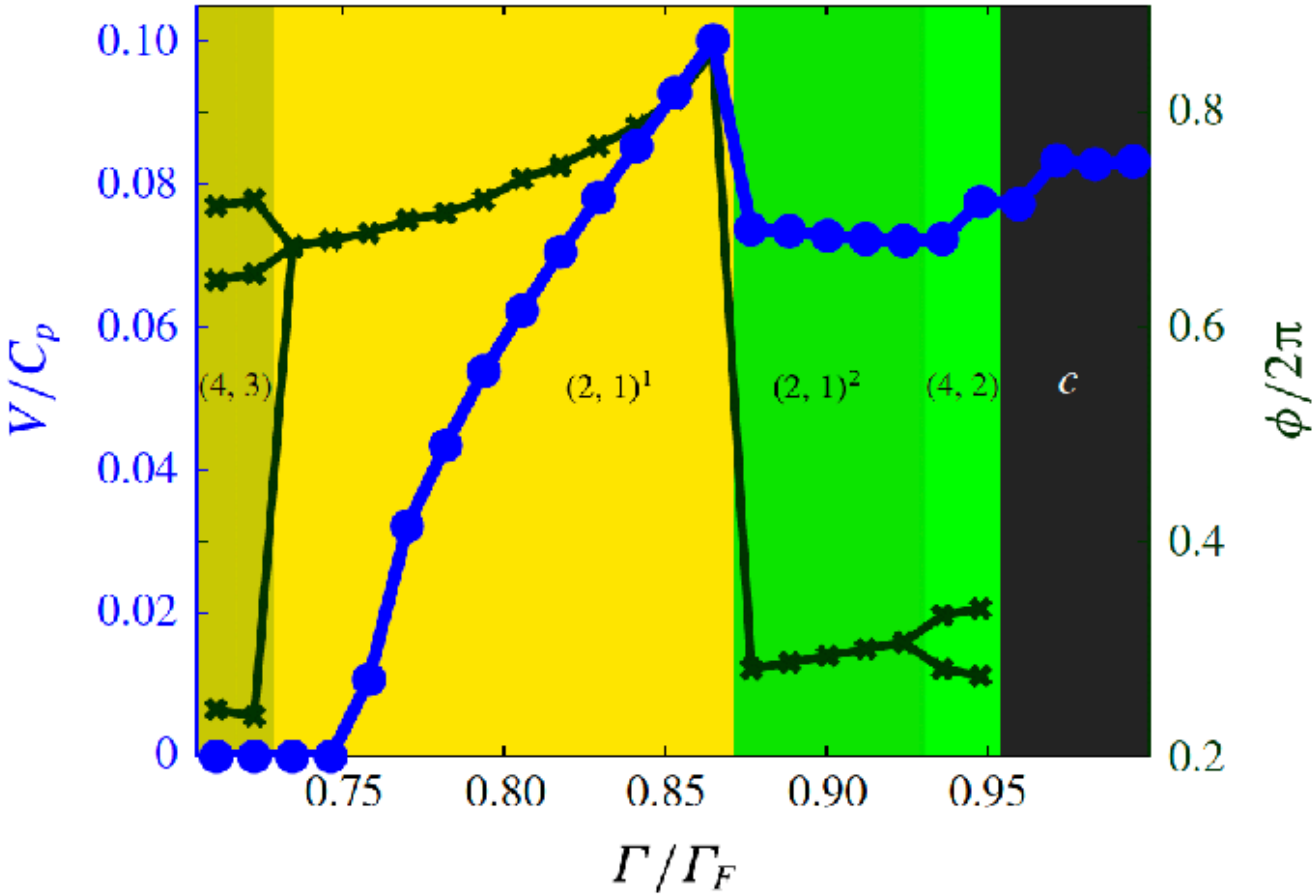
Numerical regime diagram



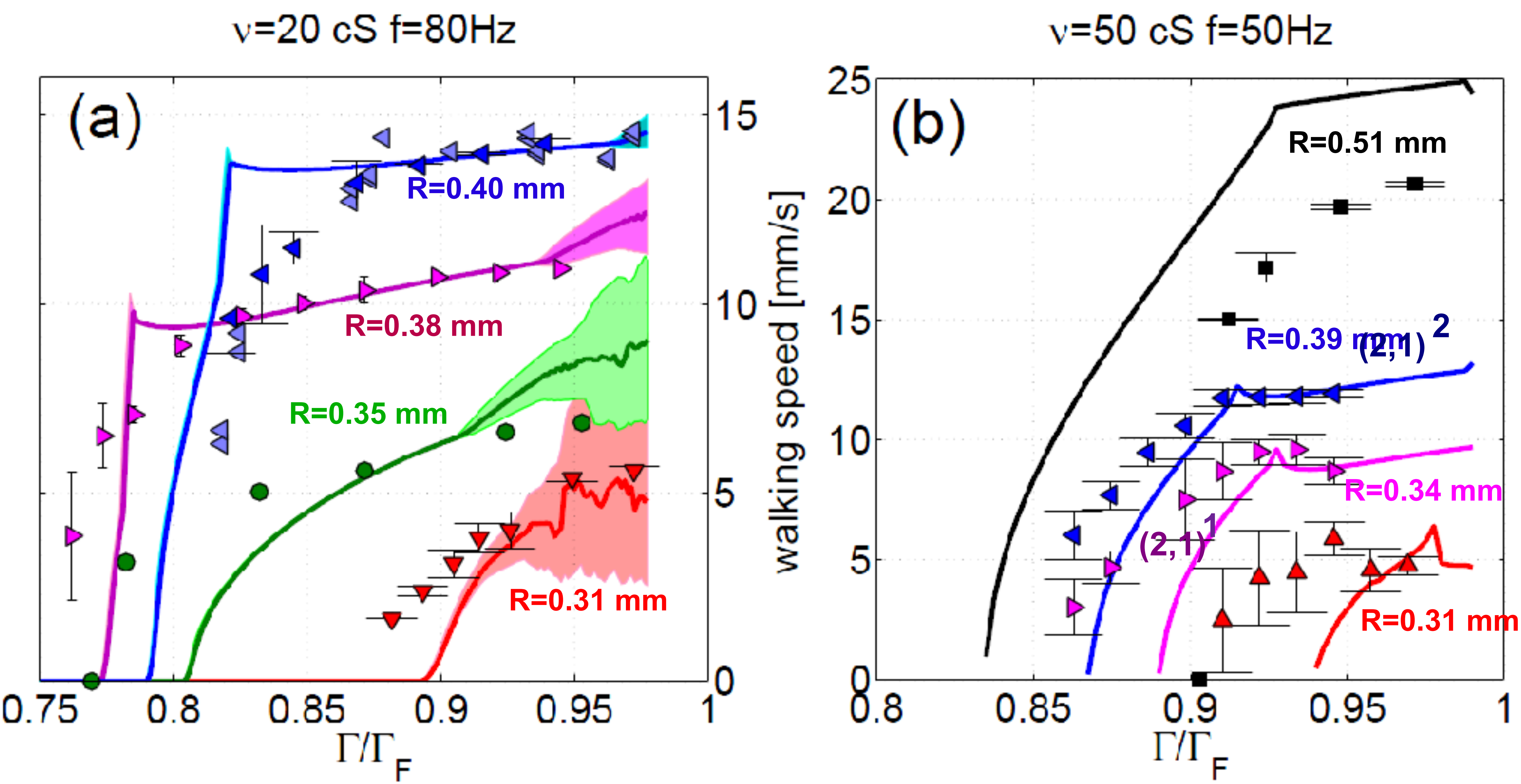
Numerical results



- variation of impact phase, speed in the walking regime



Predicted walking speeds

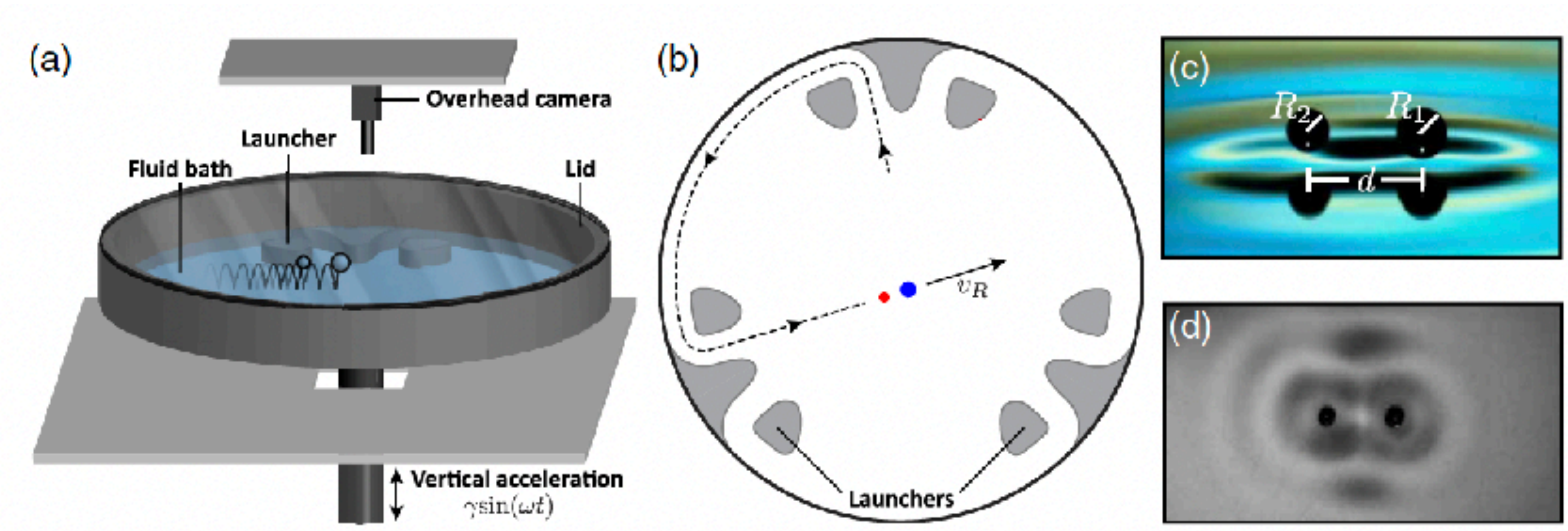


- discontinuities associated with transition to more energetic walking state:

$$(2,1)^1 \longrightarrow (2,1)^2$$

Ratcheting pairs

- optical, quantum ratchets arise when propulsion of an object is induced by interaction with a periodic or random field
- a worst case scenario: non-resonant bouncers of unequal size; transient waves
- strobe model utterly inadequate



CHAOS **28**, 096112 (2018)

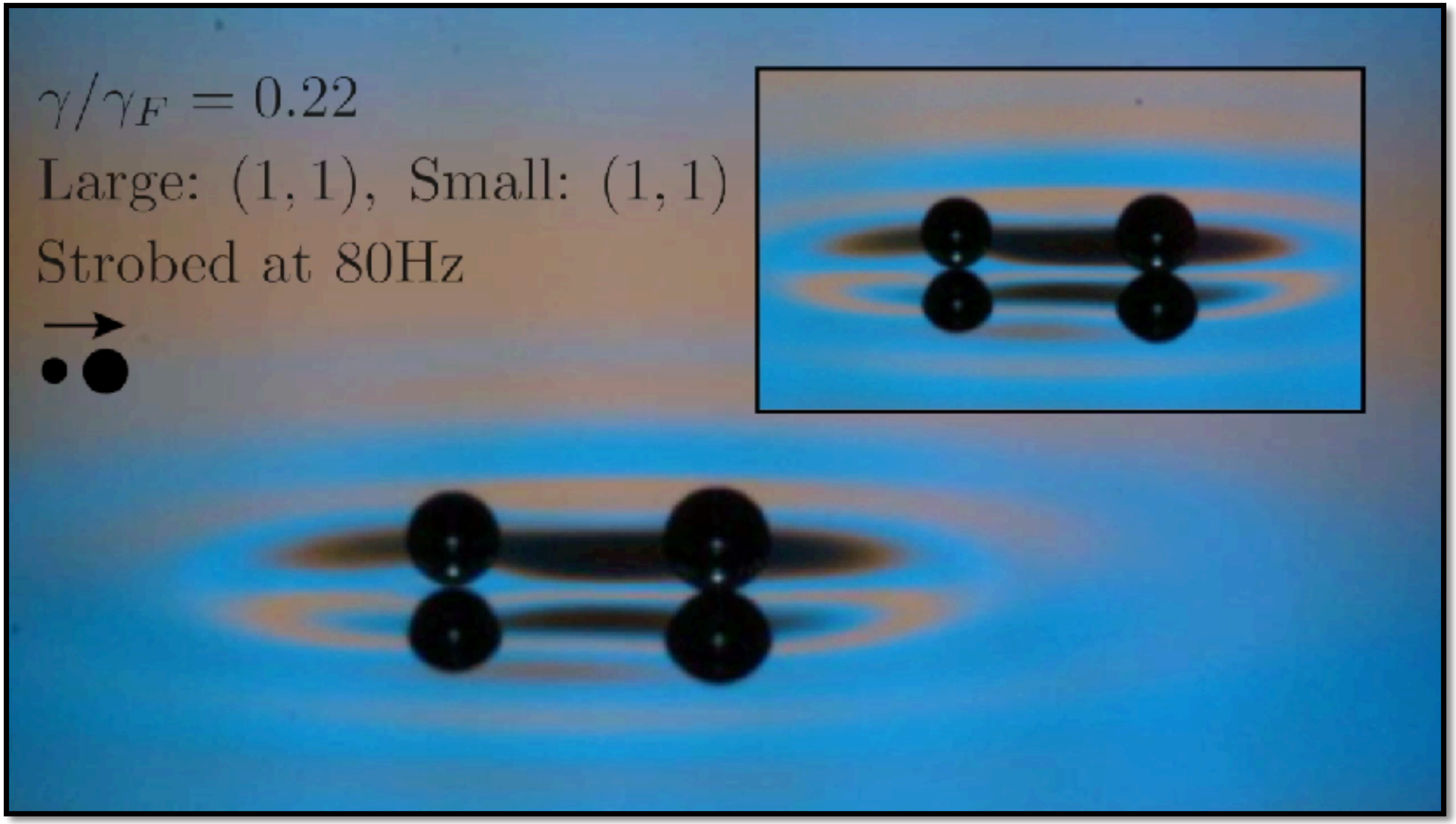
Ratcheting droplet pairs

C. A. Galeano-Rios,[1,a)] M. M. P. Couchman,[2,b)] P. Caldairou,[2] and J. W. M. Bush[2,c)]
[1]*Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, United Kingdom*
[2]*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

Ratcheting pairs

- interdrop distance depends on relative phase of bouncers
- prescribed by minima of neighbor’s wave field

Experiments

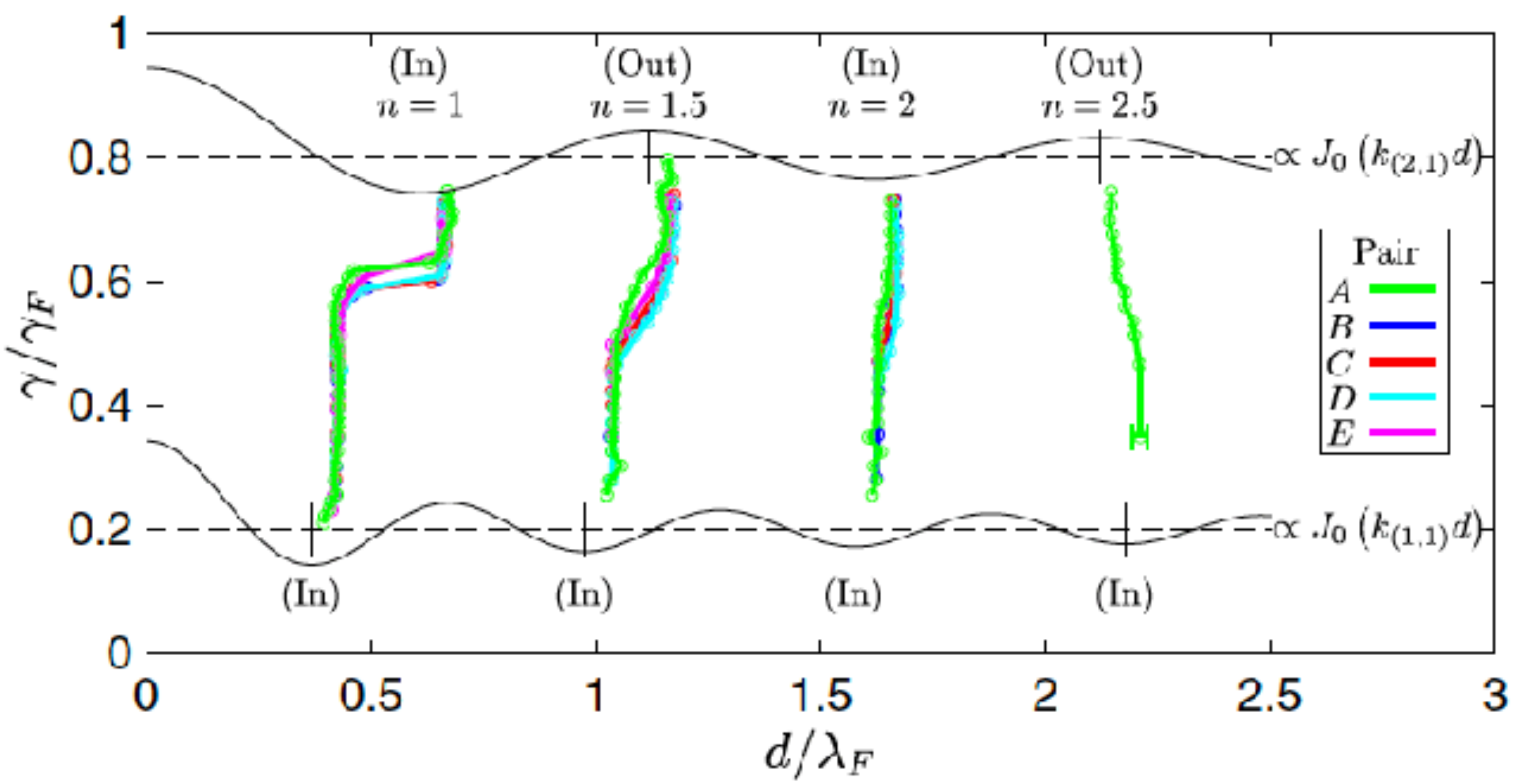


Ratcheting pairs

- interdrop distance depends on relative phase of bouncers
- prescribed by minima of neighbor’s wave field

Experiments

Pair	A	B	C	D	E
R_1 (mm)	0.444	0.406	0.400	0.383	0.406
R_2 (mm)	0.353	0.354	0.361	0.365	0.393



Ratcheting pairs

Pair	A	B	C	D	E
R_1 (mm)	0.444	0.406	0.400	0.383	0.406
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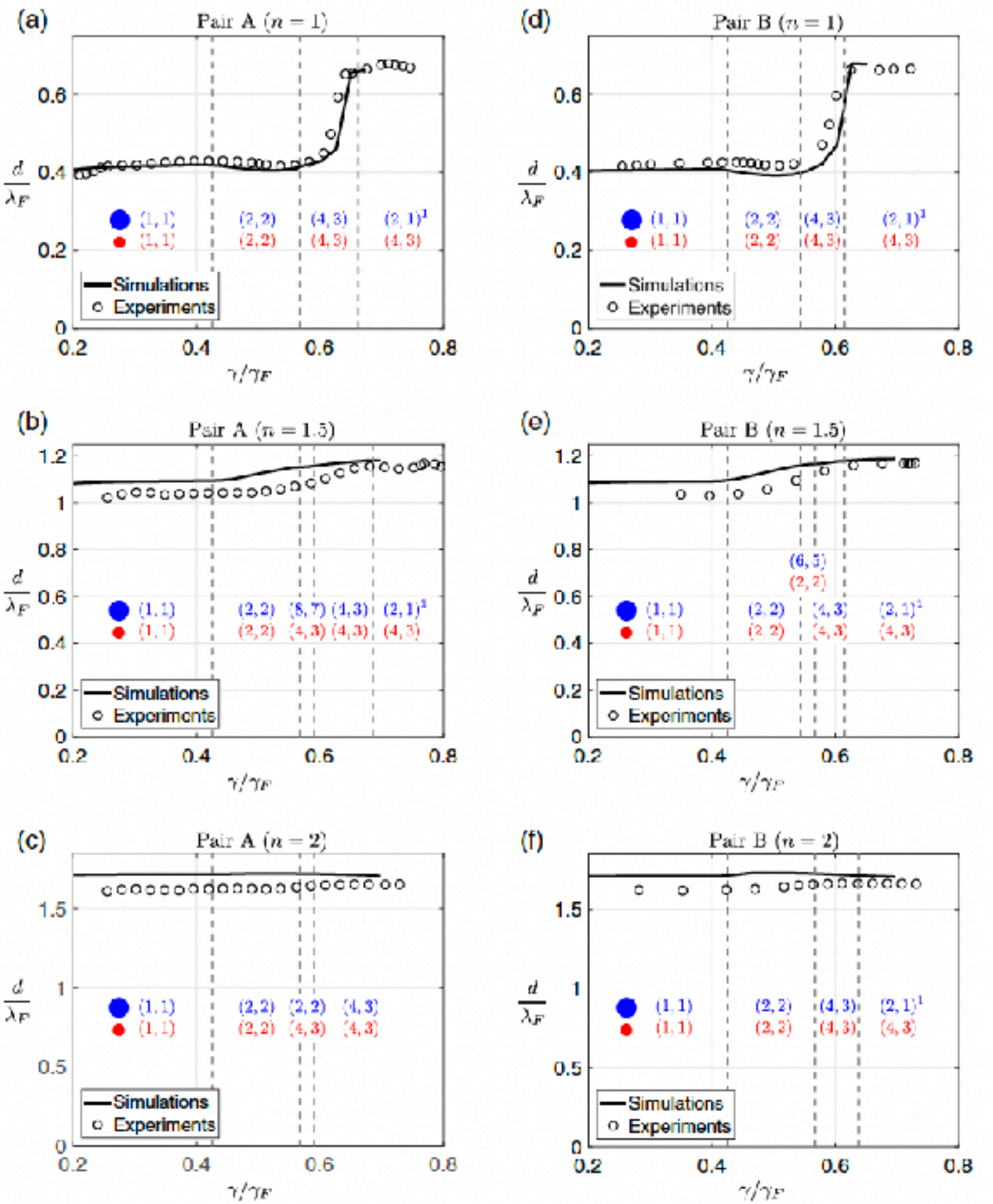
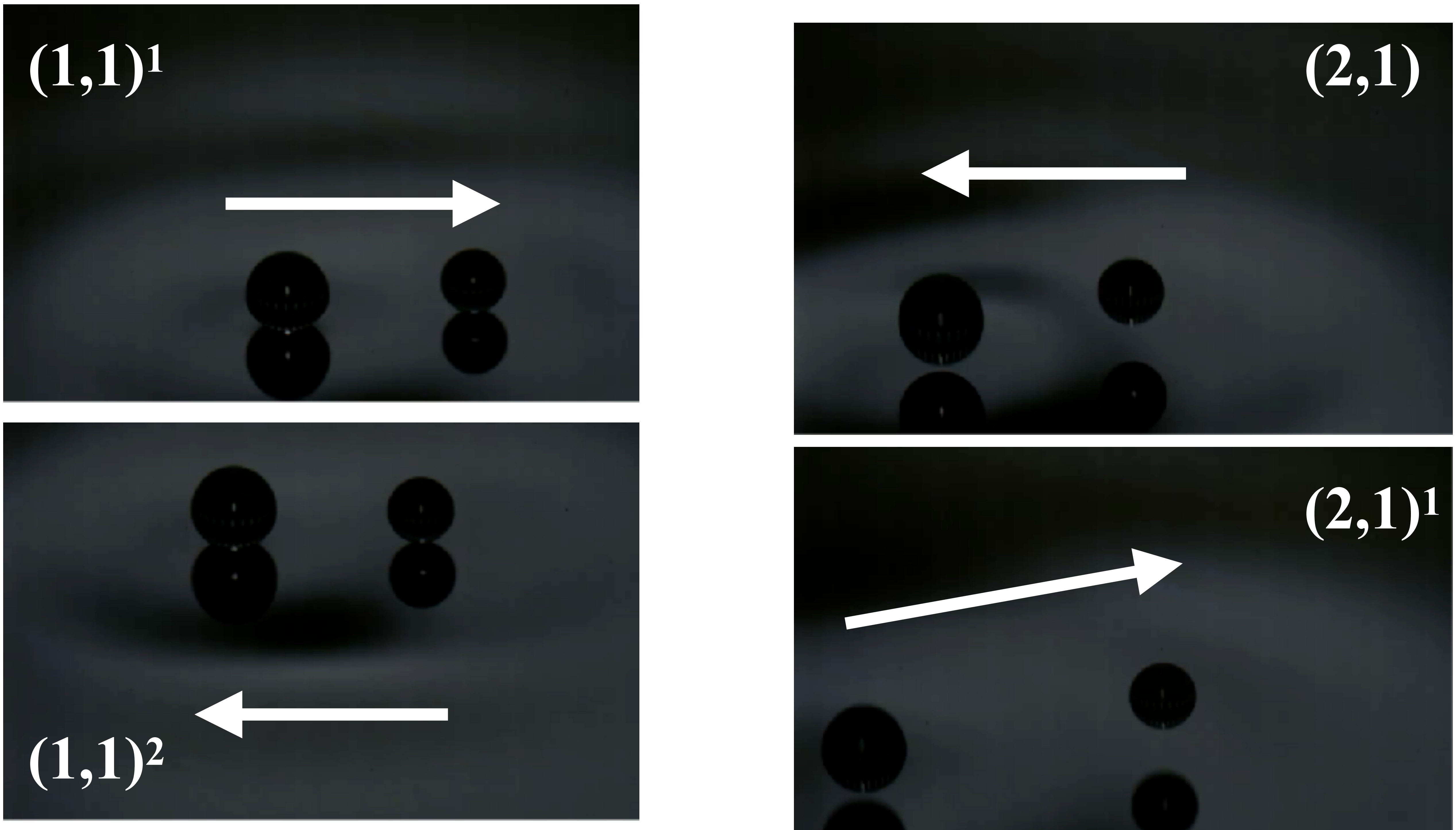
Theory



Ratcheting pairs

- as memory is increased, direction of motion may reverse up to 3 times

Experiments



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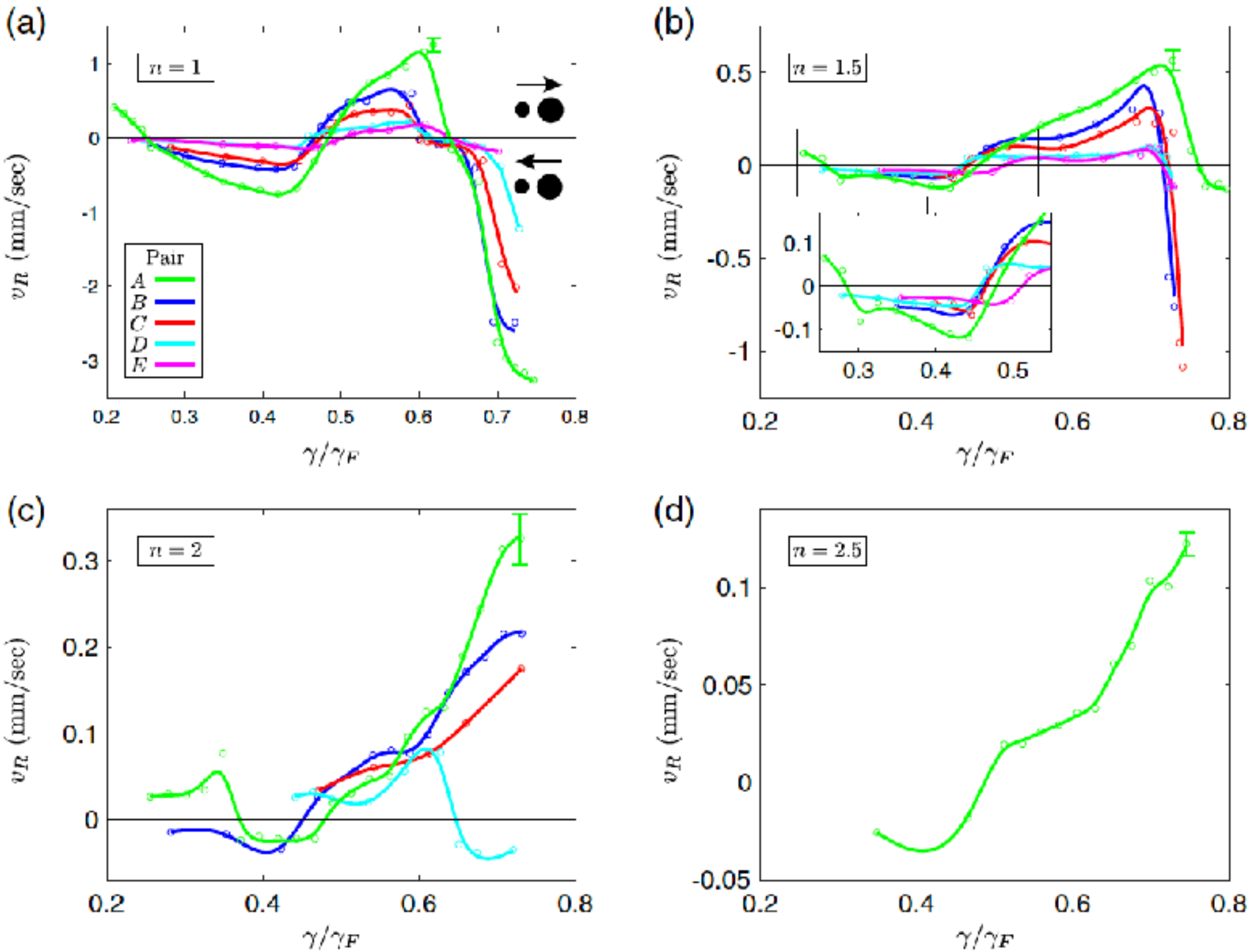
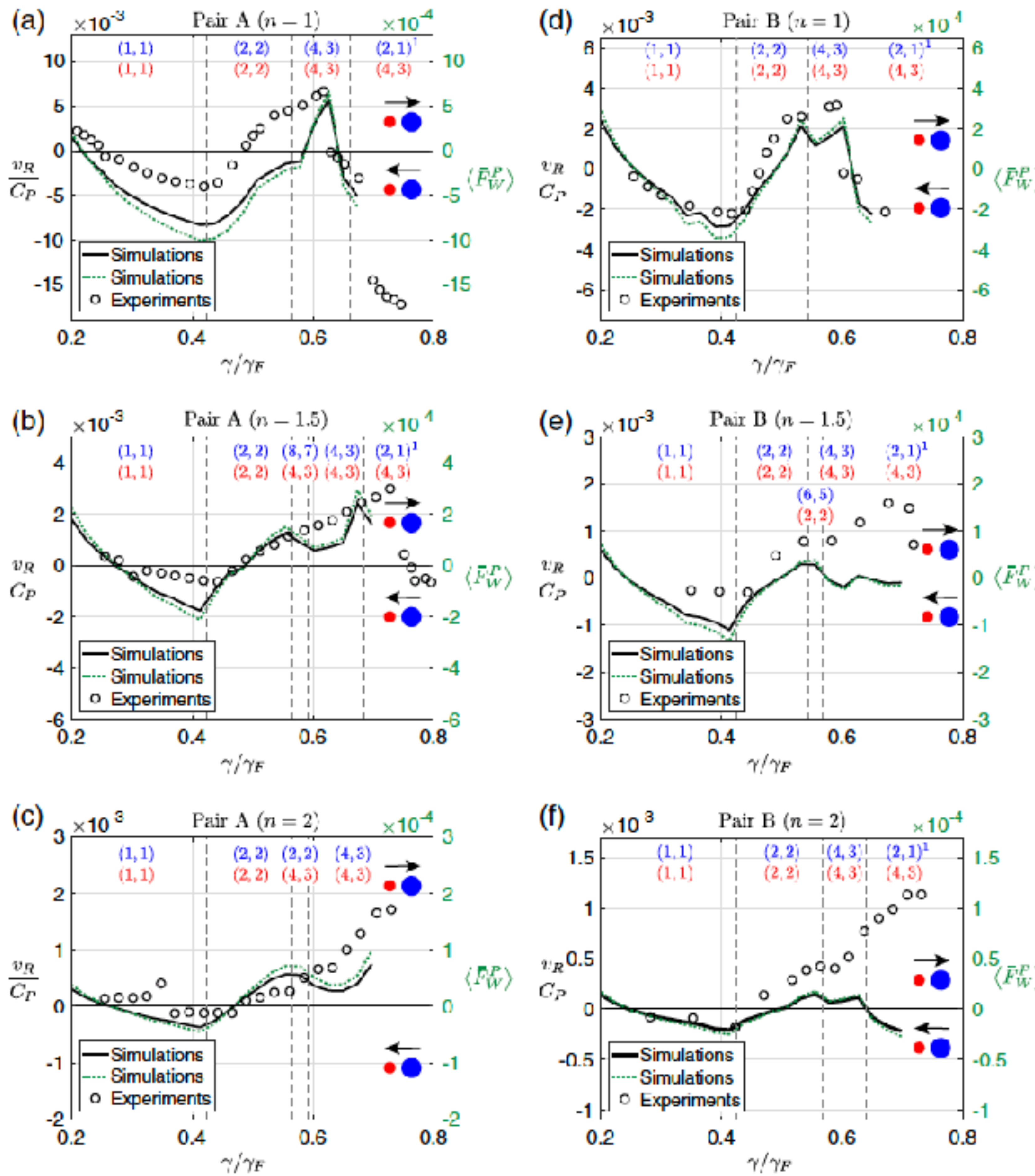


Ratcheting pairs

Theory

- highlights importance of transient wave from near neighbors



Handling variable bottom topography

- central to a number of key analog systems (diffraction, tunneling, corrals)

J. Fluid Mech. (2017), *vol.* 811, *pp.* 51–66. © Cambridge University Press 2016
doi:10.1017/jfm.2016.750

A model for Faraday pilot waves over variable topography

Luiz M. Faria†

Department of Mathematics, Massachusets Institute of Technology, Cambridge, MA 02139, USA

Long-wave approximation

Quasi-potential theory

$$\Delta\phi = 0, \qquad \text{for} \ -h(\mathbf{x}) \leqslant z \leqslant 0$$

$$\nabla\phi \cdot \mathbf{n} = 0, \qquad \text{for } z = -h(\mathbf{x})$$

$$\phi_t = -g(t)\eta + 2\nu\Delta_\perp\phi + \frac{\sigma}{\rho}\Delta_\perp\eta, \qquad \text{for } z = 0$$

$$\eta_t = \phi_z + 2\nu\Delta_\perp\eta, \qquad \text{for } z = 0$$

Long wave limit

Damped wave equation

$$\phi_t = -g(t)\eta + 2\nu\Delta\phi - \frac{1}{\rho}P_D$$

$$\eta_t = \nabla\cdot(h\nabla\phi) + 2\nu\Delta\eta$$

Milewski et al. JFM (2015)

Quasi-monochromatic approximation

Can we trade long-wave for quasi-monochromatic?
In other words, can we still use $\phi_z \approx \nabla \cdot (\bar{h} \nabla \phi)$

Waves are monochromatic! → Only need to model Faraday waves correctly!

- variations in depth modeled as a variation in λ_F, phase speed

DtN: $\mathcal{F}(\phi_z(\mathbf{x},0,t)) = |\mathbf{k}| \tanh(|\mathbf{k}| h_i) \mathcal{F}(\phi(\mathbf{x},0,t))$,

↓

$$\bar{h} = \frac{\tanh(k_{F_i} h_i)}{k_{F_i}}$$

↓

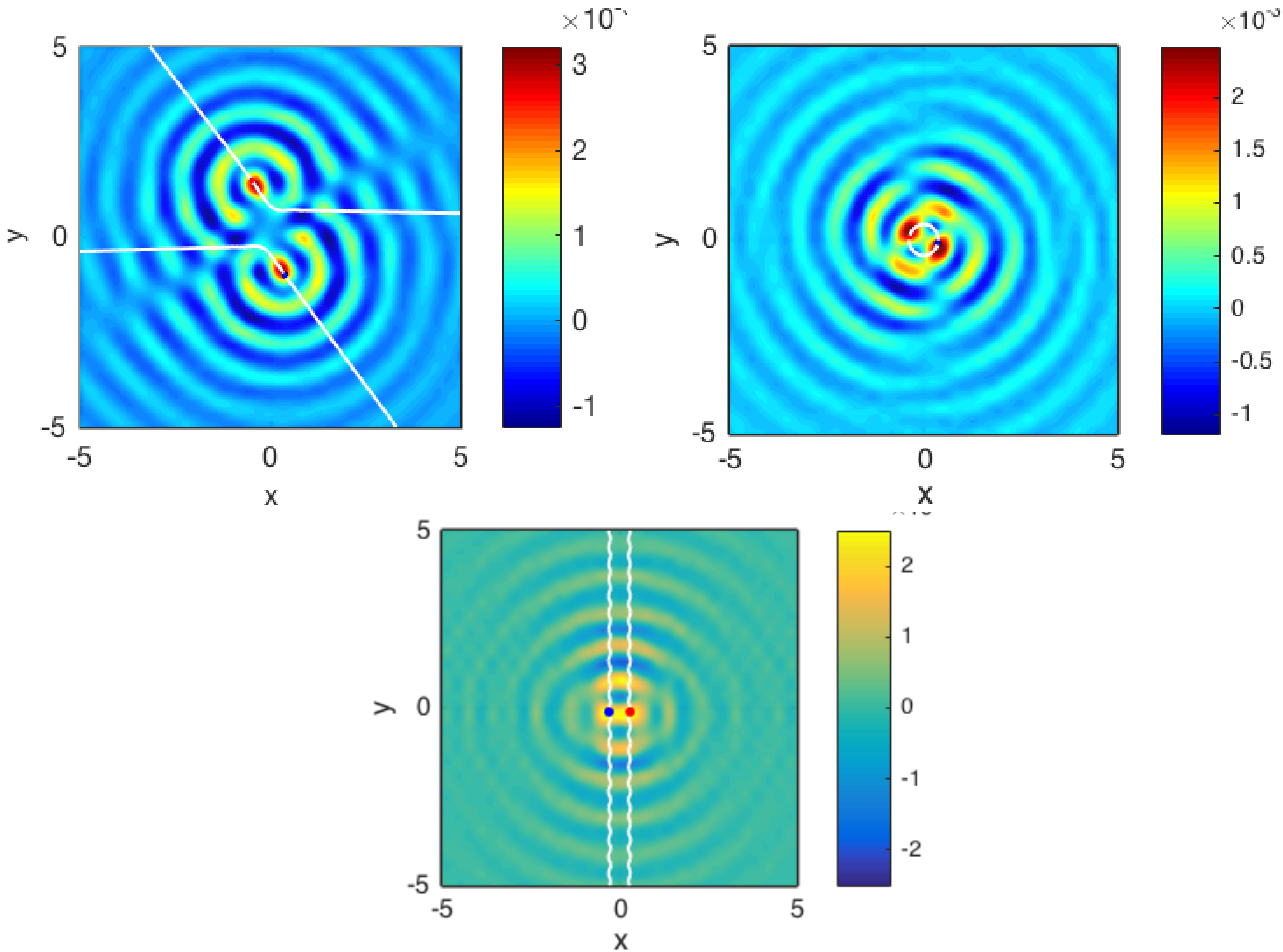
$$\phi_t = -g(t)\eta + 2\nu \Delta_\perp \phi + \frac{\sigma}{\rho} \Delta_\perp \eta,$$

$$\eta_t = -\nabla_\perp \cdot (\bar{h}(\mathbf{x}) \nabla_\perp \phi) + 2\nu \Delta_\perp \eta$$

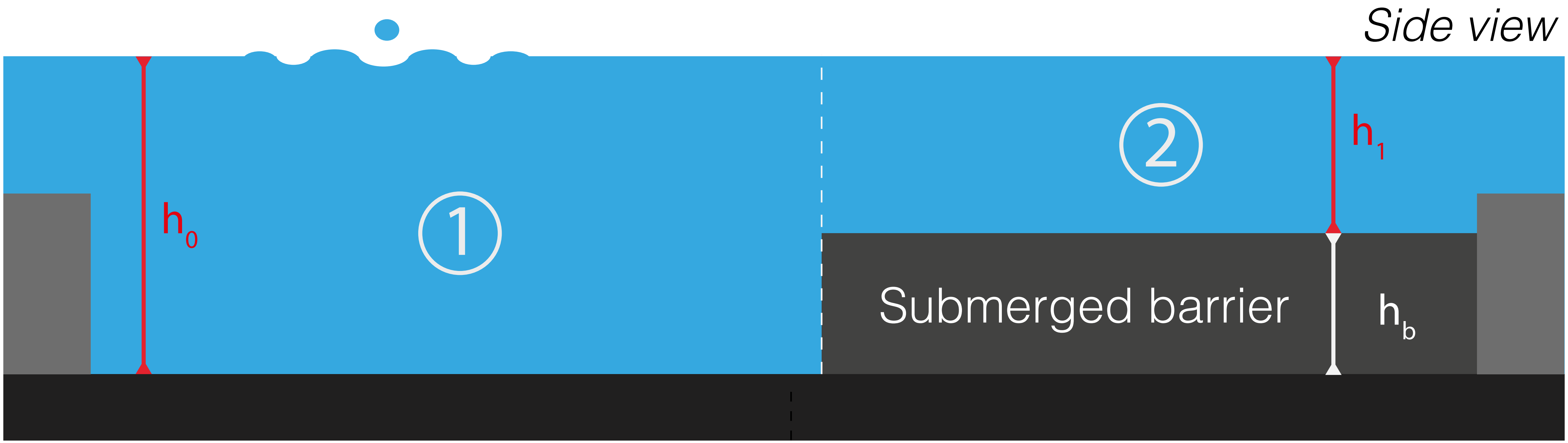
Couple to drop using MBI, MBII

- allows for first robust modeling of walker-boundary interactions

Two drops interact



Walker interacts with a step



Deep region 1

$h_0 = h_b + h_1$

$h_0 \in [6.11\text{-}6.31]$ mm

Shallow region 2

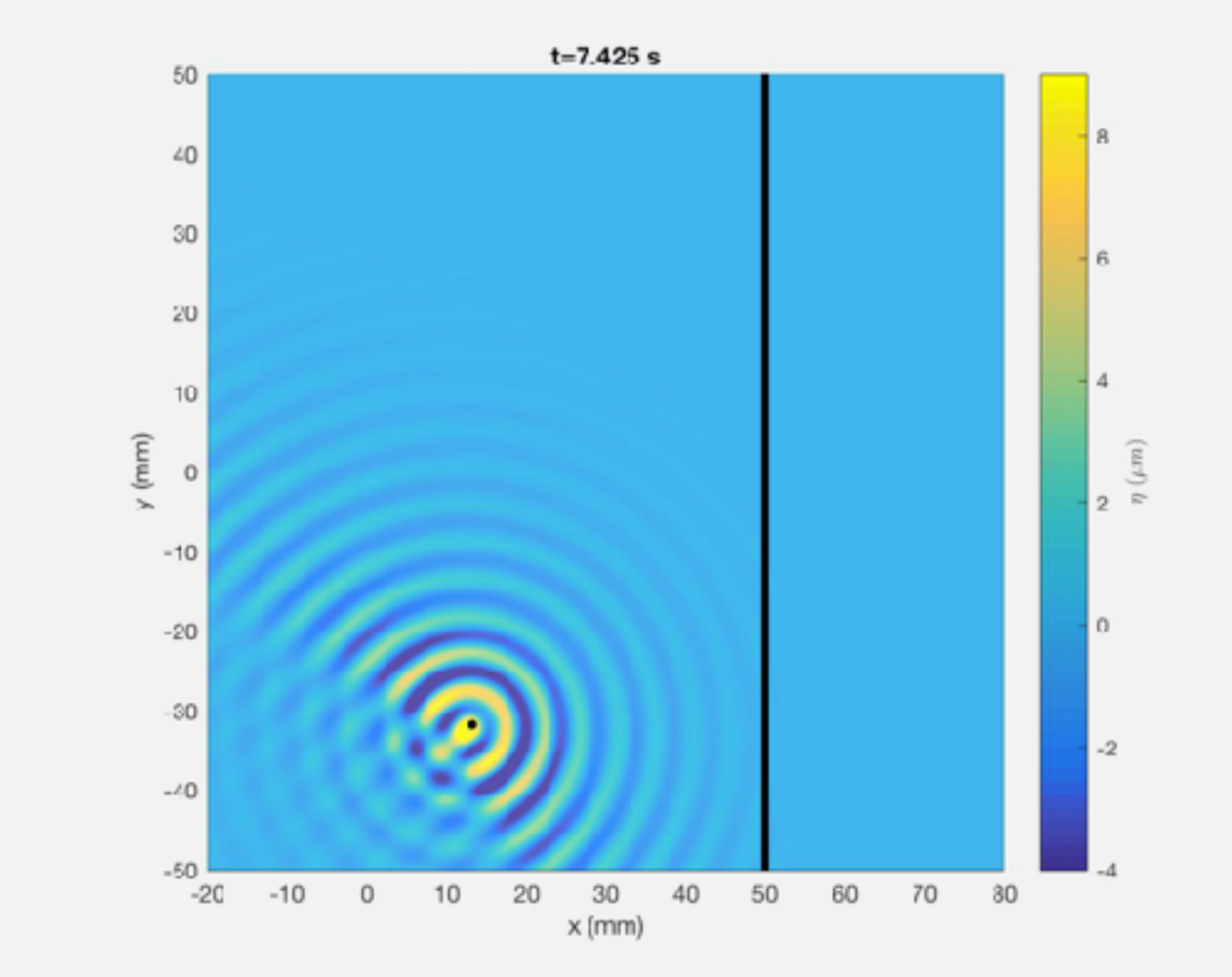
$h_b = 4.10\text{mm} \pm 0.01$

$h_1 \in [2.01\text{-}2.21]$ mm

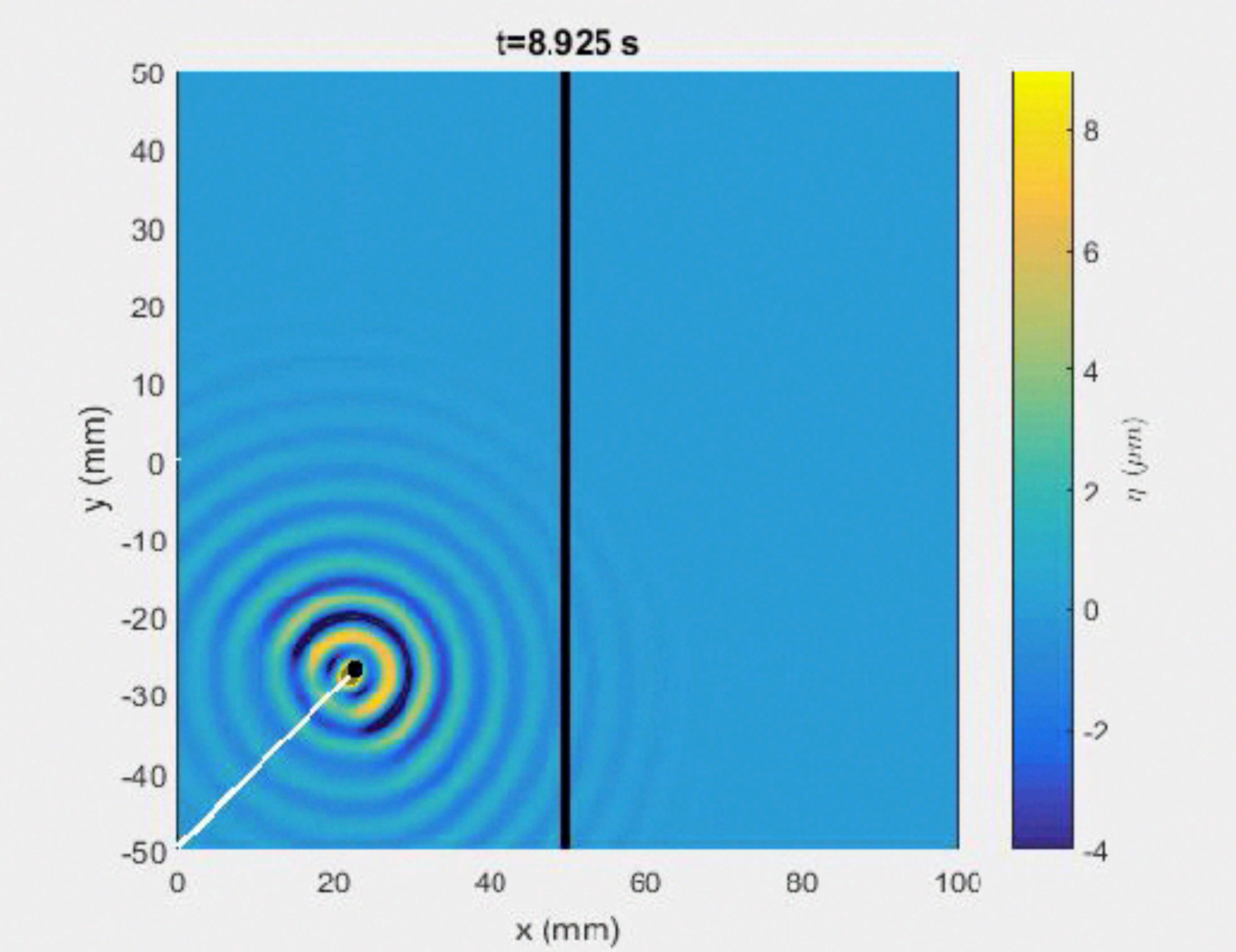
Monochromatic approximation *(Faria 2016)*

- assume only dynamically significant waves have the Faraday wavelength
- variations in depth modeled as a variation in λ_F , phase speed
- allows for robust modeling of walker-boundary interactions

Reflection from a submerged step

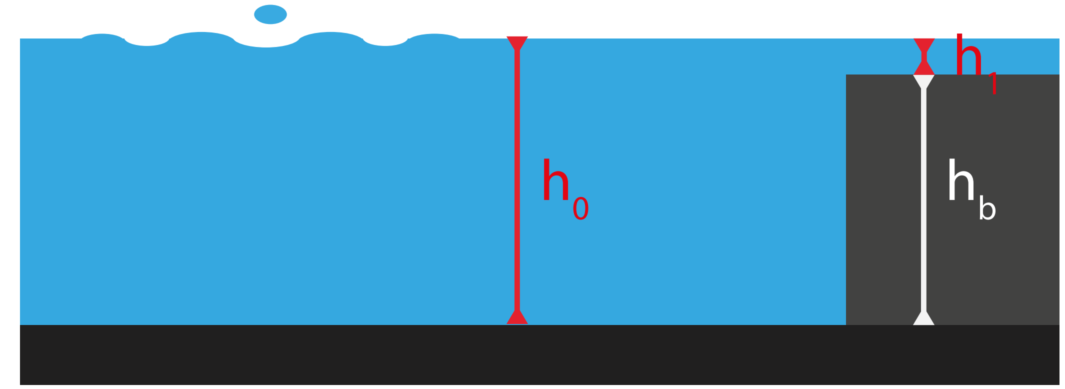


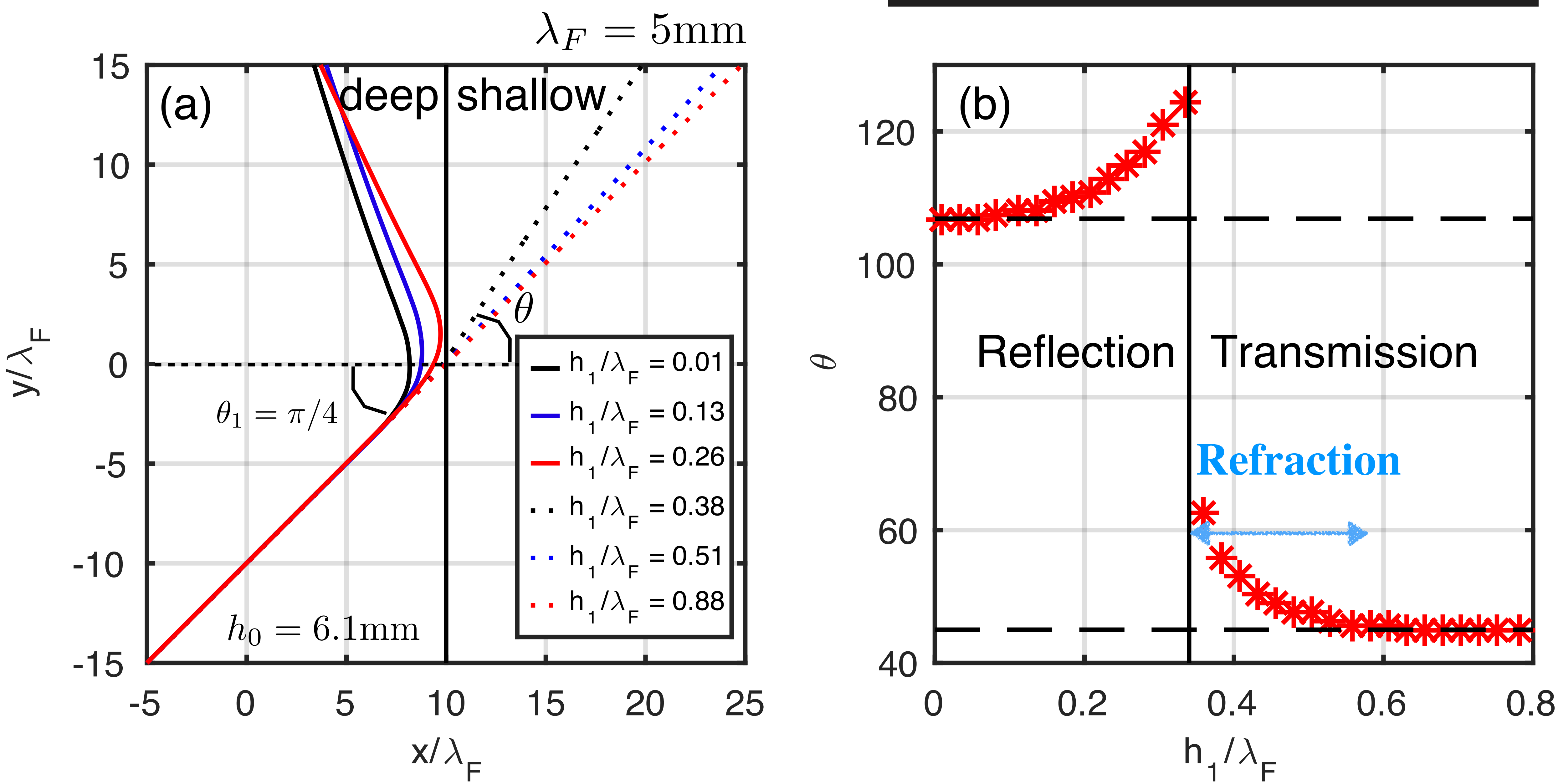
Refraction across a submerged step



From reflection to refraction

Faria (JFM, 2016)





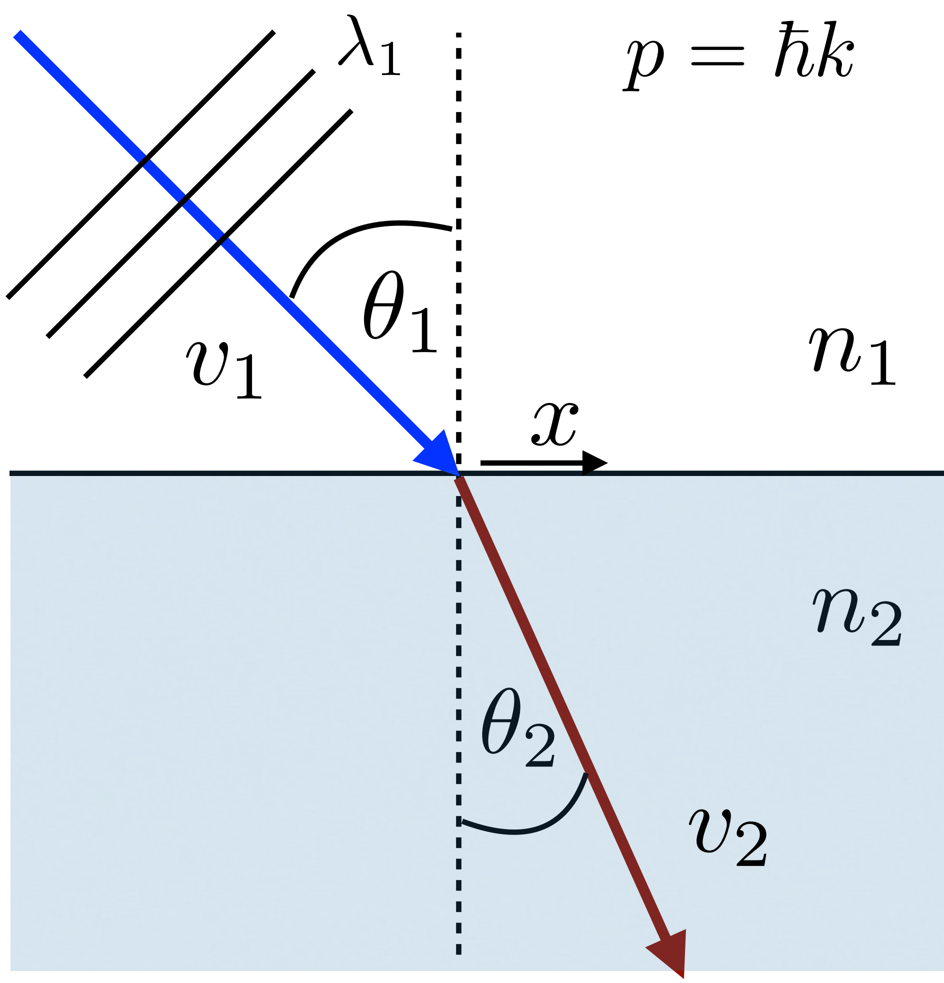
- for a small range of barrier depths (1.5 mm $< h_1 <$ 2.5 mm), transmitted walkers are refracted.

Might they satisfy something akin to Snell's Law?

Snell's law in optics

- refraction of light at interface prescribed by indices of refraction: n_1, n_2

$$\frac{\sin\theta_2}{\sin\theta_1} = \frac{v_2}{v_1} = \frac{\lambda_2}{\lambda_1} = \frac{n_1}{n_2}$$



where wave speeds $v_1 = \dfrac{c}{n_1}$, $v_2 = \dfrac{c}{n_2}$

wave numbers $k_i = 2\pi/\lambda_i = \omega_i/c$

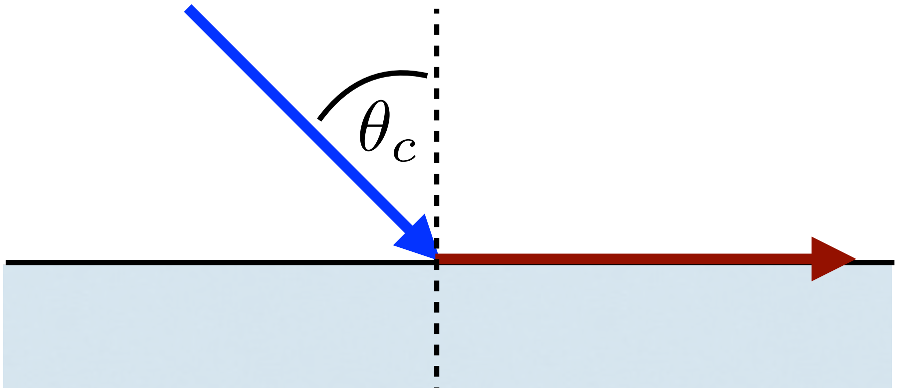
- conservation of photon's x-momentum:

$$\hbar k_1 \sin\theta_1 = \hbar k_2 \sin\theta_2$$

$$\Rightarrow \quad n_1 \sin\theta_1 = n_2 \sin\theta_2$$

- light satisfies ***Fermat's Principle of Least Time,*** *takes fastest route across interface*
- provided $n_1 > n_2$, there is an angle of total internal reflection, for which $\theta_2 = \pi/2$

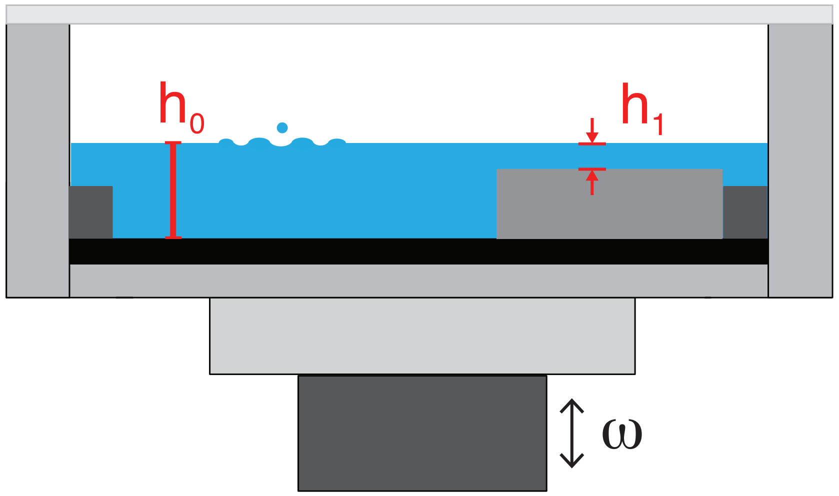
$$\theta_c = \sin^{-1}(n_2/n_1)$$

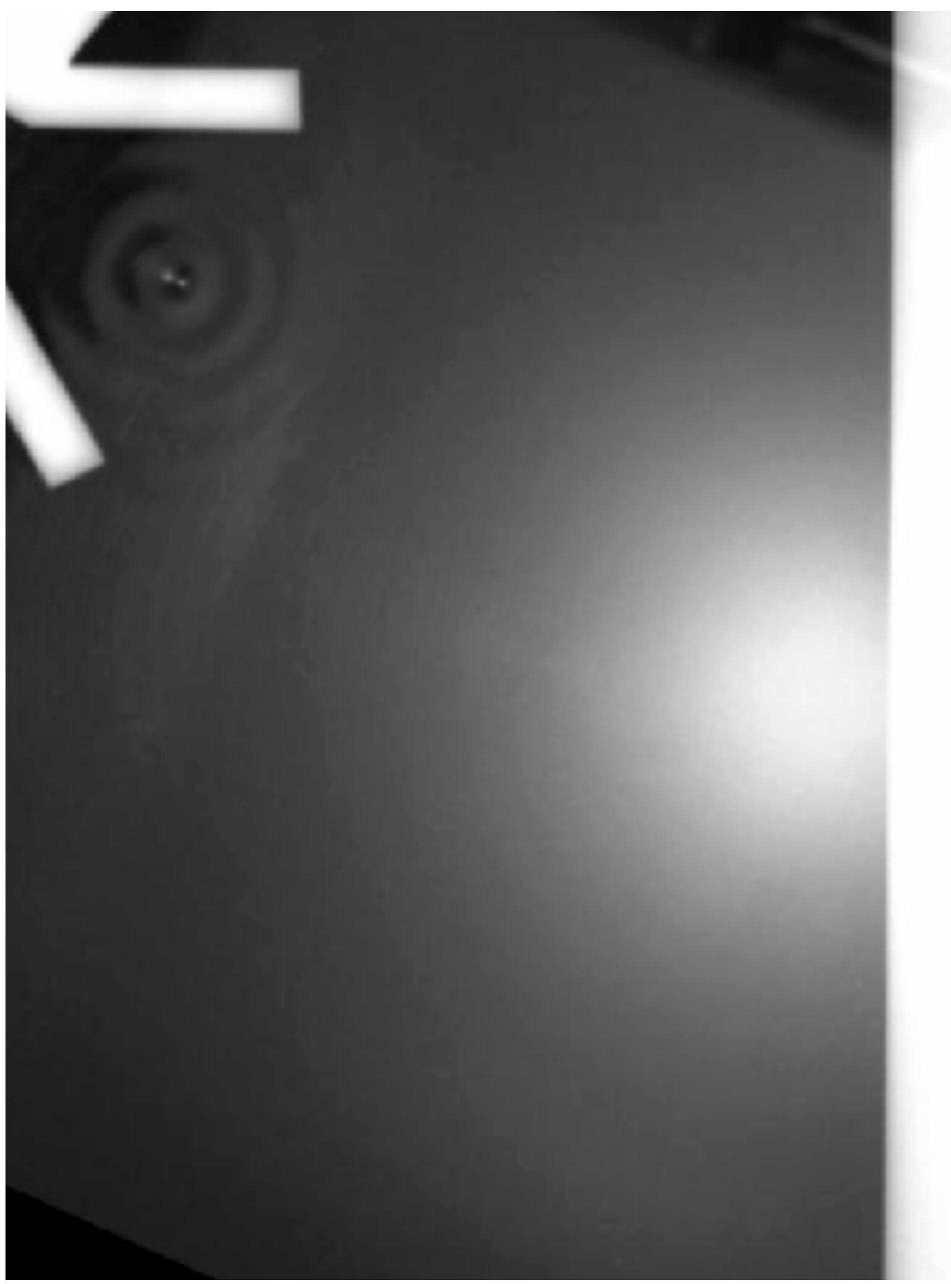


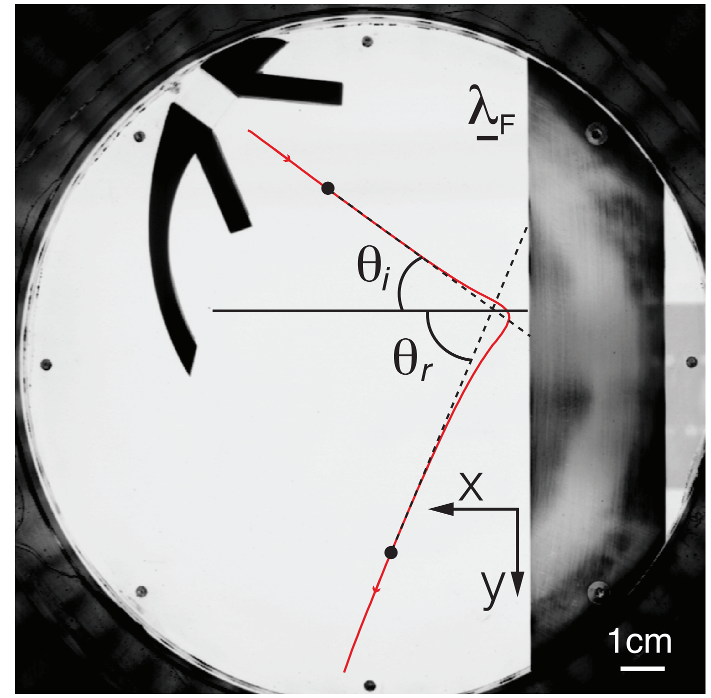
- for $\theta > \theta_c$, light reflects rather than refracts

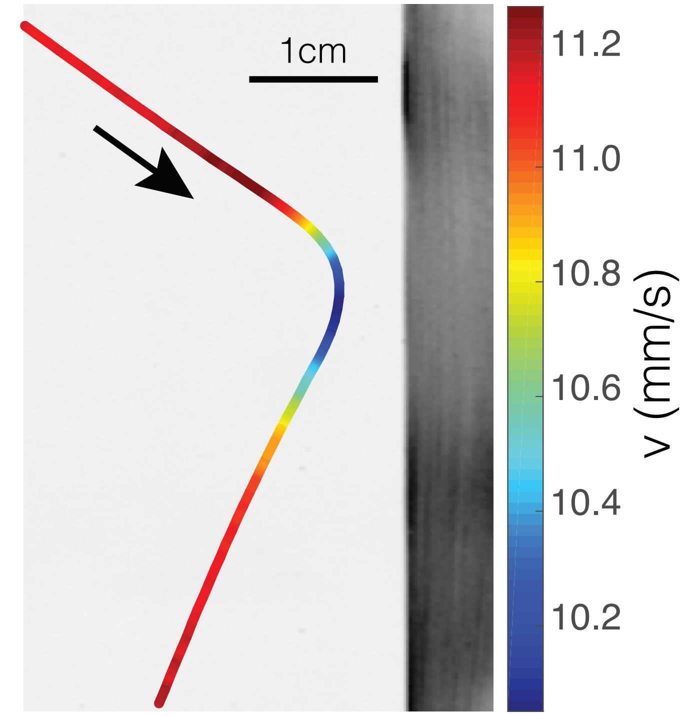
Non-specular reflection

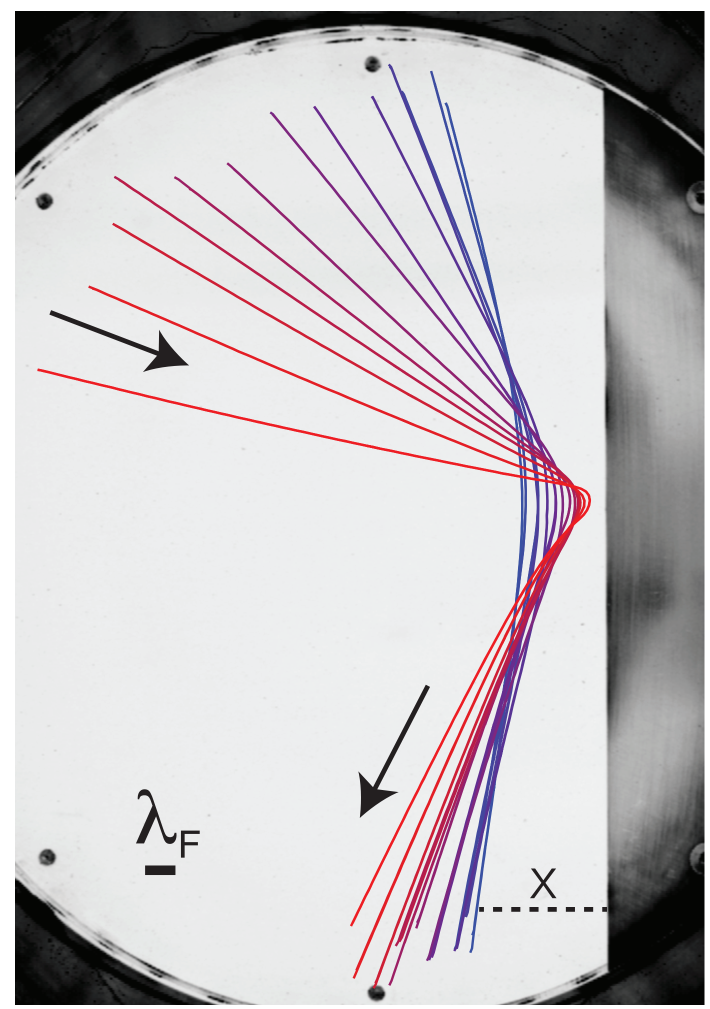
Pucci, Saenz, Faria & Bush (2016)

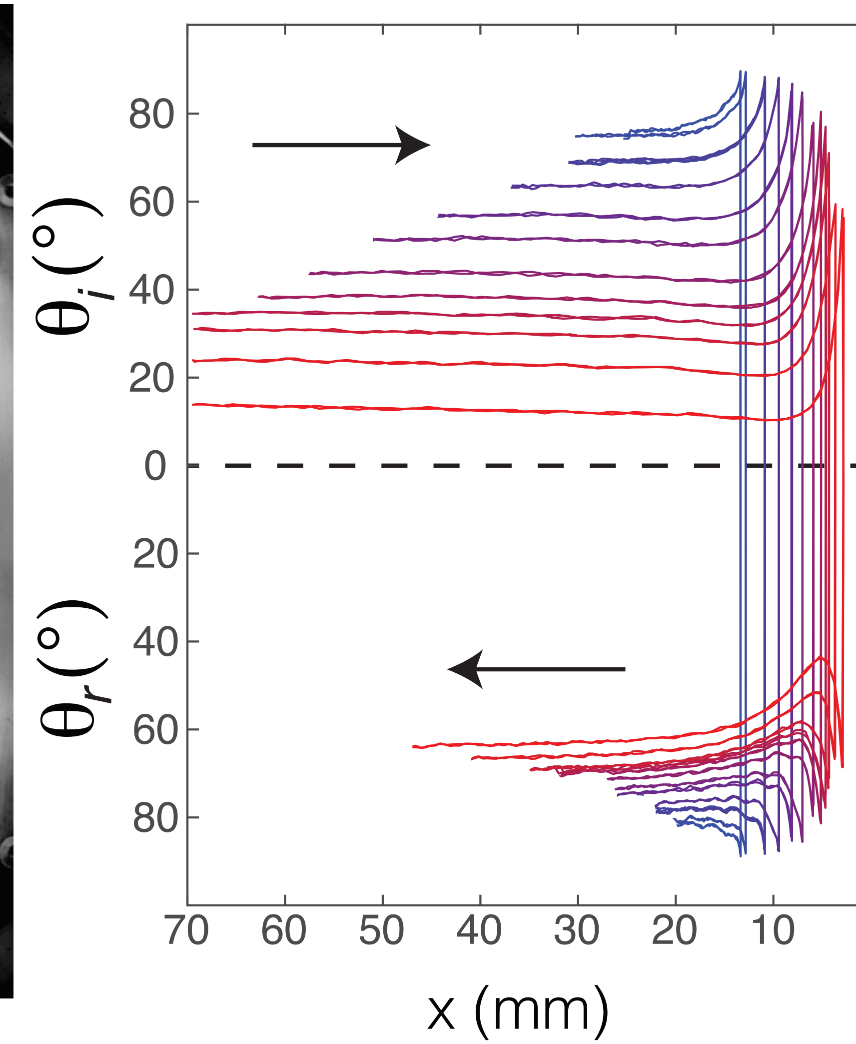












- large range of θ_i focused into small range of θ_r

The Boost equation

Bush, Oza & Molacek (JFM 2014)

In the weak-acceleration limit, the trajectory equation takes the form

$$\frac{d}{dt}\,\mathbf{p_w} + D_w\,\mathbf{v} = \mathbf{F}$$

where the walker mass $m_w = \gamma_B(v)\; m_0$ and momentum $\mathbf{p_w} = m_w \mathbf{v}$

depend on the ***hydrodynamic boost factor***: $\gamma_B = 1 + \frac{\beta}{2\kappa(1+v^2)^{3/2}}$

and a nonlinear drag $D_w = D_0\left(\frac{\mathbf{v}^2}{u_0^2} - 1\right)$ drives it to its free walking speed.

For motion at the free walking speed:

$$\frac{d}{dt}\,\mathbf{p_w} = \mathbf{F}$$

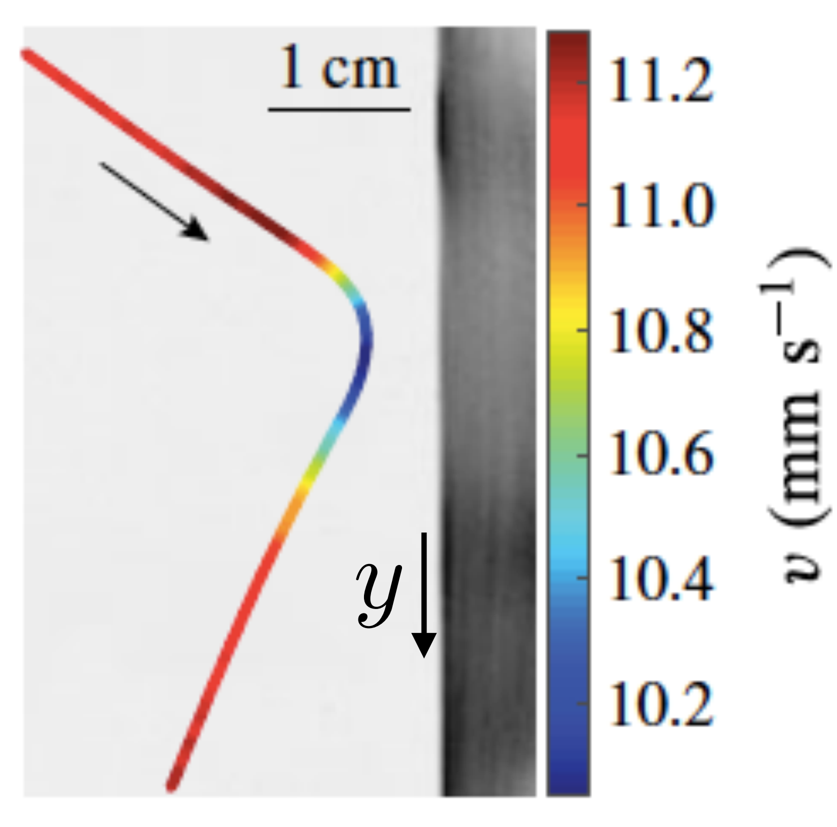
- ***the inviscid dynamics of a particle with a speed-dependent mass***

Non-specular reflection from a submerged boundary

Pucci, Saenz, Faria & Bush (JFM, 2016)

- rationalized in terms of the boost equation

$$\frac{d}{dt}\,\mathbf{p_w} + D_w\,\mathbf{v} = \mathbf{F}$$



- assume force F acts only normal to boundary, and

$$D_w = D_0\left(\frac{\mathbf{v}^2}{u_0^2} - 1\right) < 0 \quad , \quad \text{since } |\mathbf{v}| < v_0$$

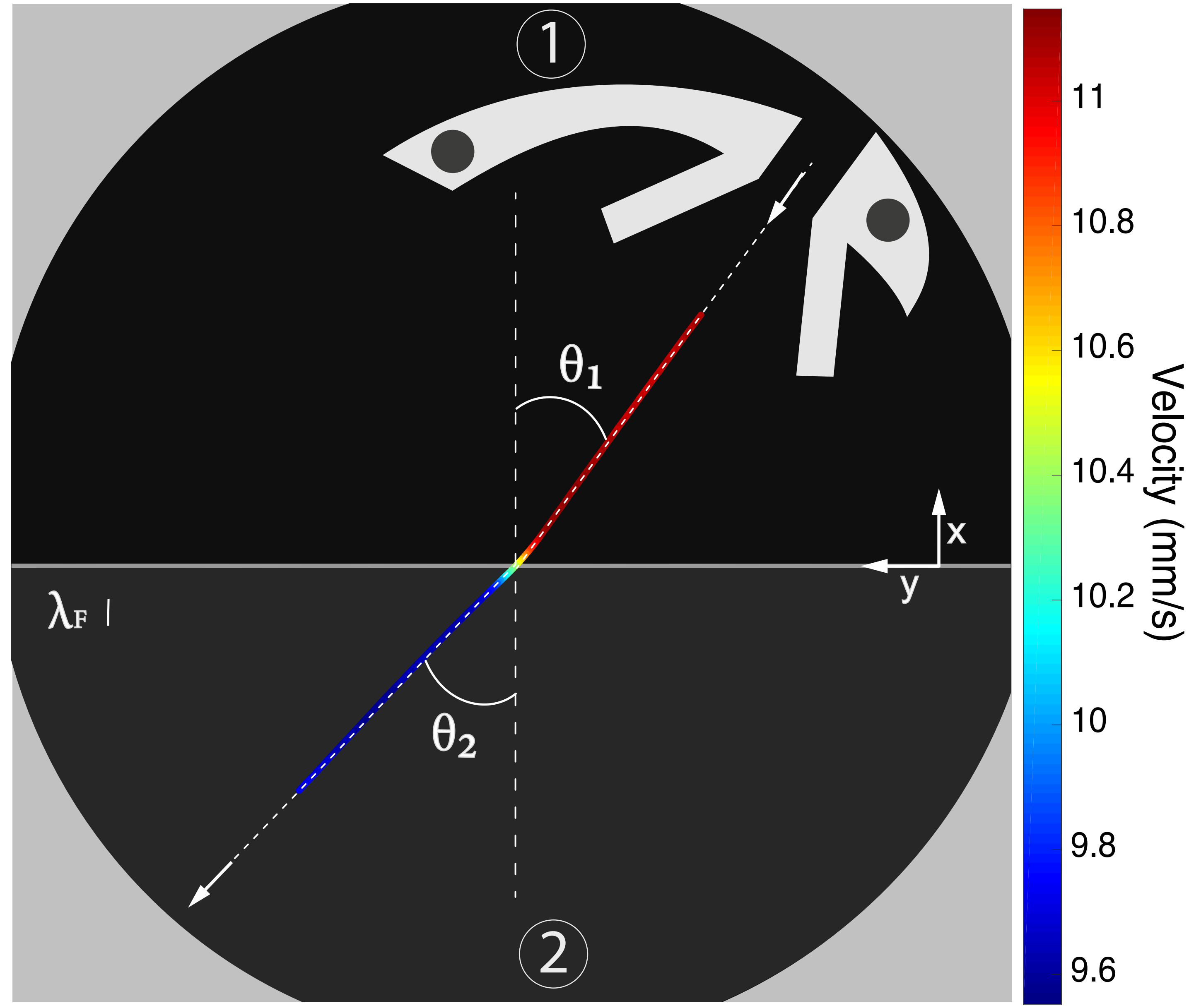
during wall interaction.

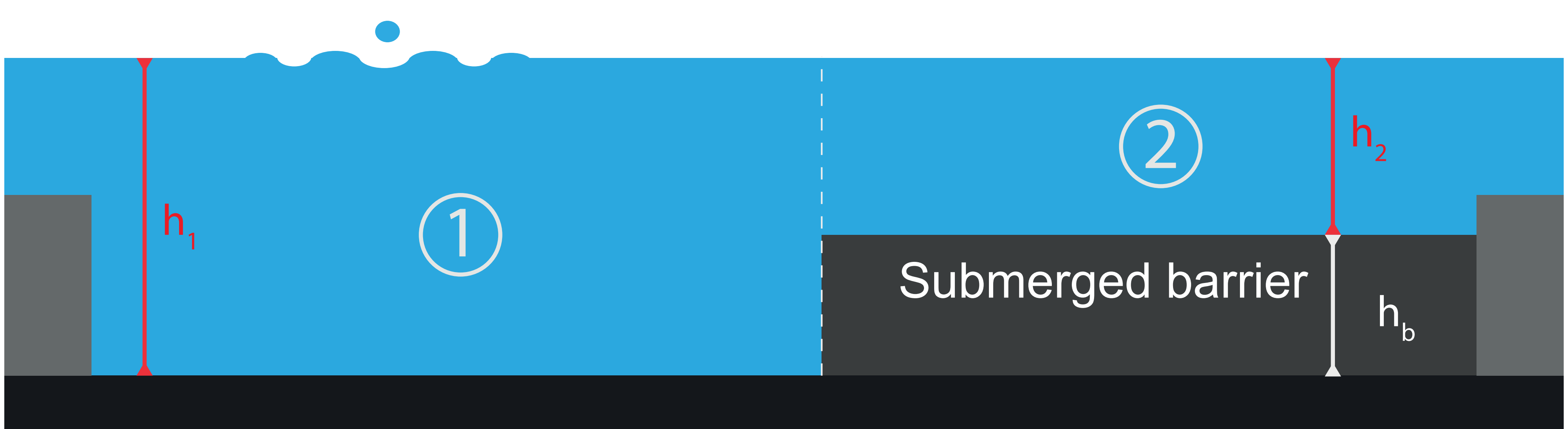
Tangential momentum imparted during reflection:

$$\Delta p_y = \int_{-\infty}^{\infty} -D_w(\mathbf{v})\,\mathbf{v}\,dt \quad > 0$$

- tangential momentum not conserved owing to boost effect: the wave field accelerates the walker towards its free walking speed.

Experiments: Refraction across a submerged boundary

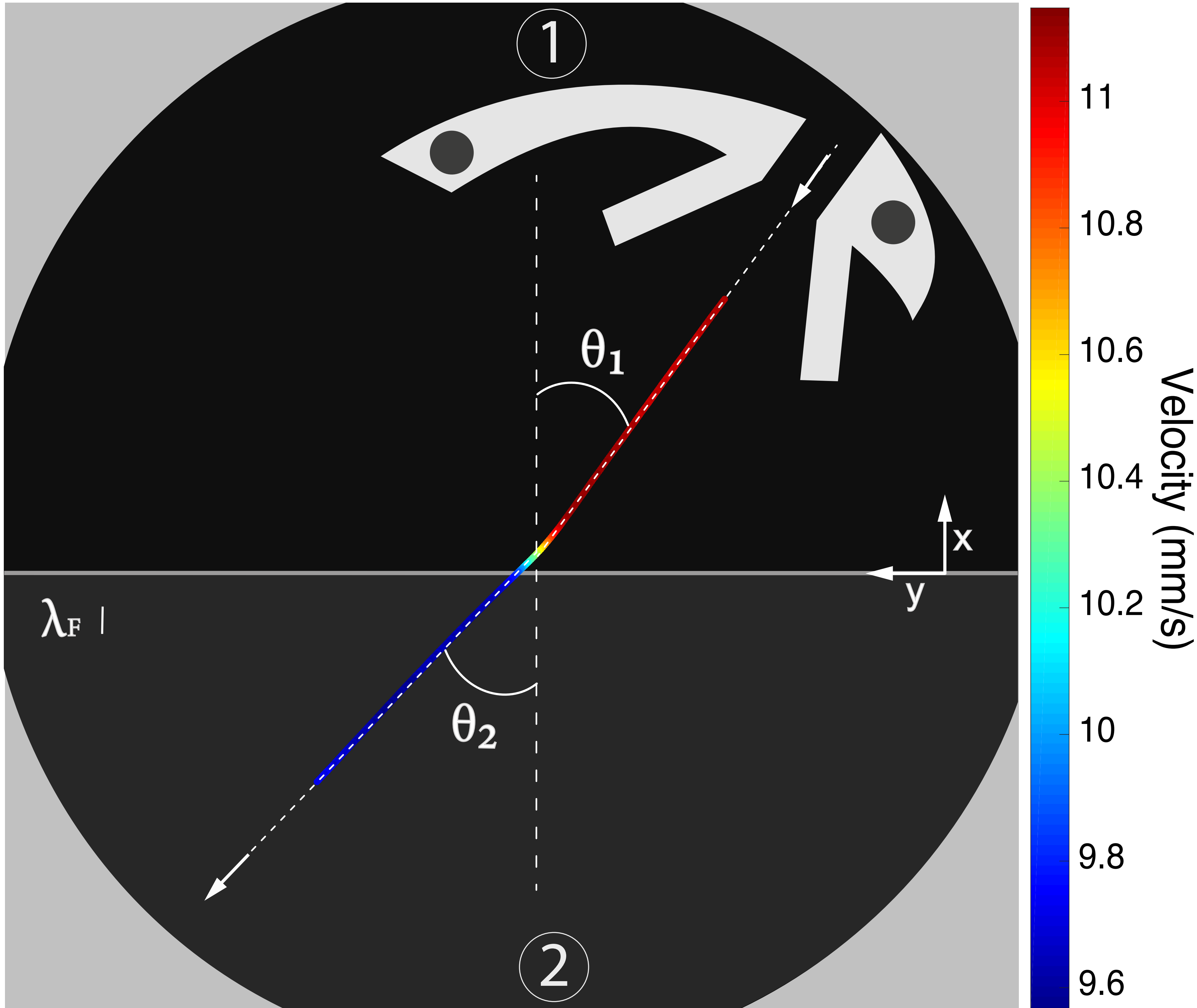




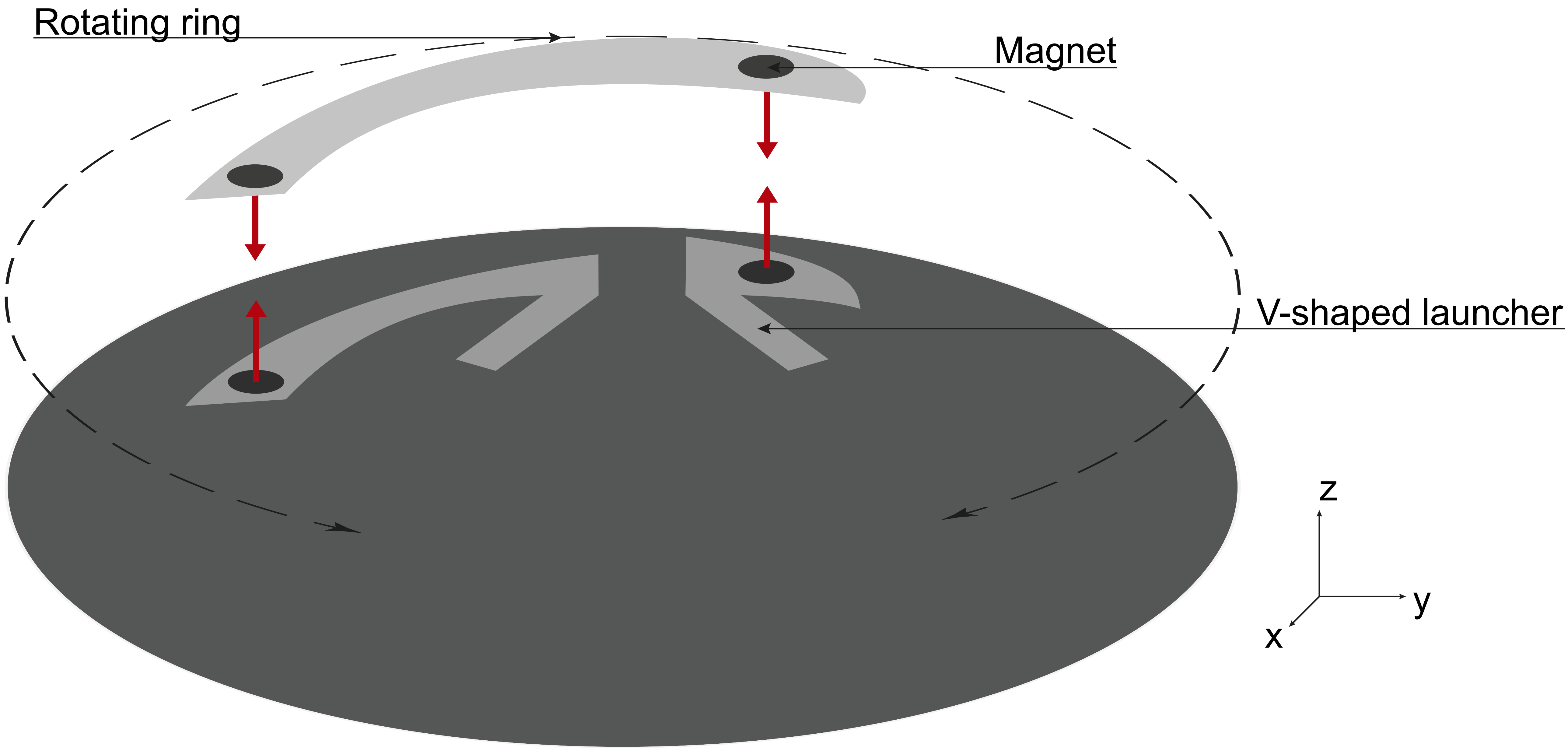
$$h_1 = (6.1 - 6.3)\ \text{mm}$$
$$h_b = 4.1\ \text{mm}$$
$$h_2 = (2.0 - 2.2)\ \text{mm}$$

Snell's Law Set-up

Angles of incidence and refraction

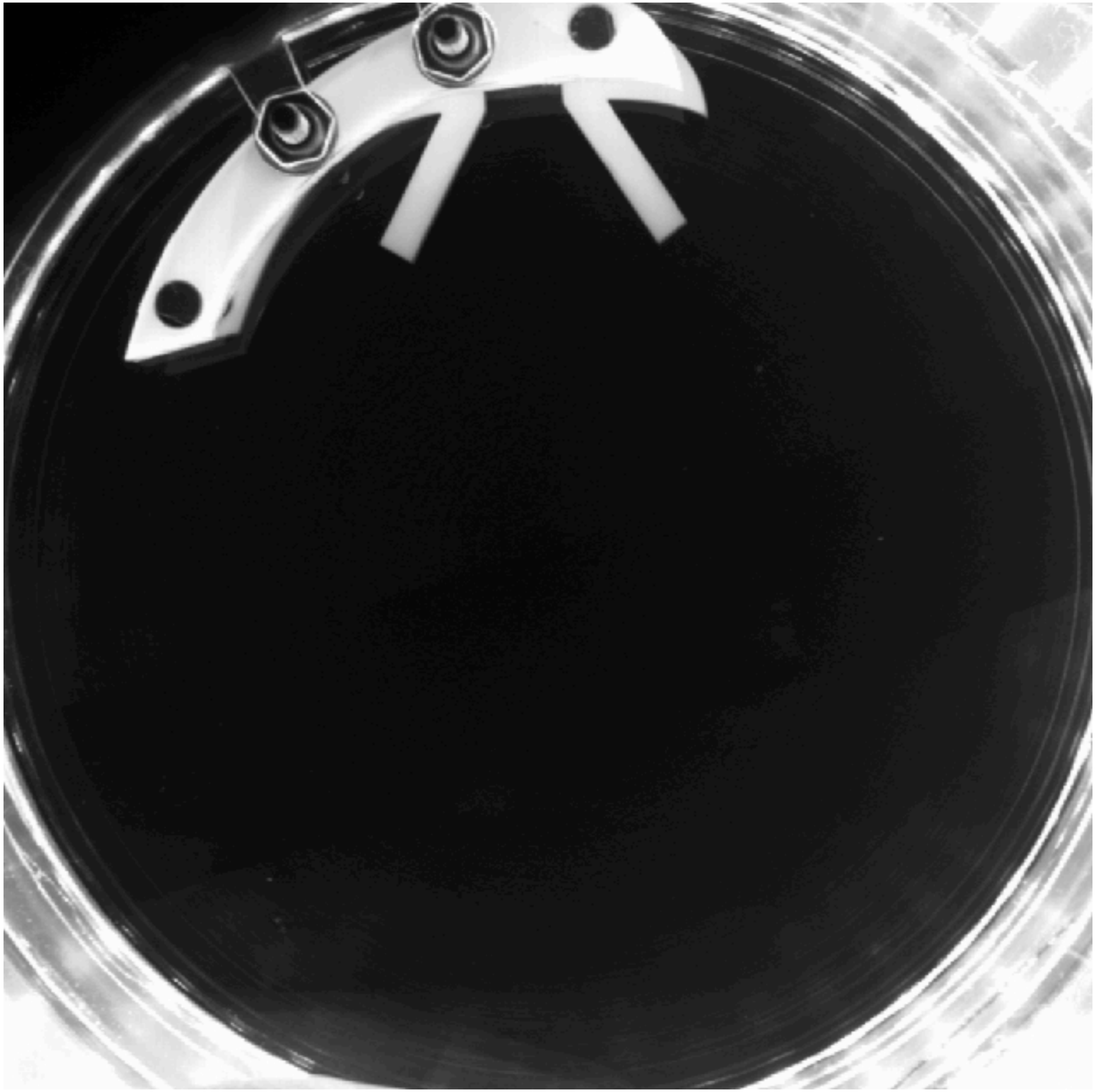


Magnetic launcher

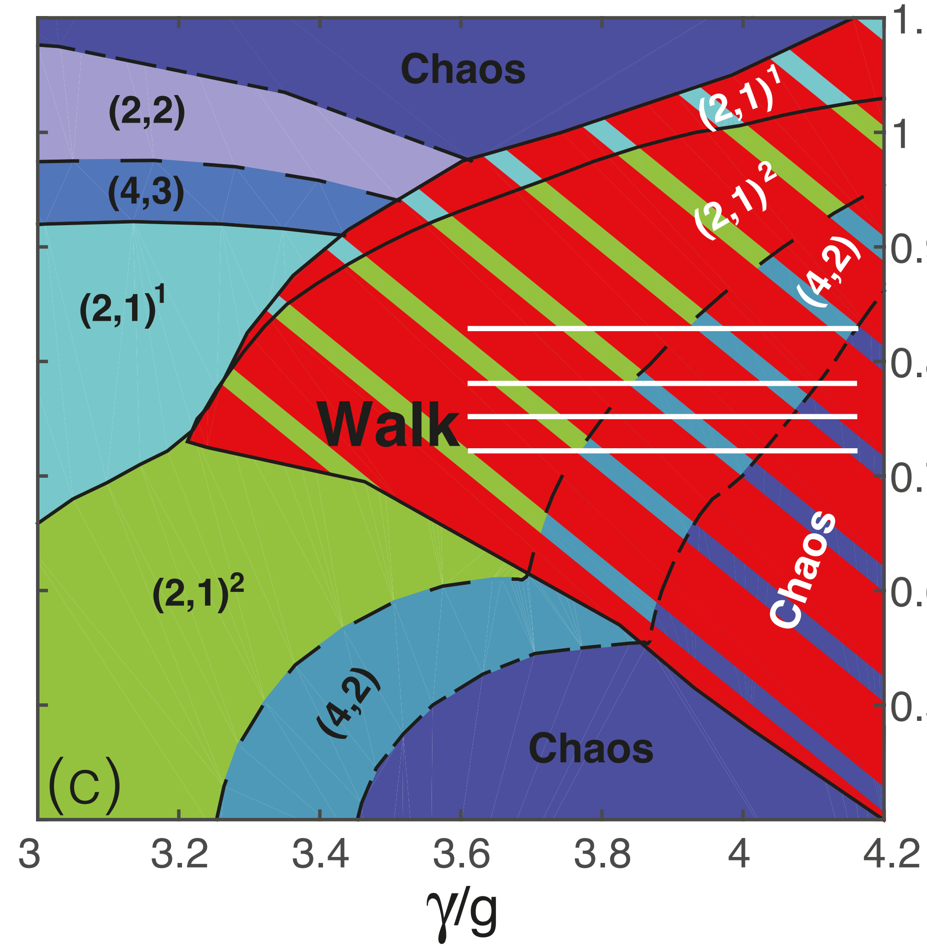


- repeatable experiments require isolation of walkers from air currents - a lid
- launcher allows for continuous running of experiments without opening lid

Magnetic launcher



Parameter regime explored



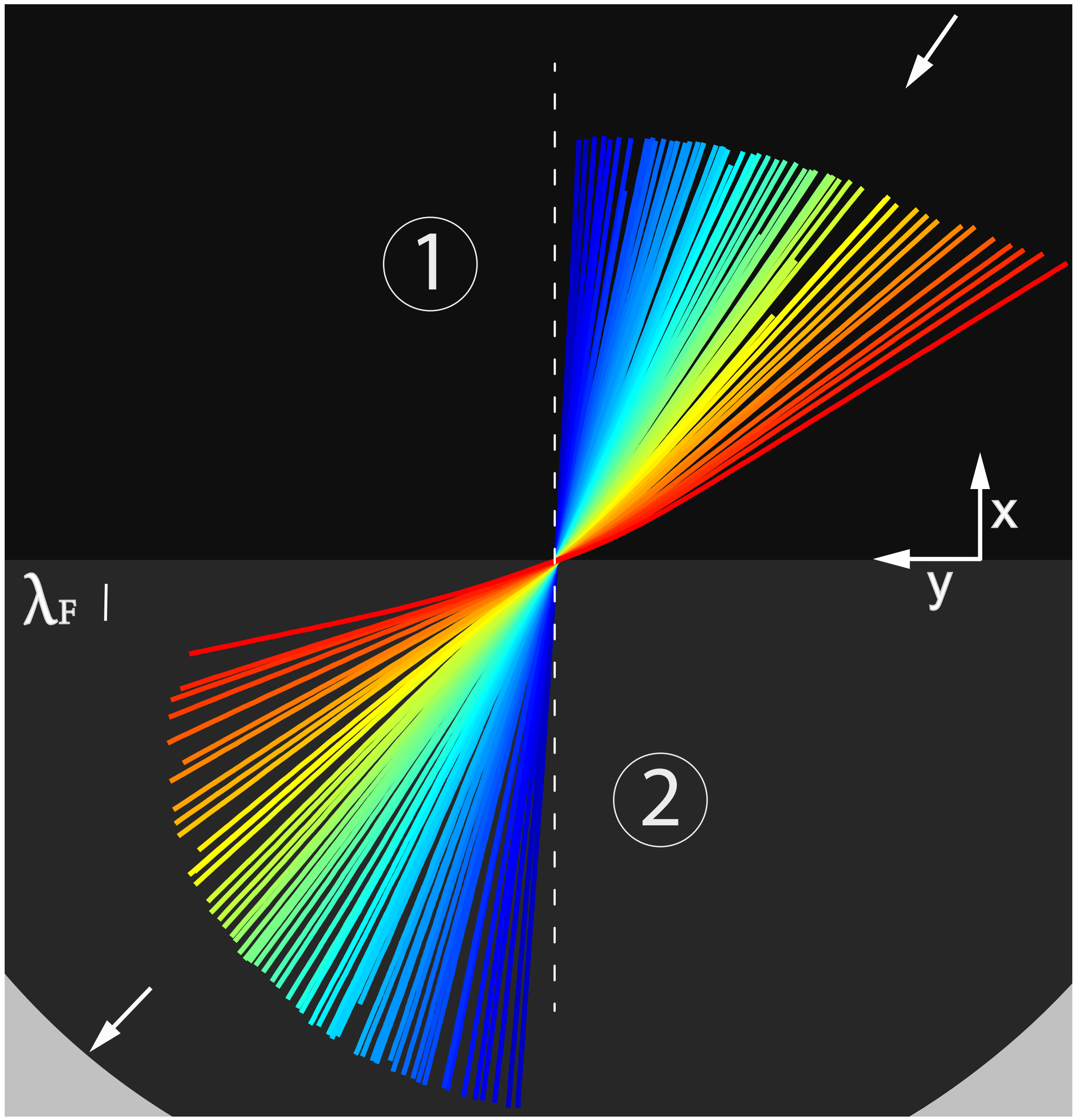
$$V_i = \frac{\omega}{\sqrt{\frac{\sigma}{\rho \cdot R^3}}}$$

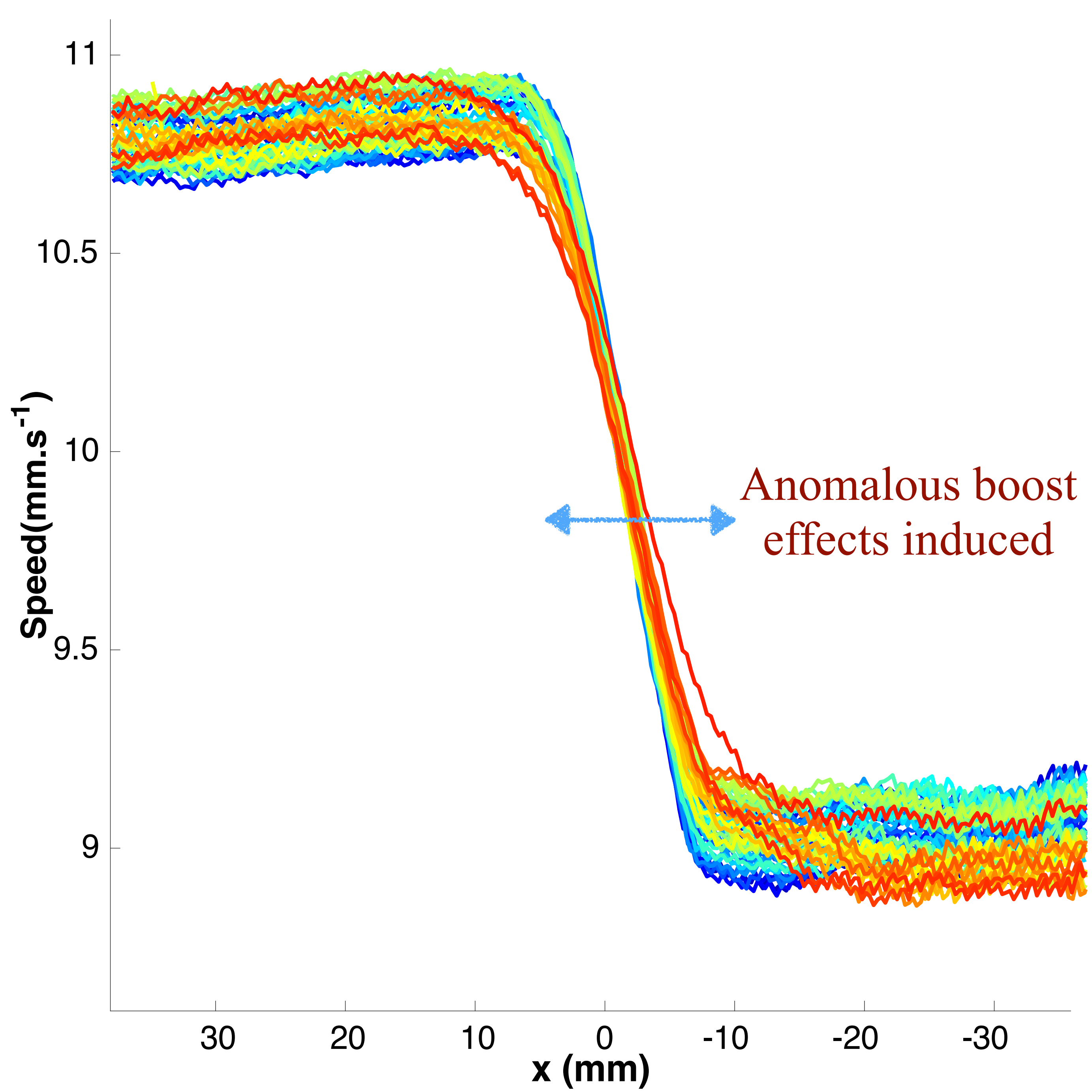
➡ ~40 trajectories per dataset
≈ **1200 Trajectories**

Drop radius : R (mm)	Shallow region depth: h2 (mm)	Memory
0.39	**2.01**	**0.85-0.90-0.95-0.99**
-	**2.11**	**0.85-0.90-0.95-0.99**
-	**2.21**	**0.85-0.90-0.95-0.99**
0.375	2.01	**0.90-0.95-0.99**
-	2.11	**0.85-0.90-0.95-0.99**
-	2.21	**0.90-0.95-0.99**
0.365	2.01	**0.90-0.95-0.99**
-	2.11	**0.90-0.95-0.99**
-	2.21	**0.90-0.95-0.99**
0.355	2.11	**0.99**
-	2.21	**0.95-0.99**

A single data set

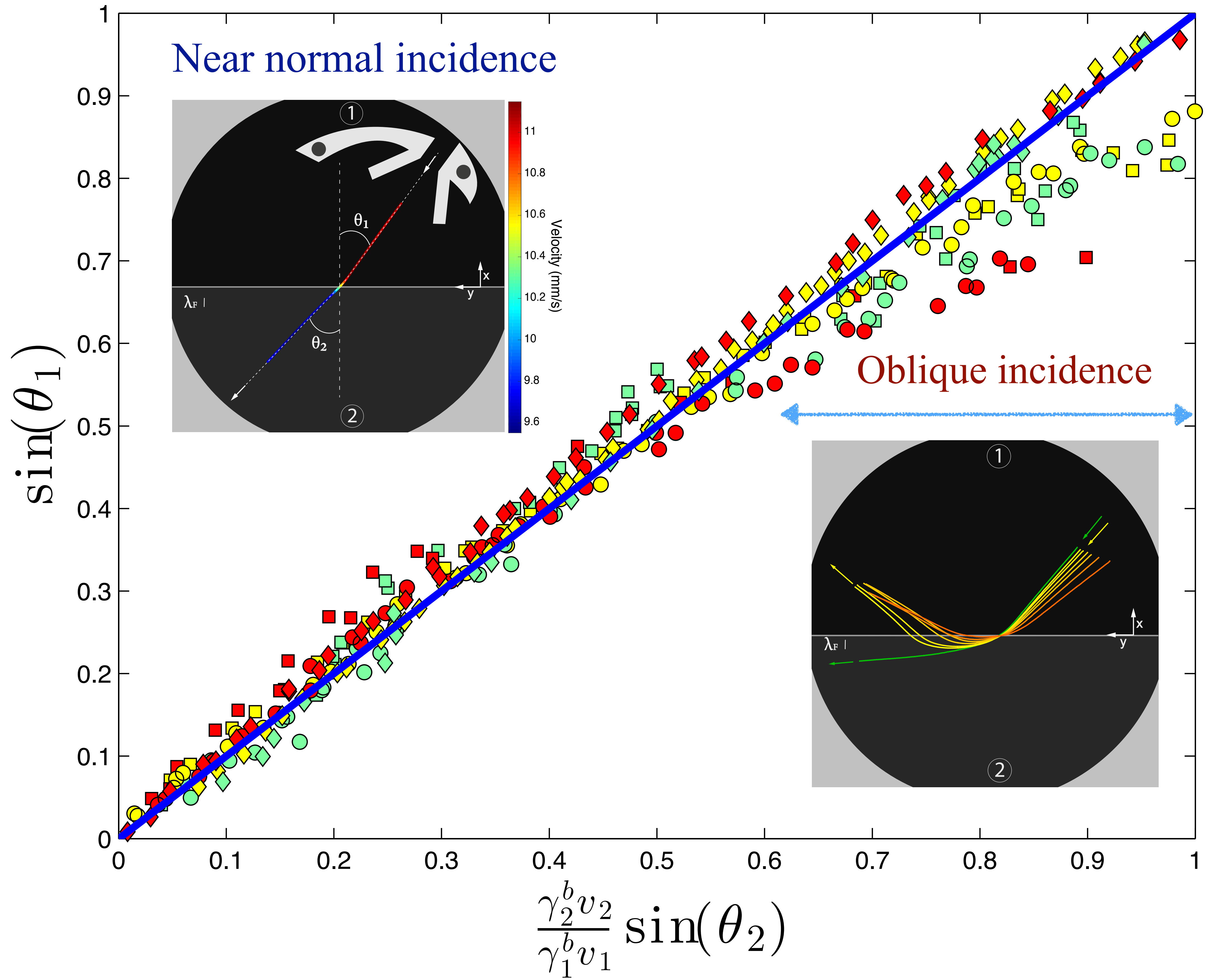
R=0.39mm / Mem=0.95 / h_1=2.21mm





- walker speed decreases by approximately 20% over a distance ~1cm
- boost effects associated with speed variations less significant in refraction

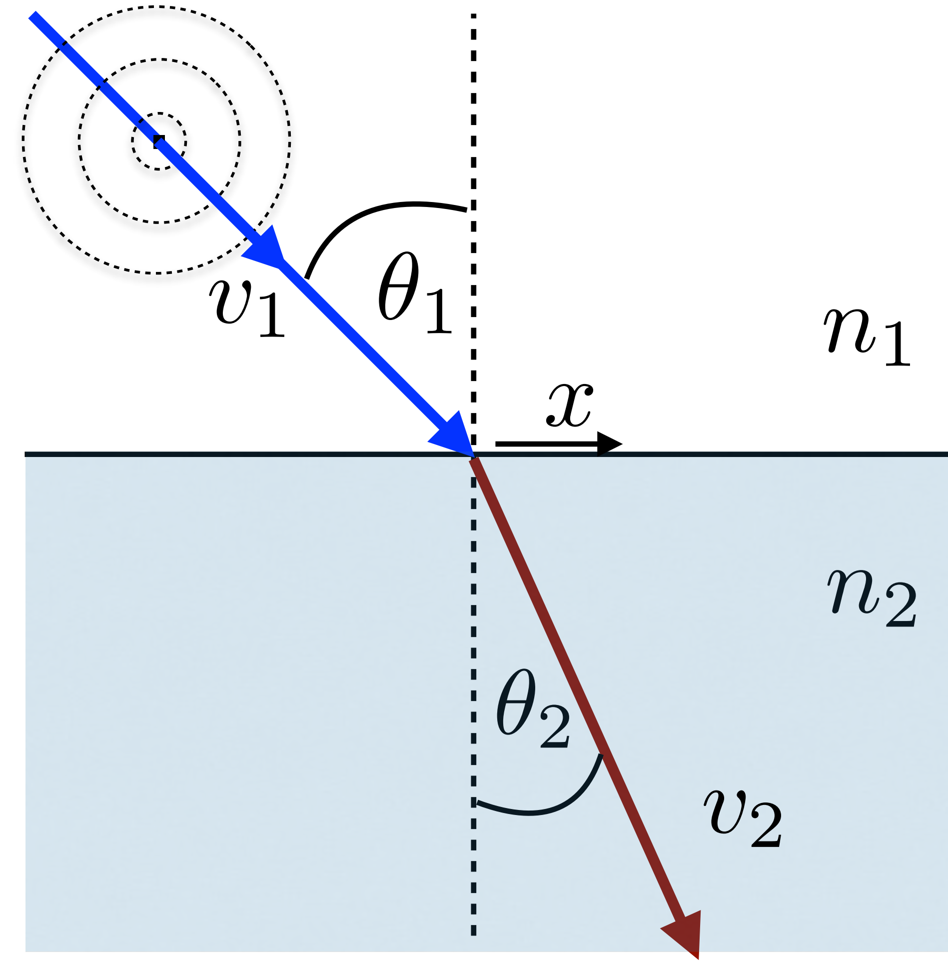
Snell's law for walking drops



Snell's law for walkers

- conservation of walker's x-momentum requires:

$$\gamma_1 m \; v_1 \sin\theta_1 \;\; = \;\; \gamma_2 m \; v_2 \sin\theta_2$$



- effective Snell's Law:

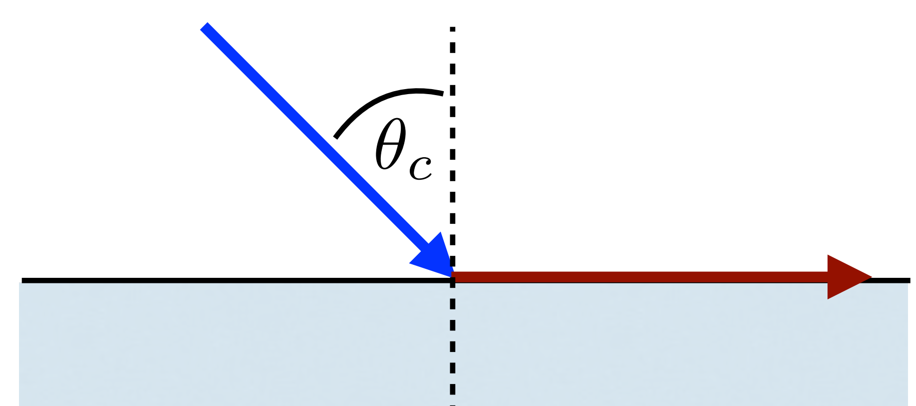
$$\frac{\sin\theta_2}{\sin\theta_1} = \frac{\gamma_1 v_1}{\gamma_2 v_2} = \frac{n_1}{n_2}$$

- effective index of refraction: $n = \gamma_B v$

- consistent with Maupertuis's Principle of Least Action: $I \; = \; \int \gamma_B m v^2 \; dt$

- provided $n_1 > n_2$, there is an angle of total internal reflection, for which $\theta_2 = \pi/2$

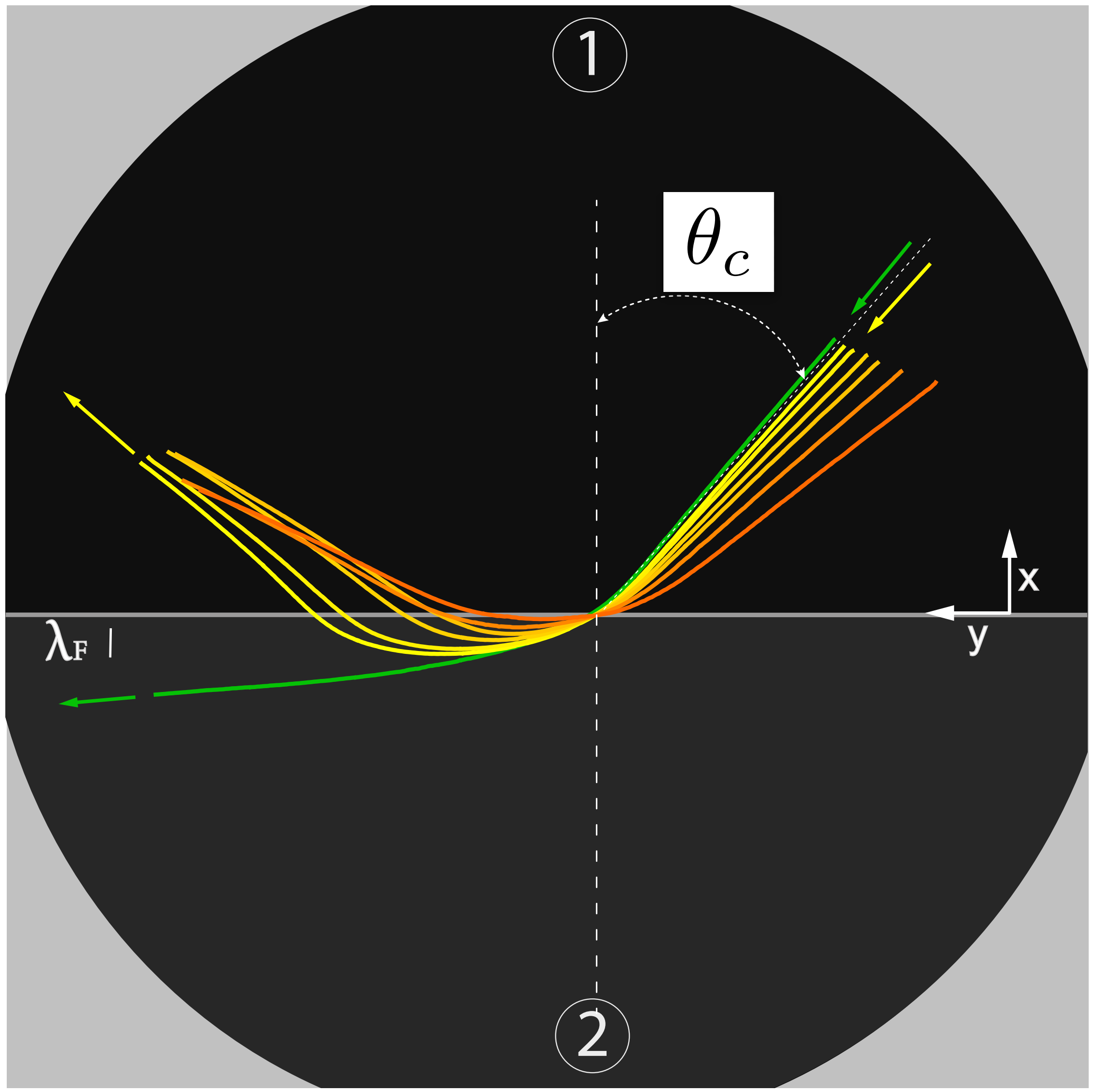
$$\theta_c \;\; = \;\; \sin^{-1}\frac{\gamma_2 v_2}{\gamma_1 v_1}$$

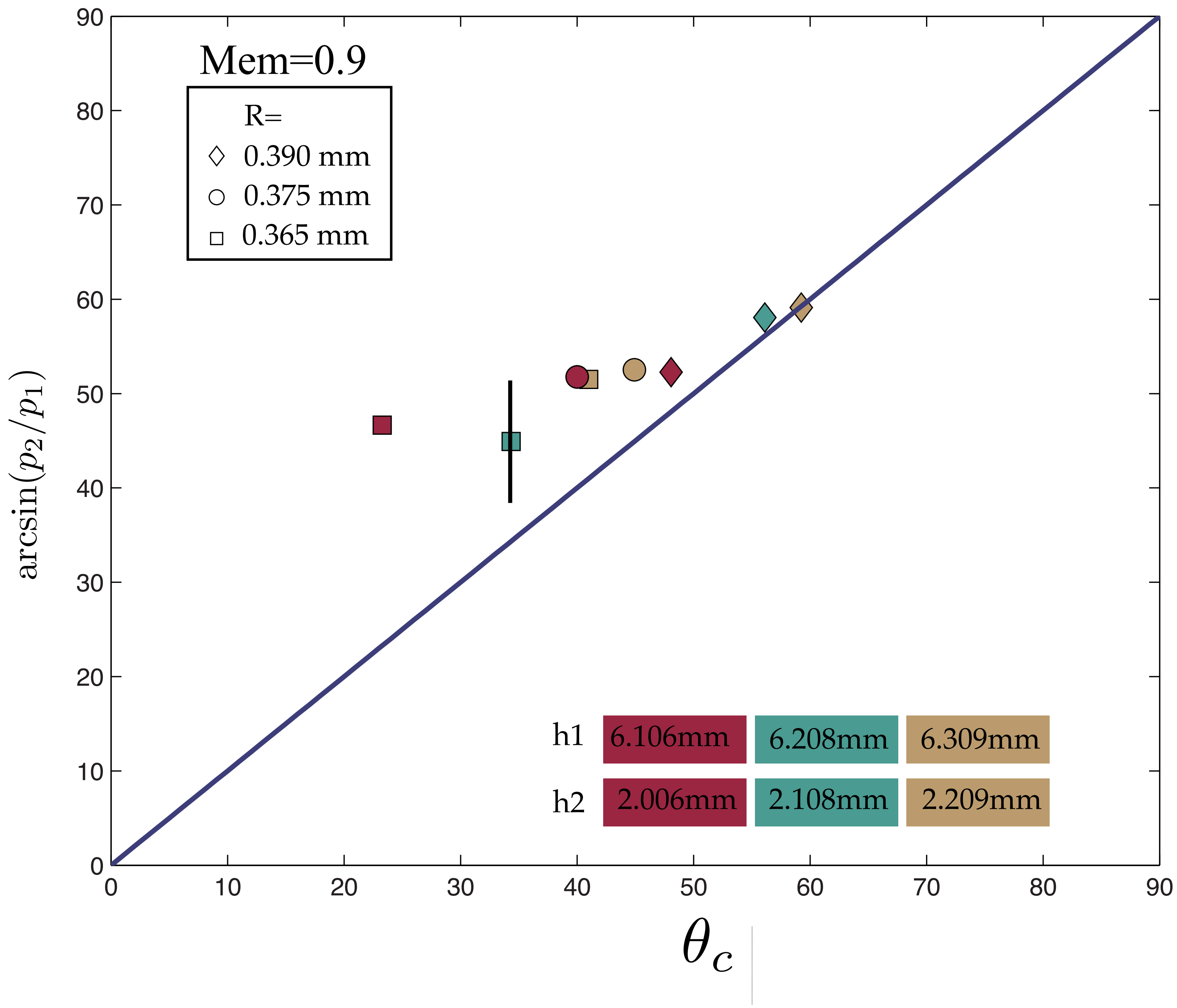


- for $\theta > \theta_c$, walkers reflect rather than refract

Total internal reflection

One anticipates: $$\theta_c = \sin^{-1}\frac{n_2}{n_1} = \sin^{-1}\frac{\gamma_2 v_2}{\gamma_1 v_1}$$

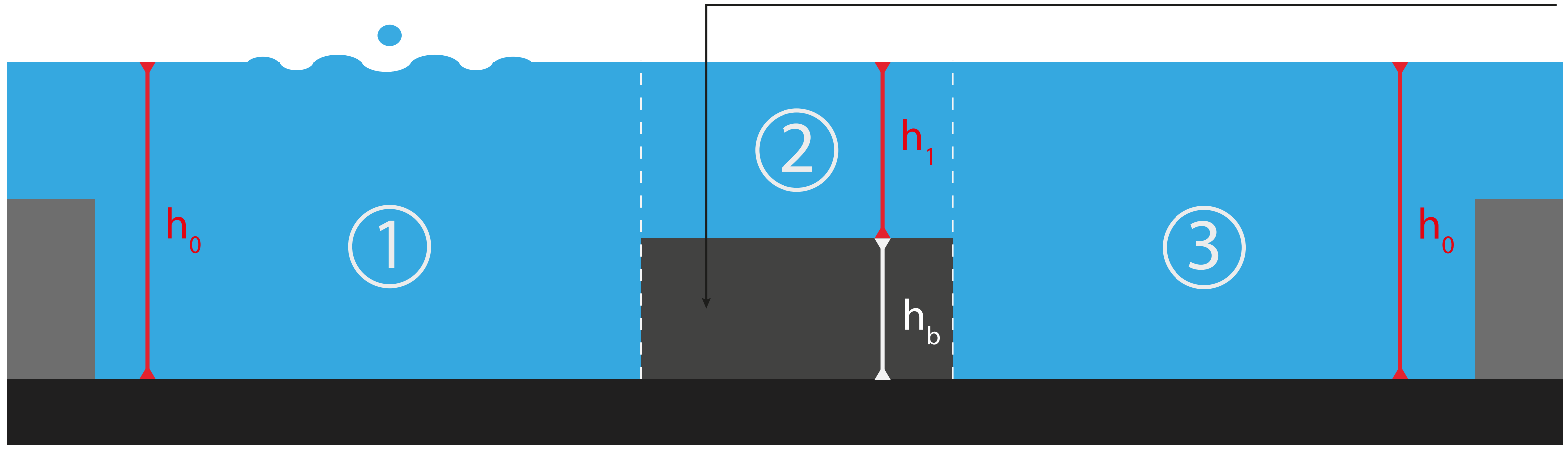




- discrepancy attributable to anomalous Boost effect arising during oblique approach

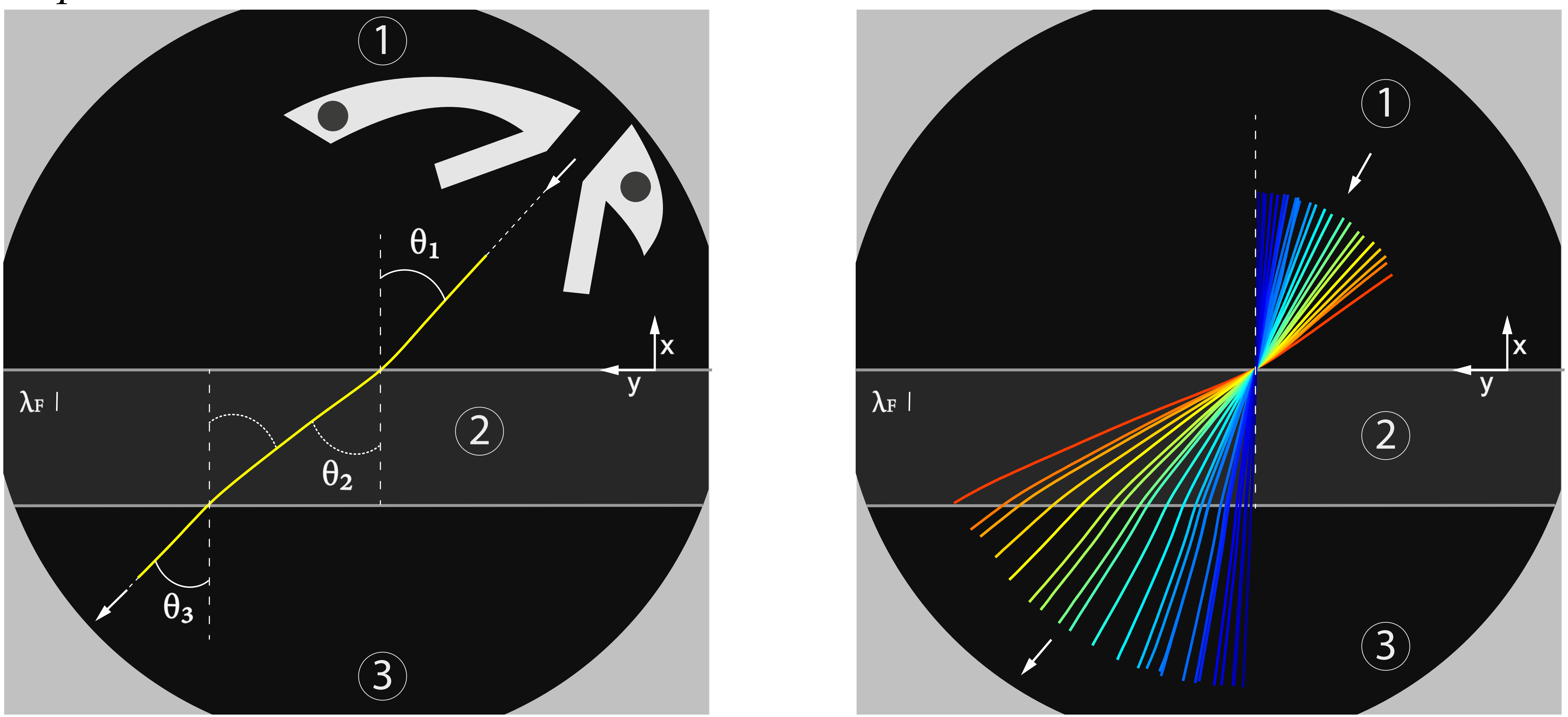
Reciprocity

Side view

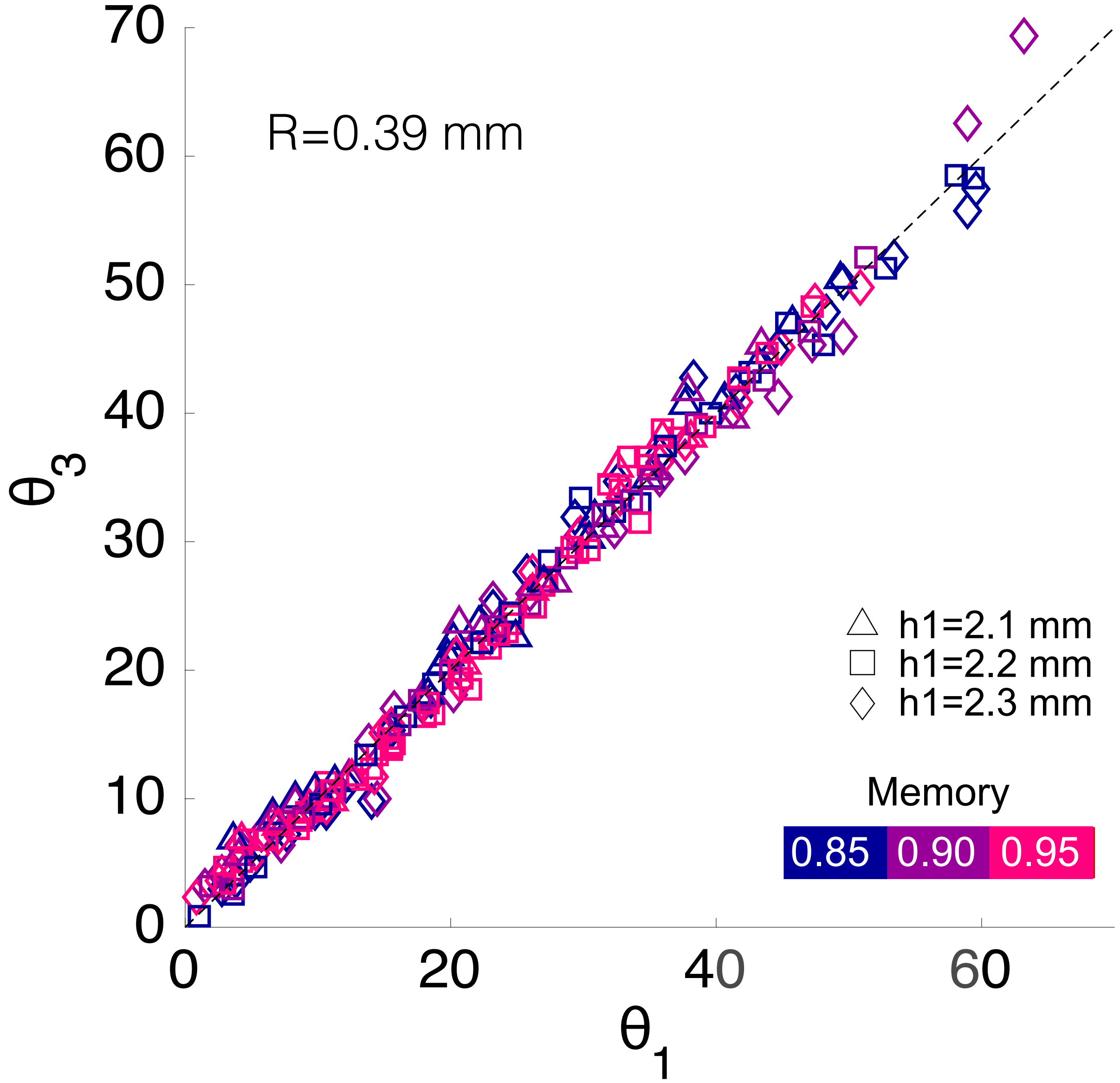


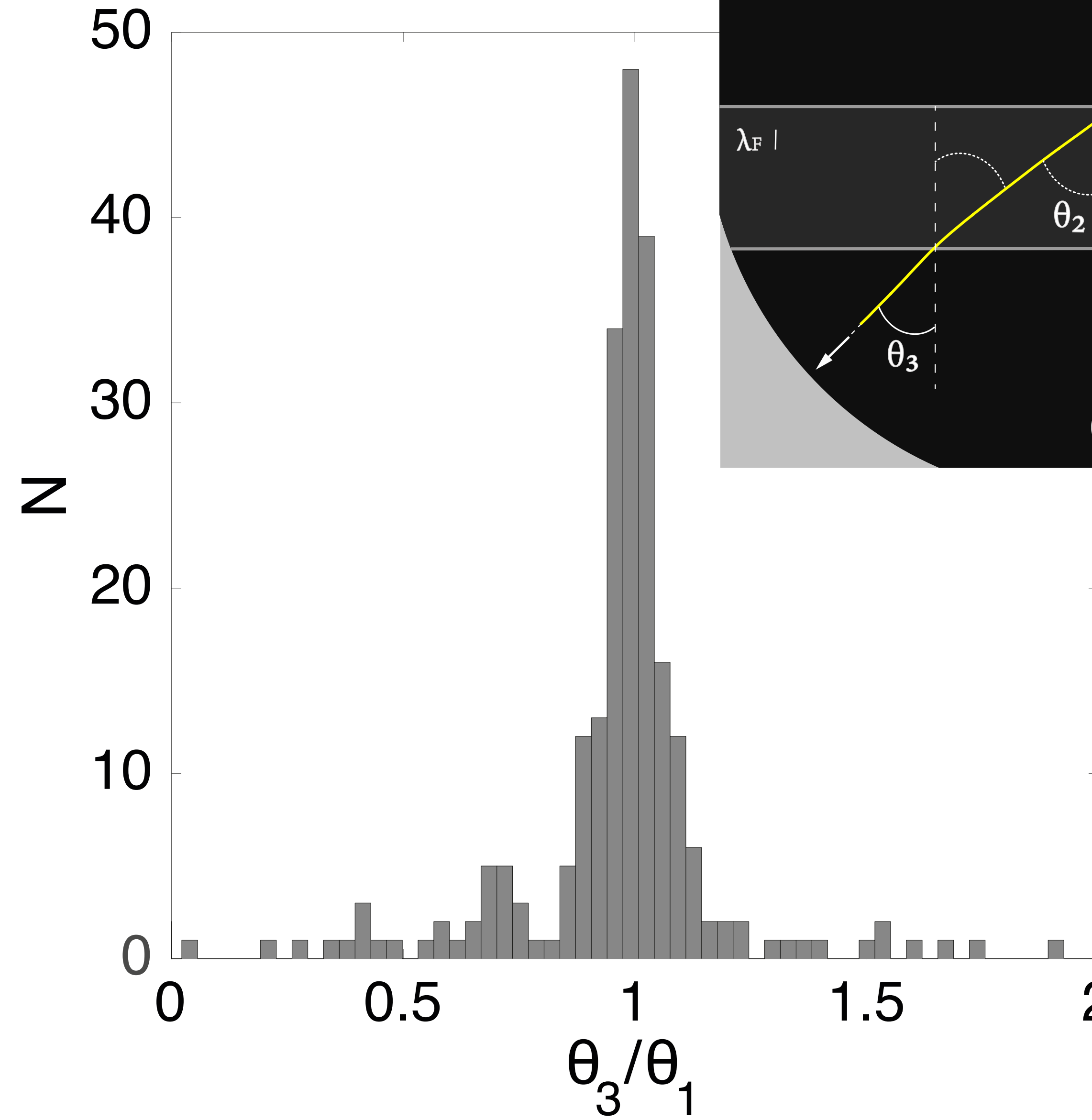
Acrylic barrier
≃ 6 λ_F

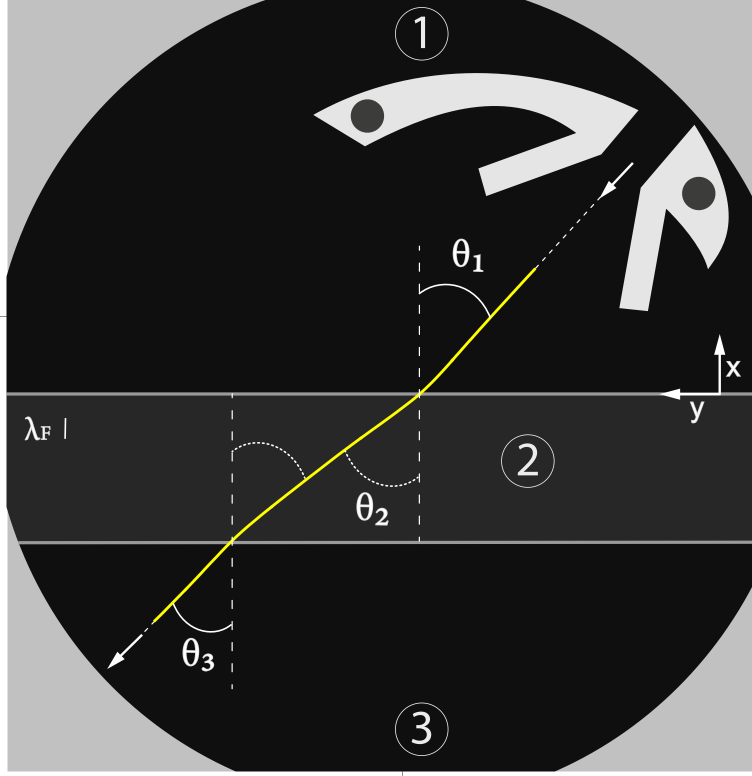
Top view



Reciprocity





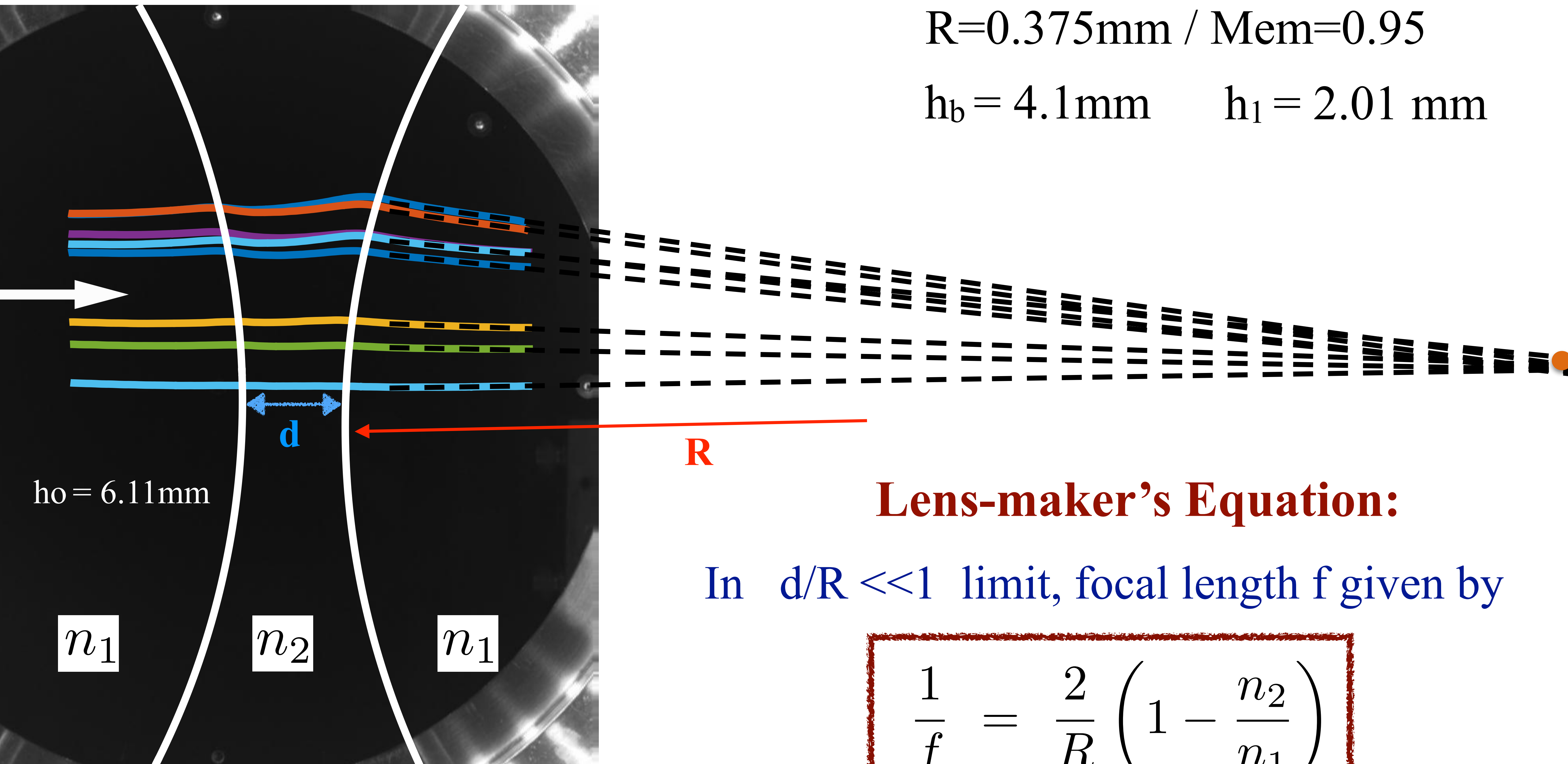


$$\boxed{\theta_3 = \theta_1}$$

$$\frac{\sin\theta_2}{\sin\theta_1} = \frac{n_1}{n_2} = \frac{\sin\theta_2}{\sin\theta_3}$$

Topographic lensing

Biconcave submerged acrylic barrier



R=0.375mm / Mem=0.95

$h_b = 4.1$mm $h_1 = 2.01$ mm

Lens-maker's Equation:

In d/R <<1 limit, focal length f given by

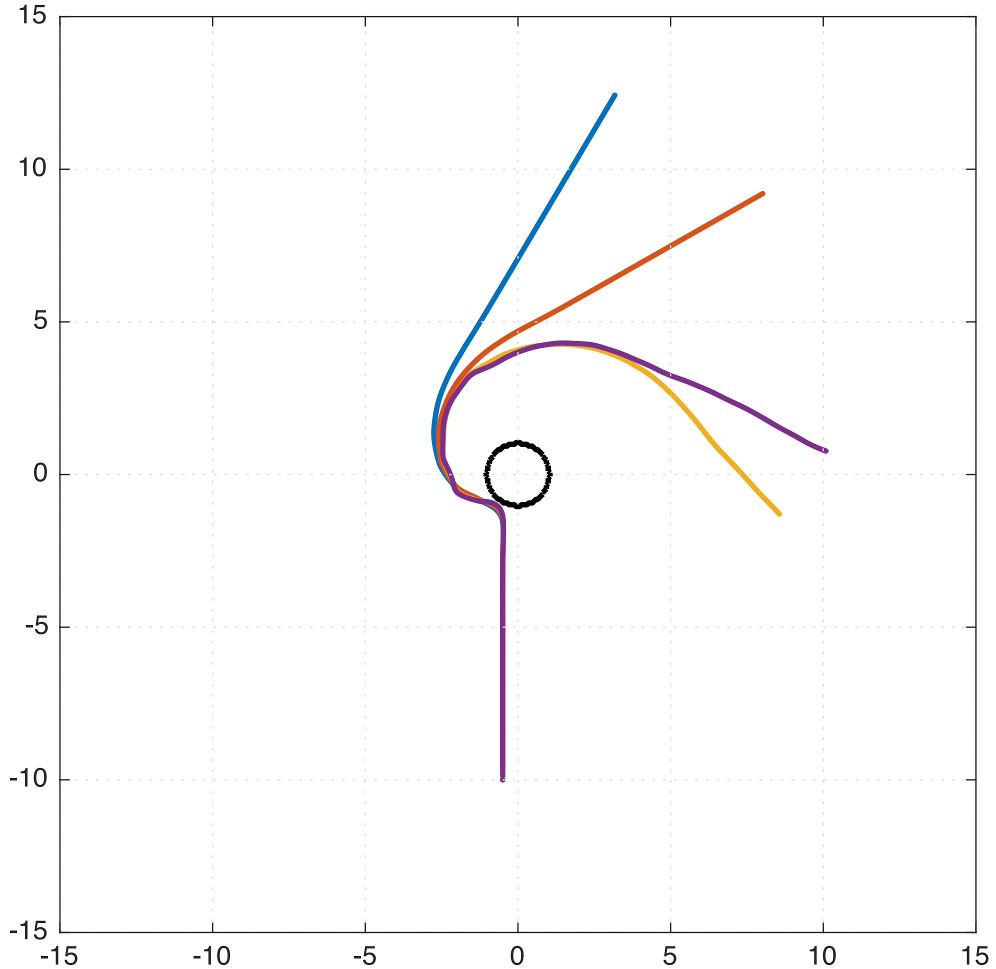
$$\frac{1}{f} = \frac{2}{R}\left(1 - \frac{n_2}{n_1}\right)$$

- provides means of focusing walker motion using bottom topography

Monochromatic approximation (Faria 2016)

- assume only dynamically significant waves have the Faraday wavelength
- variations in depth modeled as a variation in λ_F, phase speed
- allows for first robust modeling of walker-boundary interactions

The Logarithmic Spiral



Single-slit diffraction