# HQA Lecture 15

## I. Limitations of the stroboscopic model

— dynamic bound states

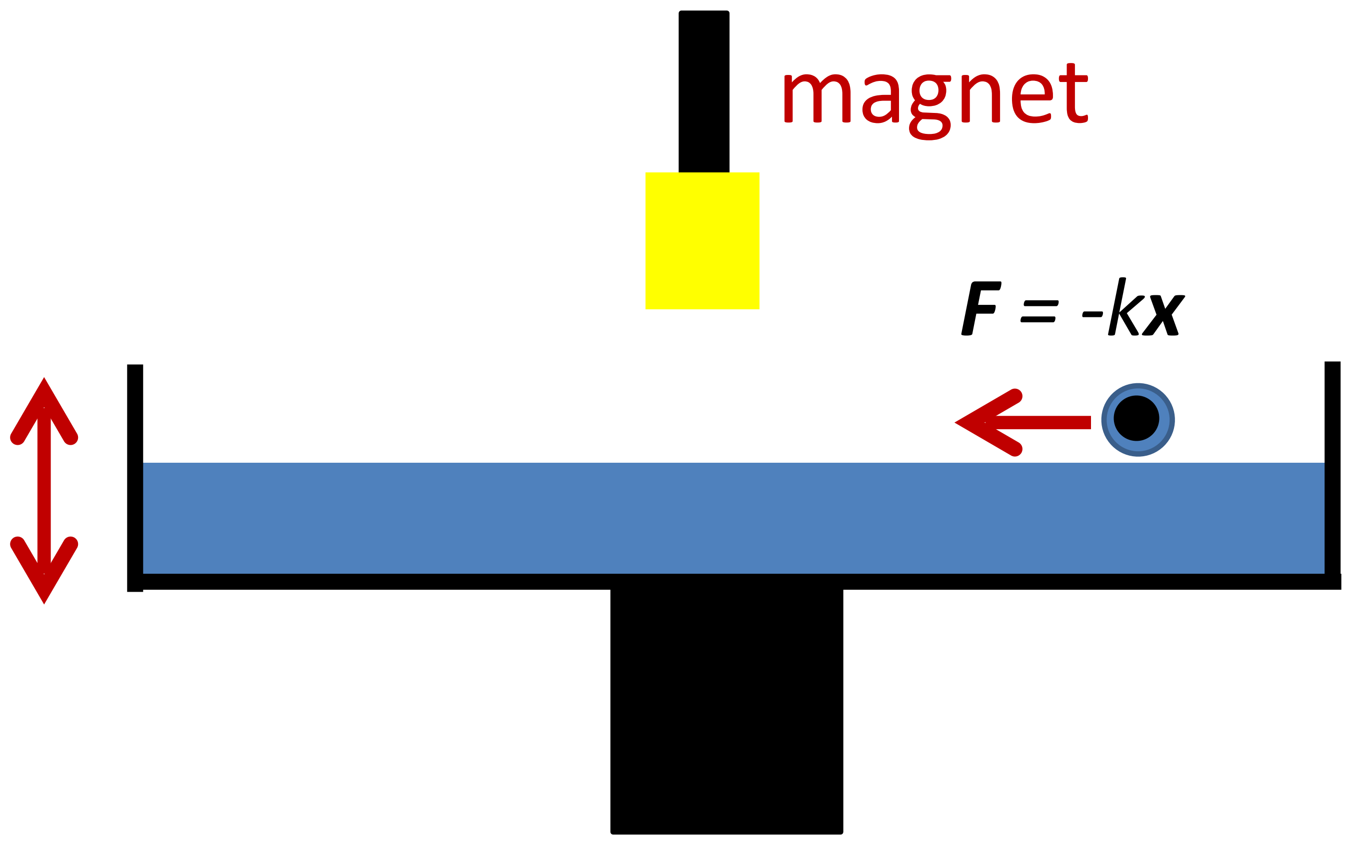
## II. The boost model

— scattering off a pillar

# Pilot-wave harmonic oscillator with strobe model

$$m\ddot{\mathbf{x}} + D\dot{\mathbf{x}} = \frac{W}{T_F}\int_{-\infty}^{t} J_1\left(k_F\left|\mathbf{x}(t)-\mathbf{x}(s)\right|\right)\frac{\mathbf{x}(t)-\mathbf{x}(s)}{\left|\mathbf{x}(t)-\mathbf{x}(s)\right|}\mathrm{e}^{-(t-s)/T_M}\,\mathrm{d}s - k\mathbf{x}$$

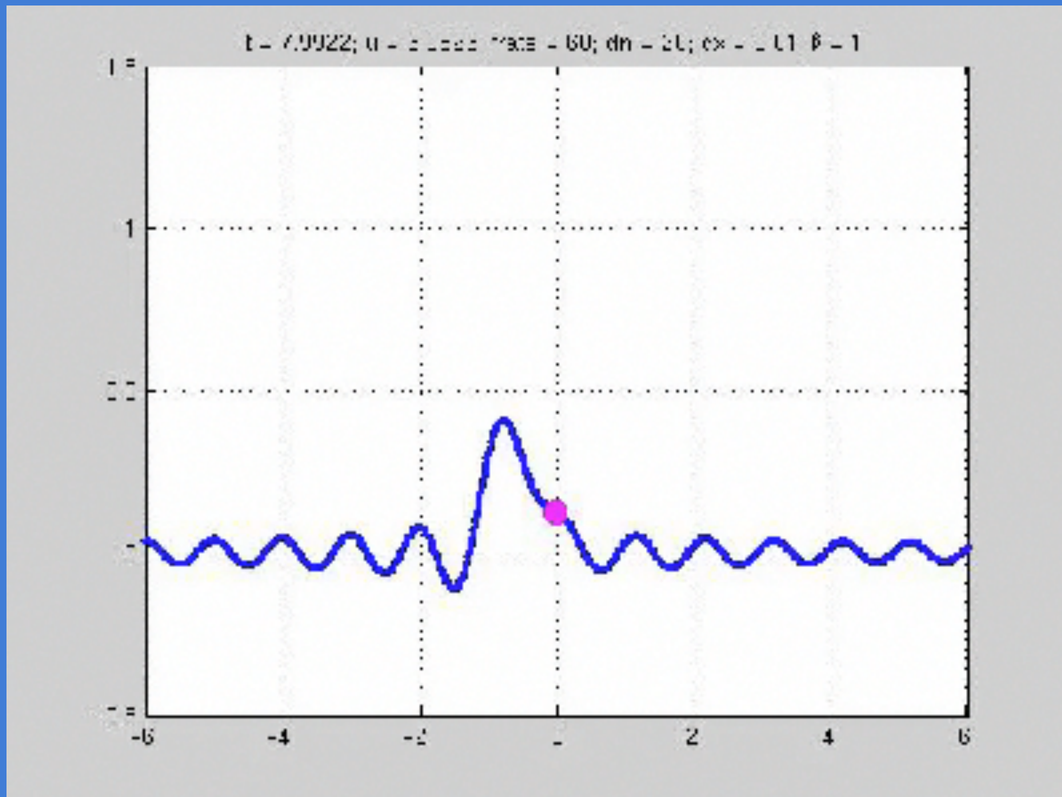
spring force



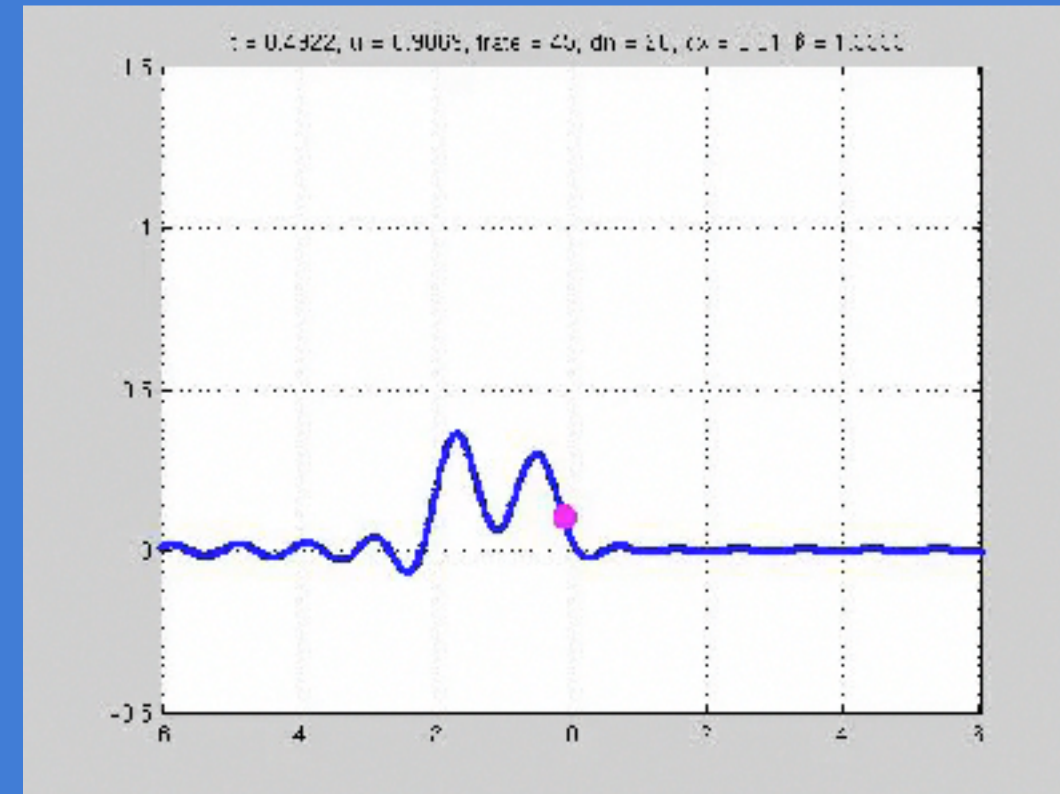

→ simulate in 1D

# Pilot-wave dynamics and the 1D harmonic oscillator

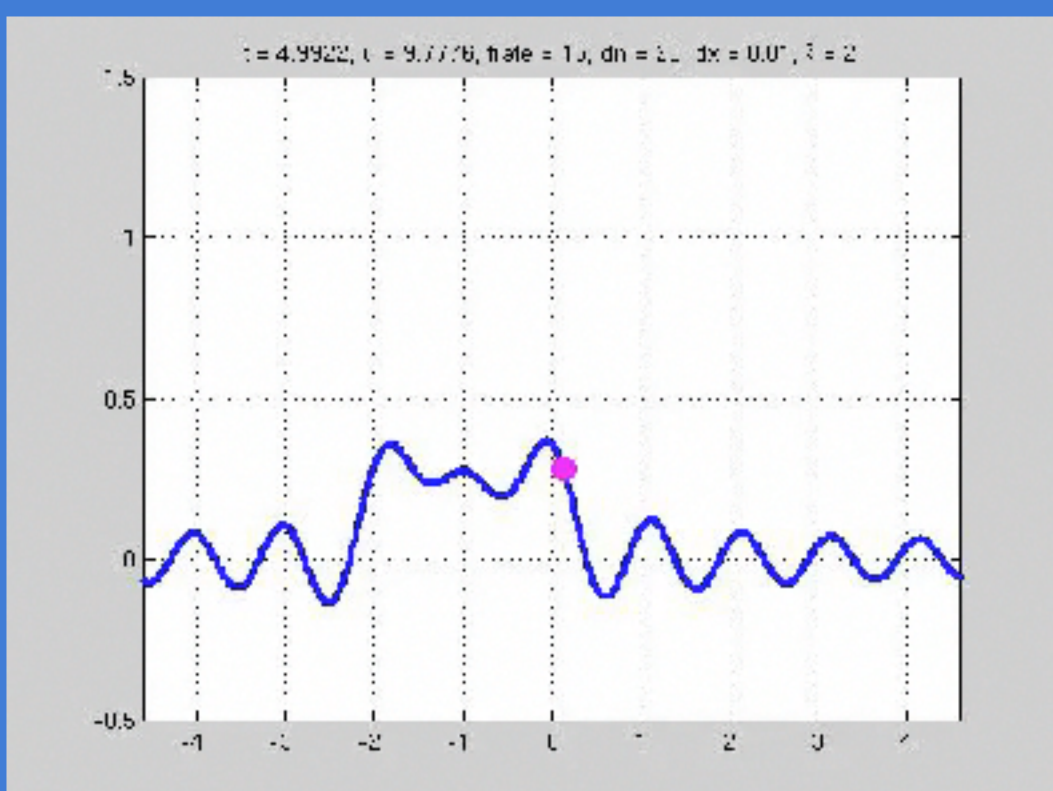
$\gamma_F = 4.1$



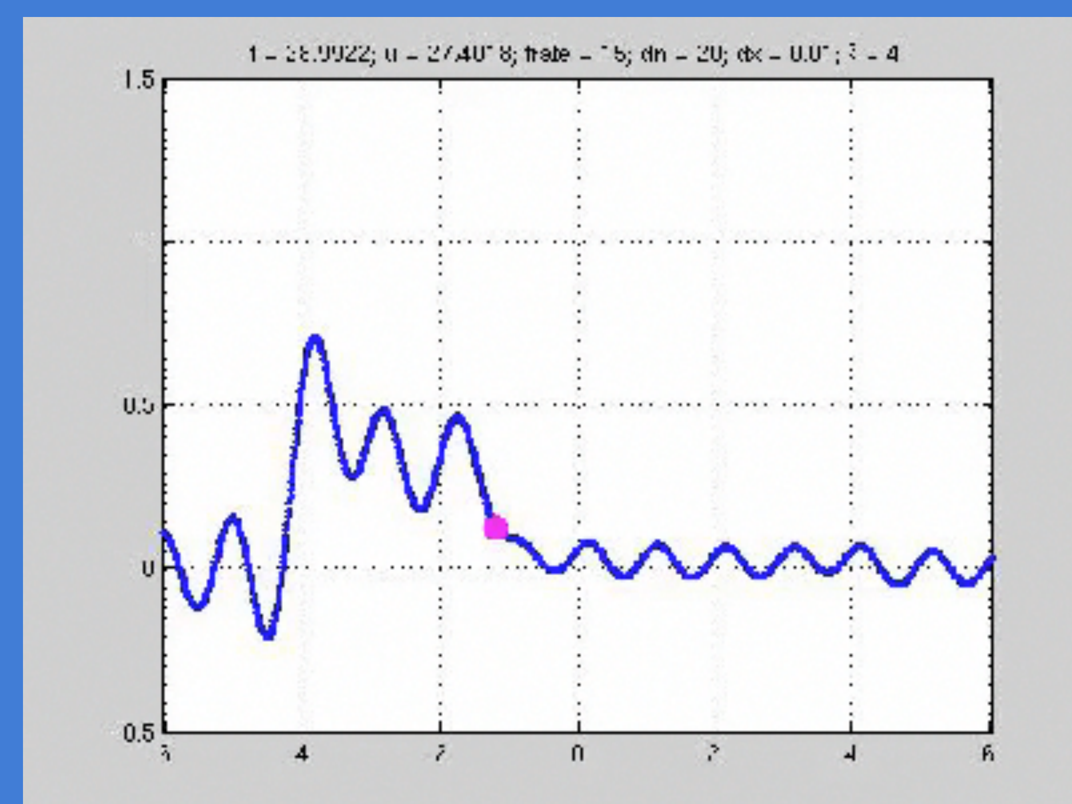
Low memory $\gamma = 3.7$



Low-mid memory $\gamma = 3.8$

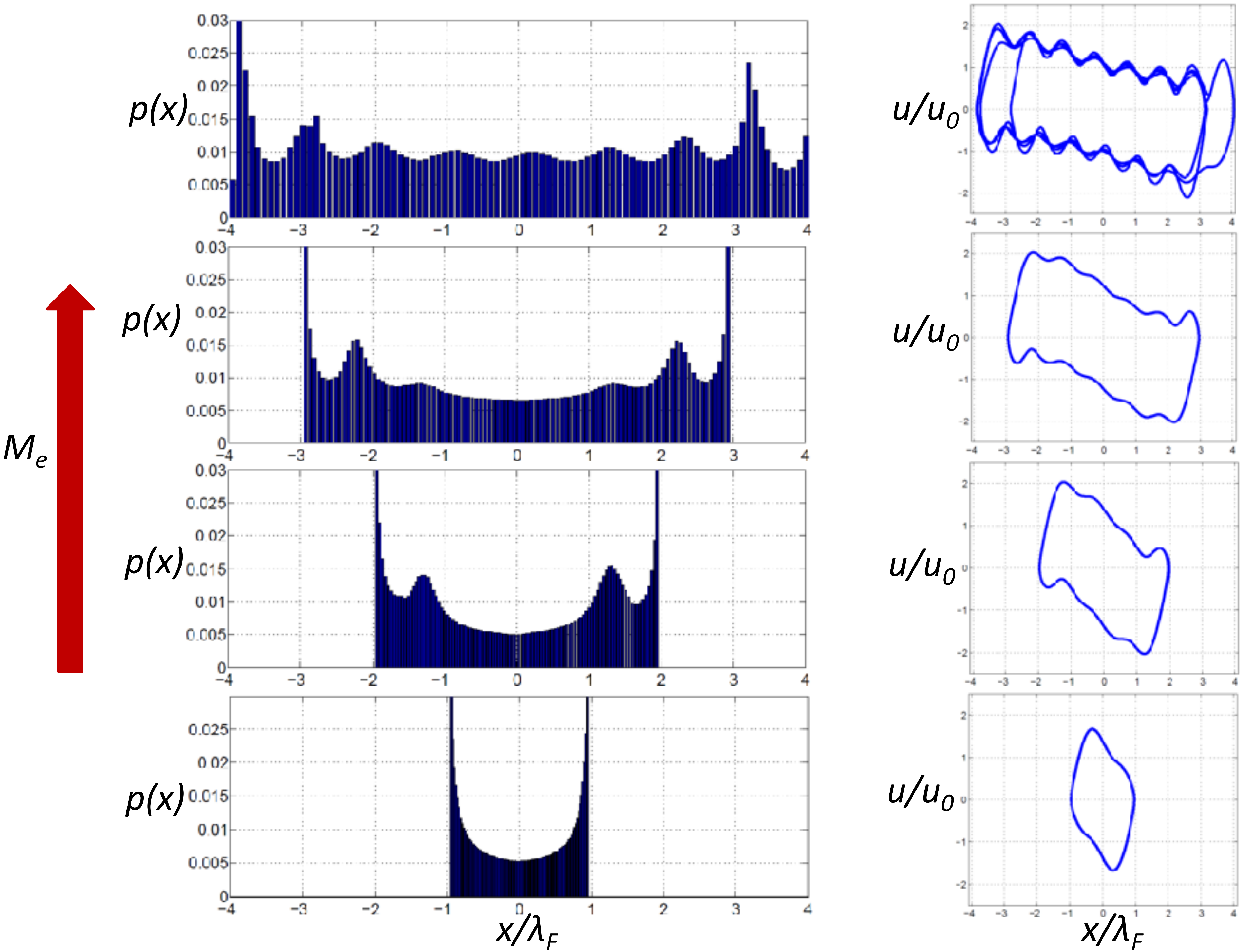


Mid memory $\gamma = 3.9$

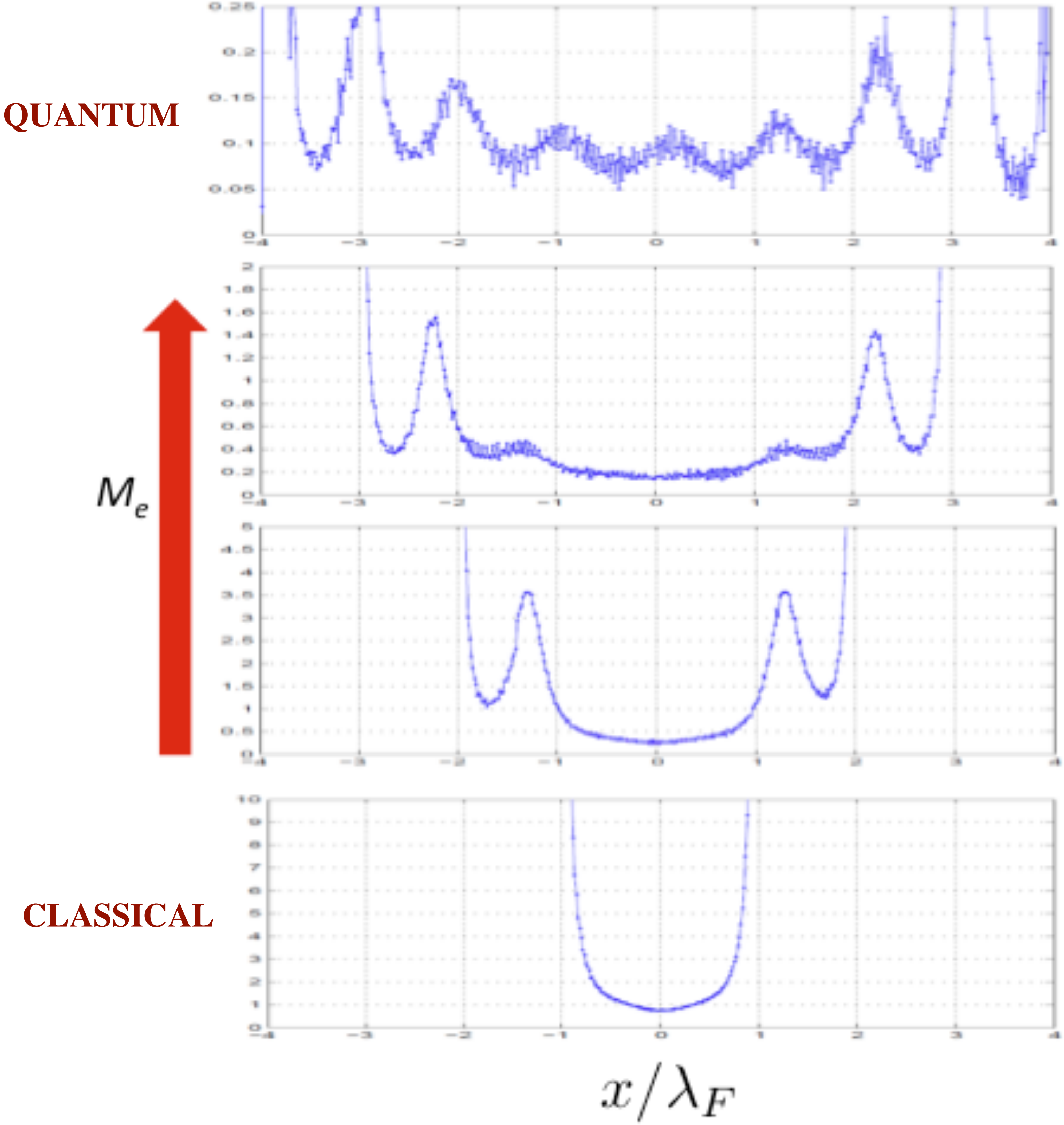


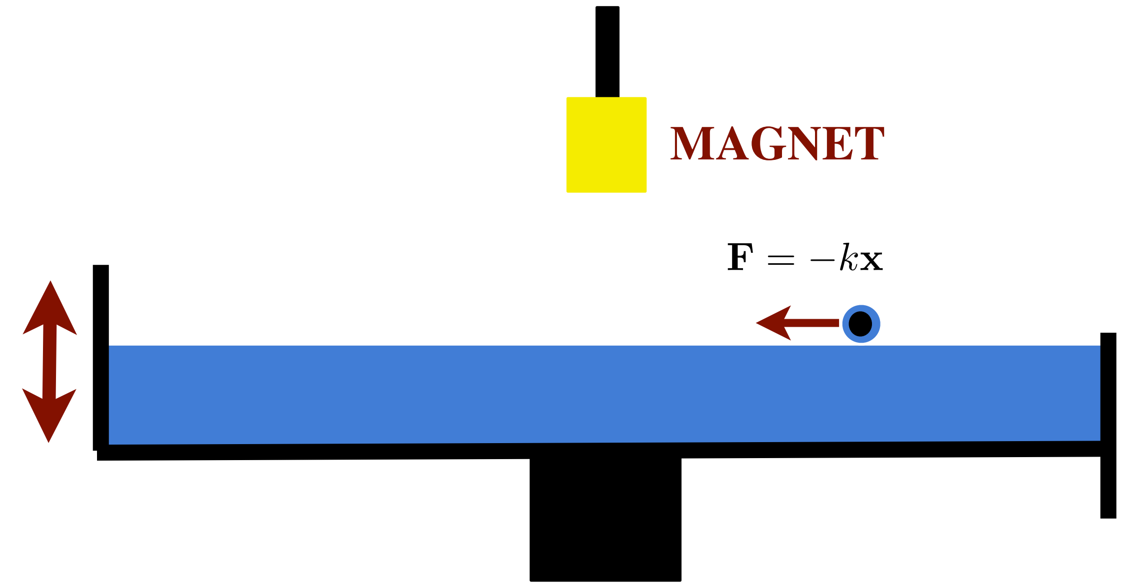
High memory $\gamma = 4.0$

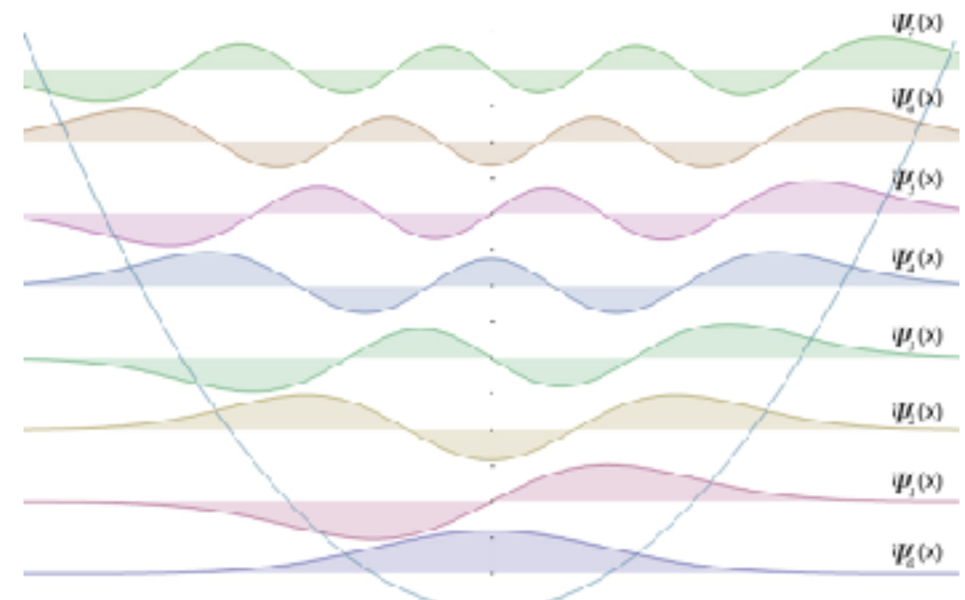
# Probability distribution function



# Walking along a line in a central force field



Wavefunctions of quantum harmonic oscillator

- coherent, wave-like statistics emerge from an underlying pilot-wave dynamics
- structure rooted in in-line speed oscillations with the Faraday wavelength

# Stroboscopic model

- we have seen the success of the stroboscopic model in capturing the behavior of single-particle systems

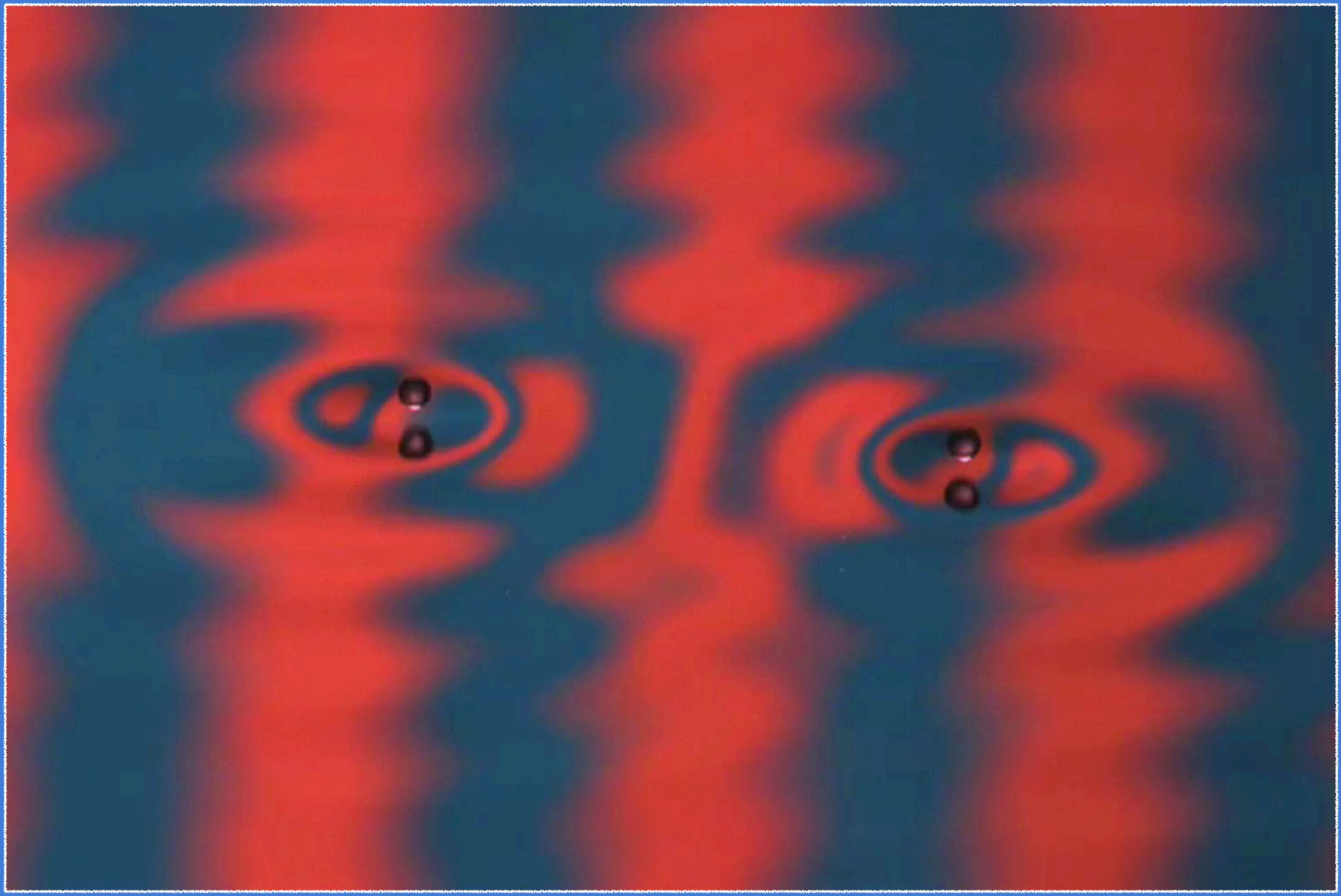
  — there is a single fudge factor, the impact phase

- we proceed by enumerating its shortcomings in several multiple-particle systems

  — here, consideration must be given to the far-field form of the wave

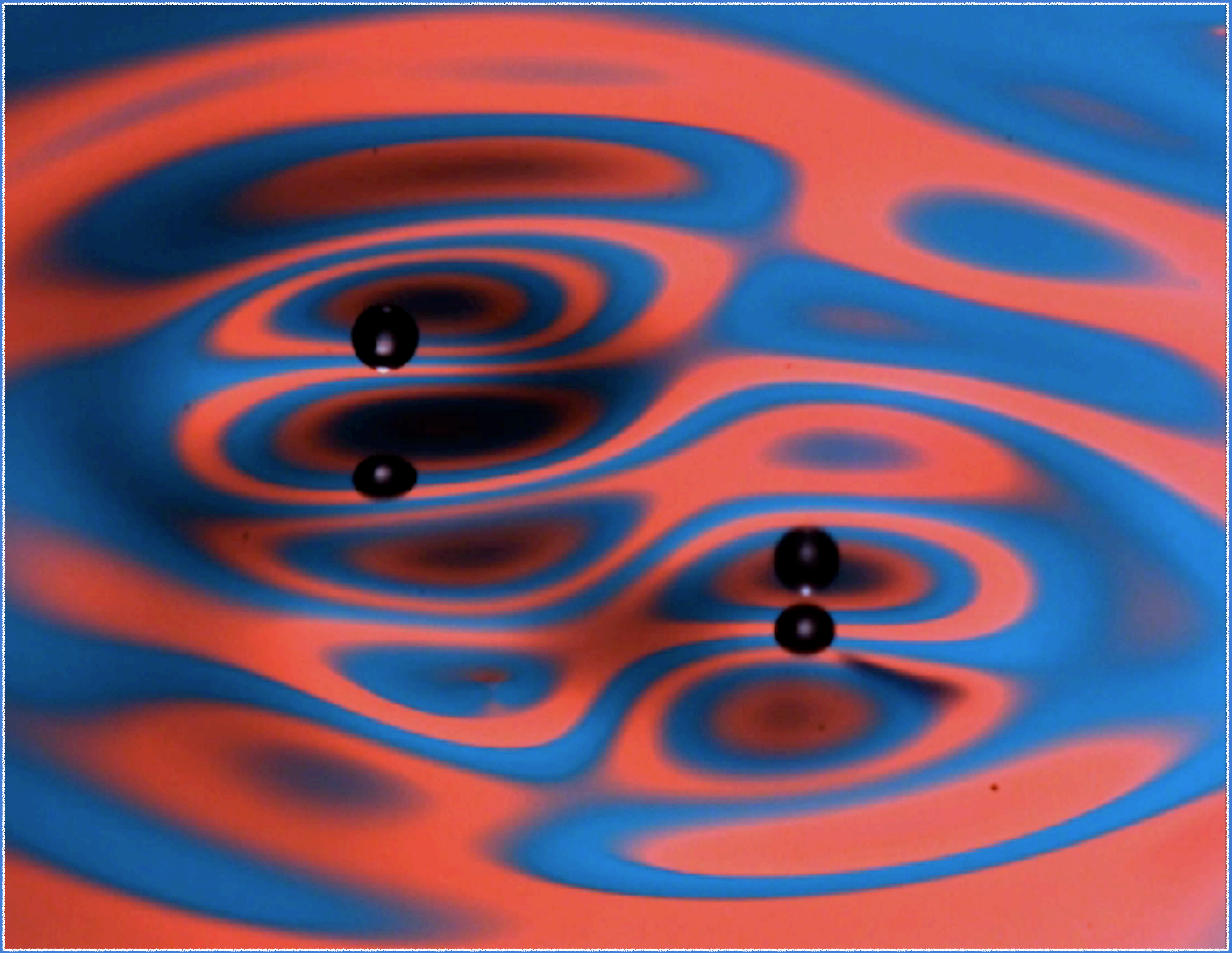
# Dynamic bound states: orbiting pairs

- impinging pairs may scatter or lock into orbit



*Couder et al. (2005), Protiere et al. (2006ab)*
*Oza et al. (2017), Tadrist & Gilet (2018)*

# Orbiting pairs



*Couder et al. (2005), Protiere et al. (2006ab)*
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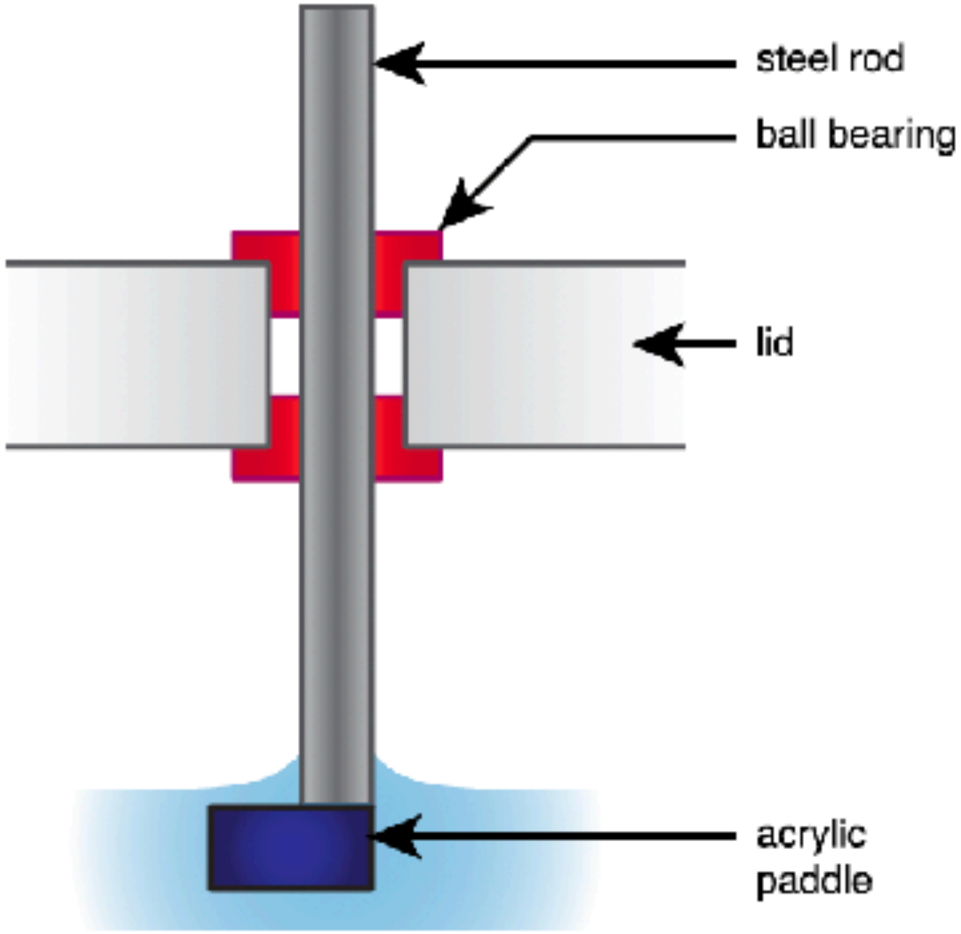
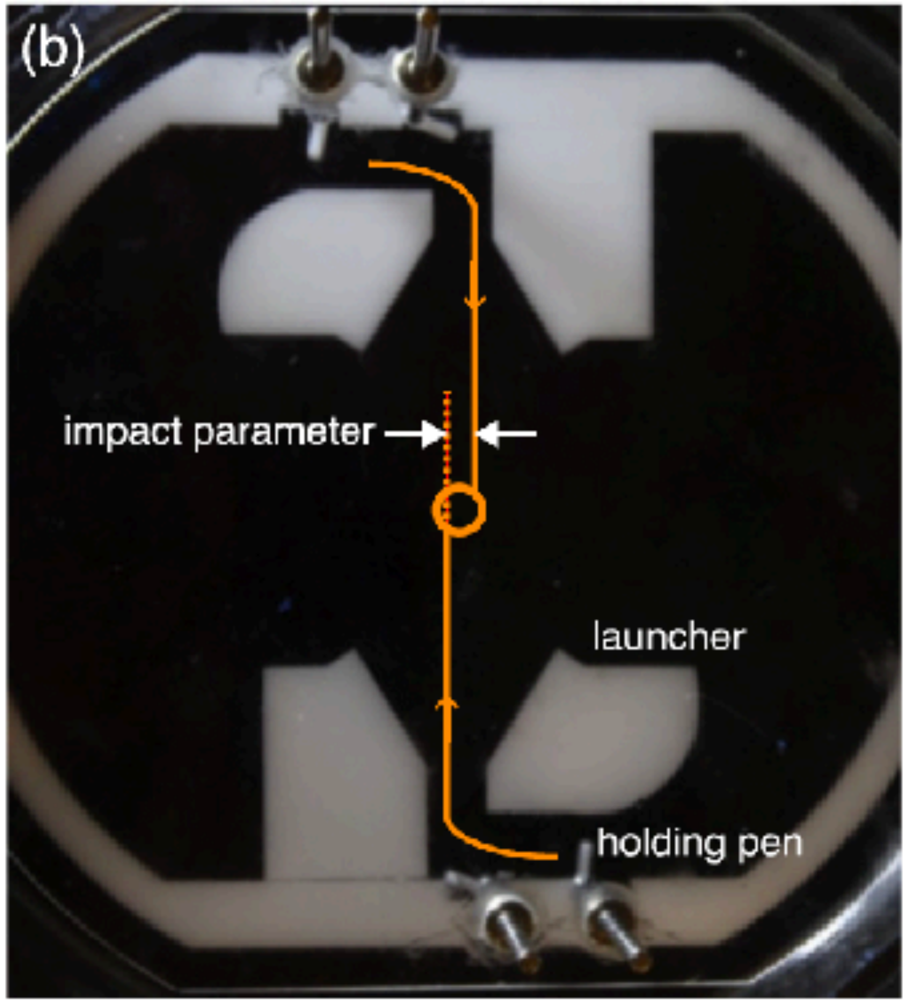
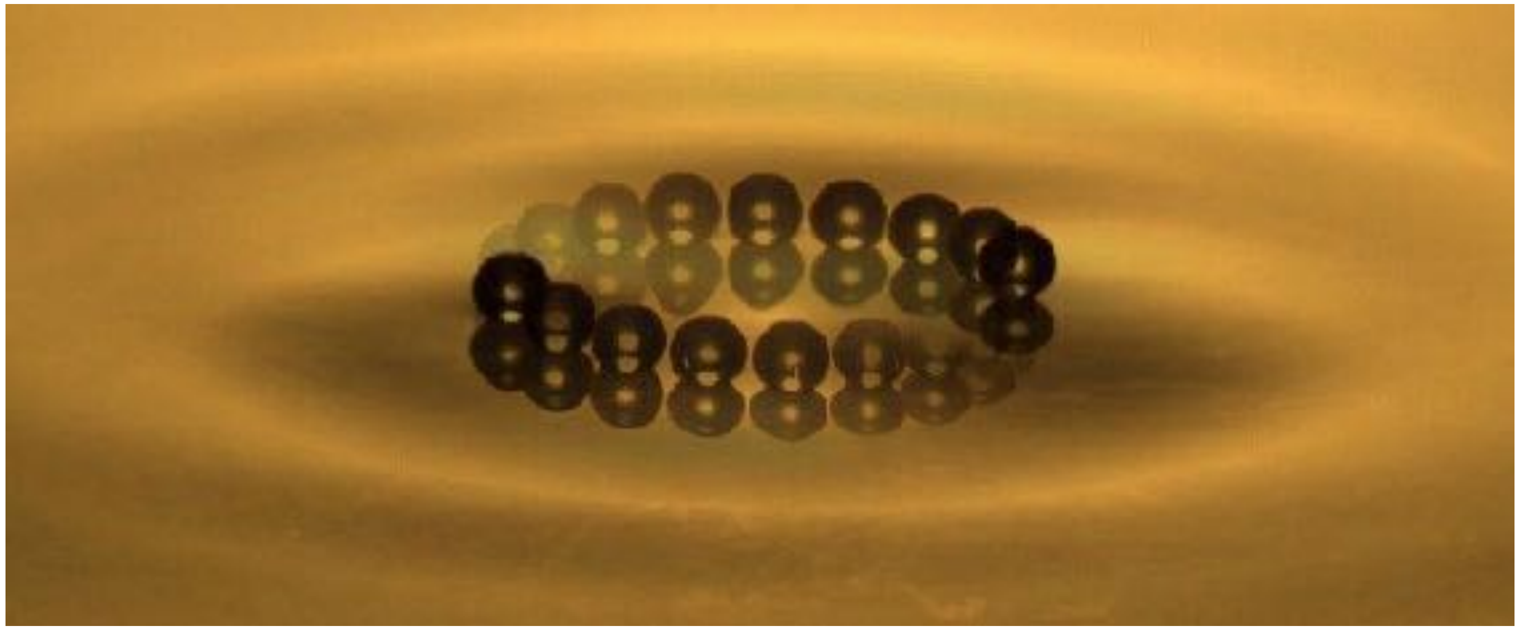
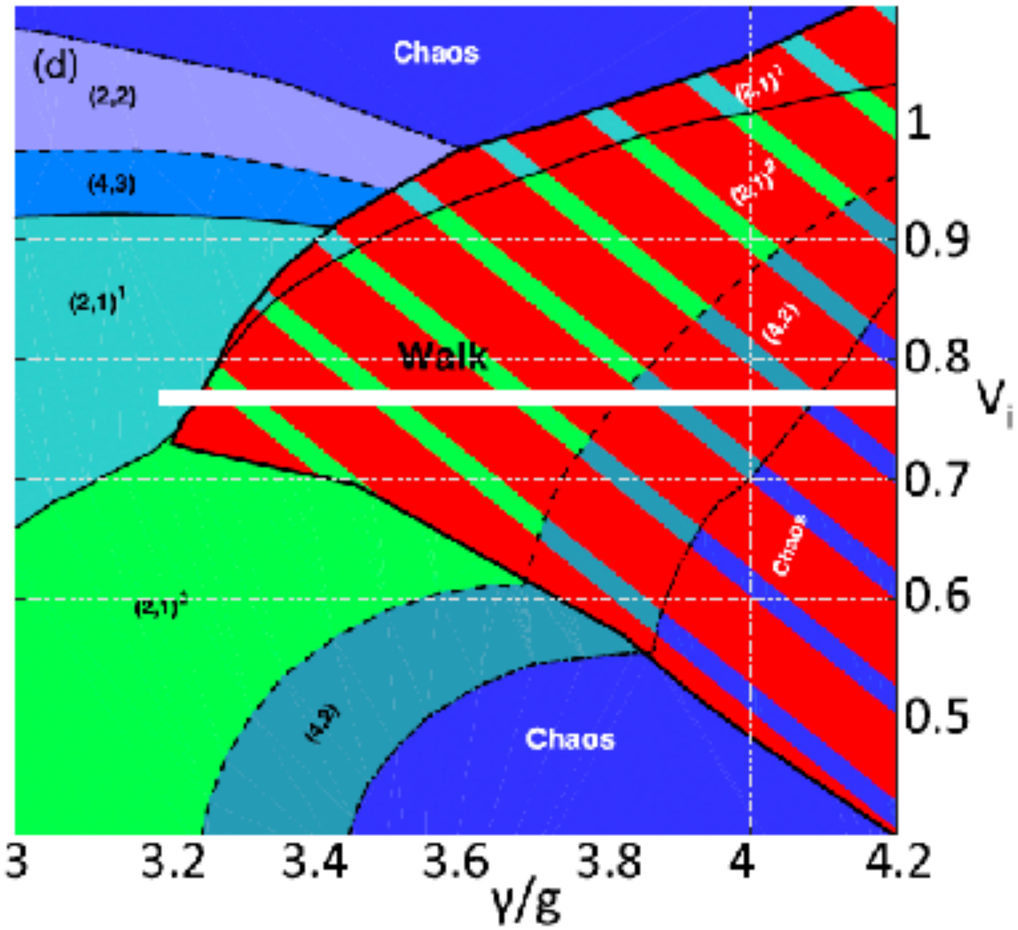
# Orbiting pairs



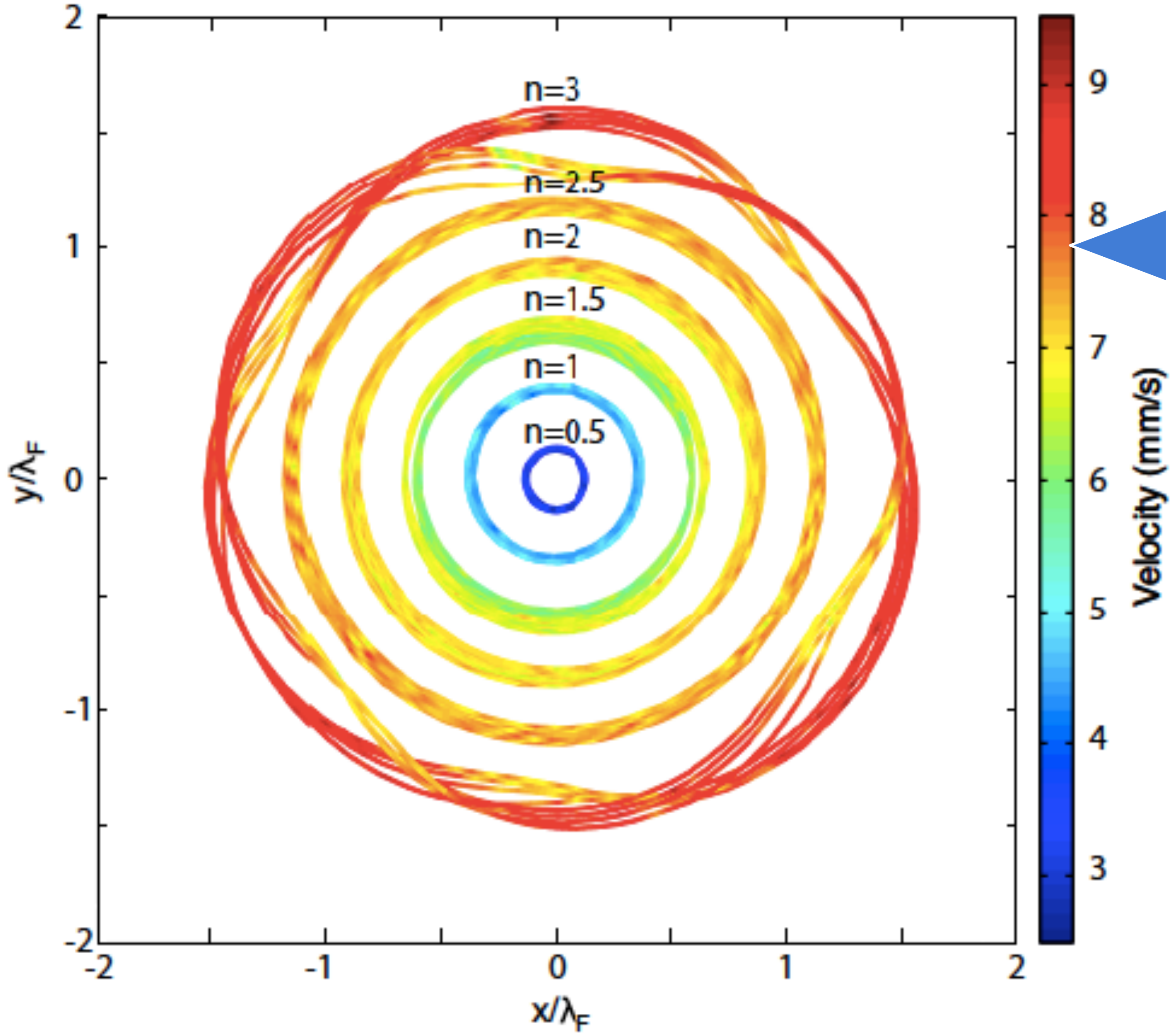



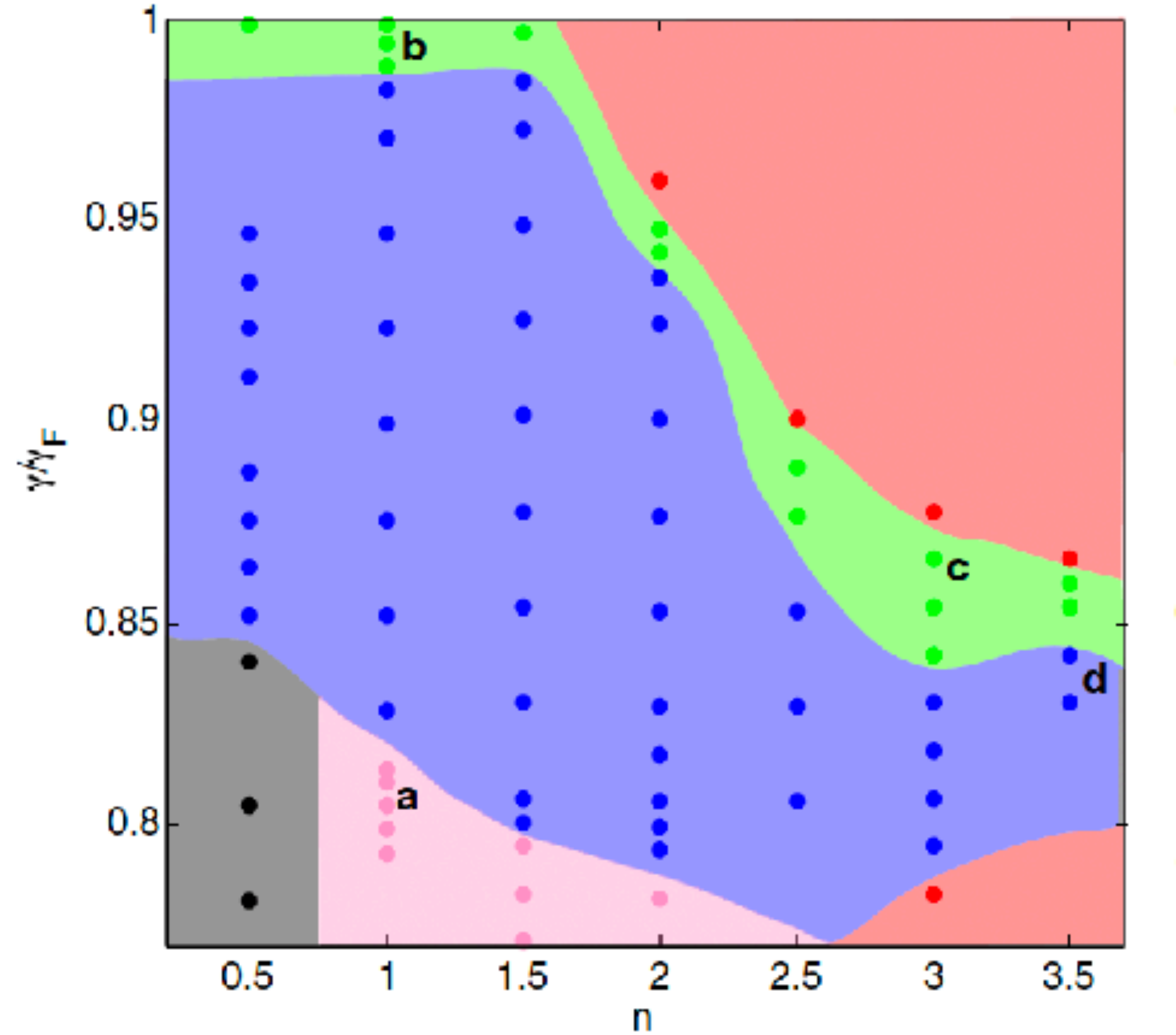



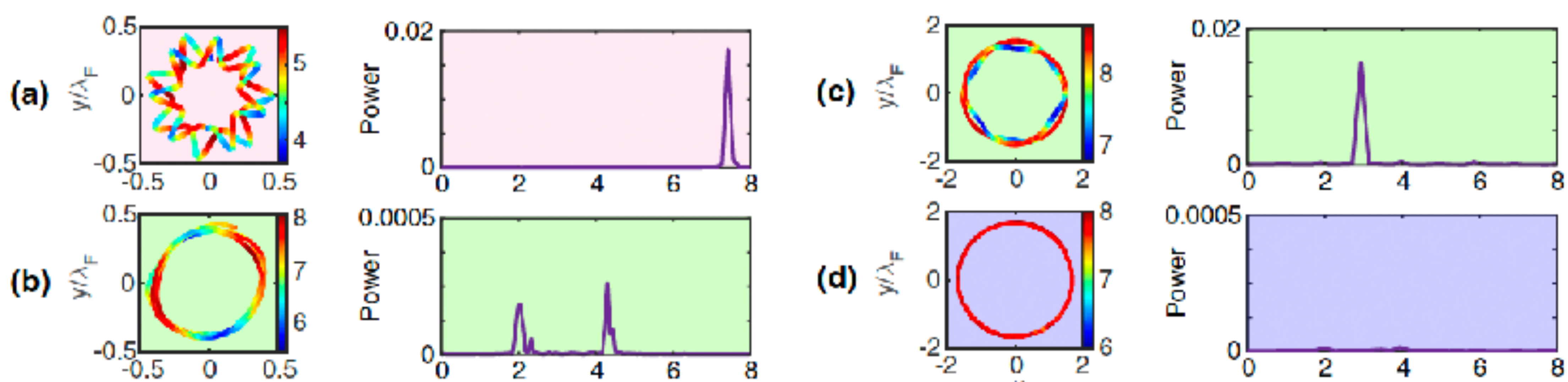


# Orbiting pairs

- different orbital modes accessible according to relative bouncing phase of walkers



# Theoretical model: 2-particle stroboscopic model

- drops move in response to the wave field produced by themselves and their neighbors

## Trajectory equation

$$m\ddot{\boldsymbol{x}}_i + D\dot{\boldsymbol{x}}_i = -mg\mathcal{S}(h_i(\boldsymbol{x}_i,t))\nabla h_i(\boldsymbol{x}_i,t)$$

## Wave field

$$h_1(\boldsymbol{x},t) = \frac{A}{T_F}\int_{-\infty}^{t}[f(|\boldsymbol{x}-\boldsymbol{x}_1(s)|,t-s) + \varsigma f(|\boldsymbol{x}-\boldsymbol{x}_2(s)|,t-s)]\,ds$$

where $f(r,t) = J_0(k_F r)\mathrm{e}^{-\alpha r^2/(t+T_F)}\mathrm{e}^{-t/(T_F M_e)}$. is the wave kernel

SPATIO-TEMPORAL DAMPING

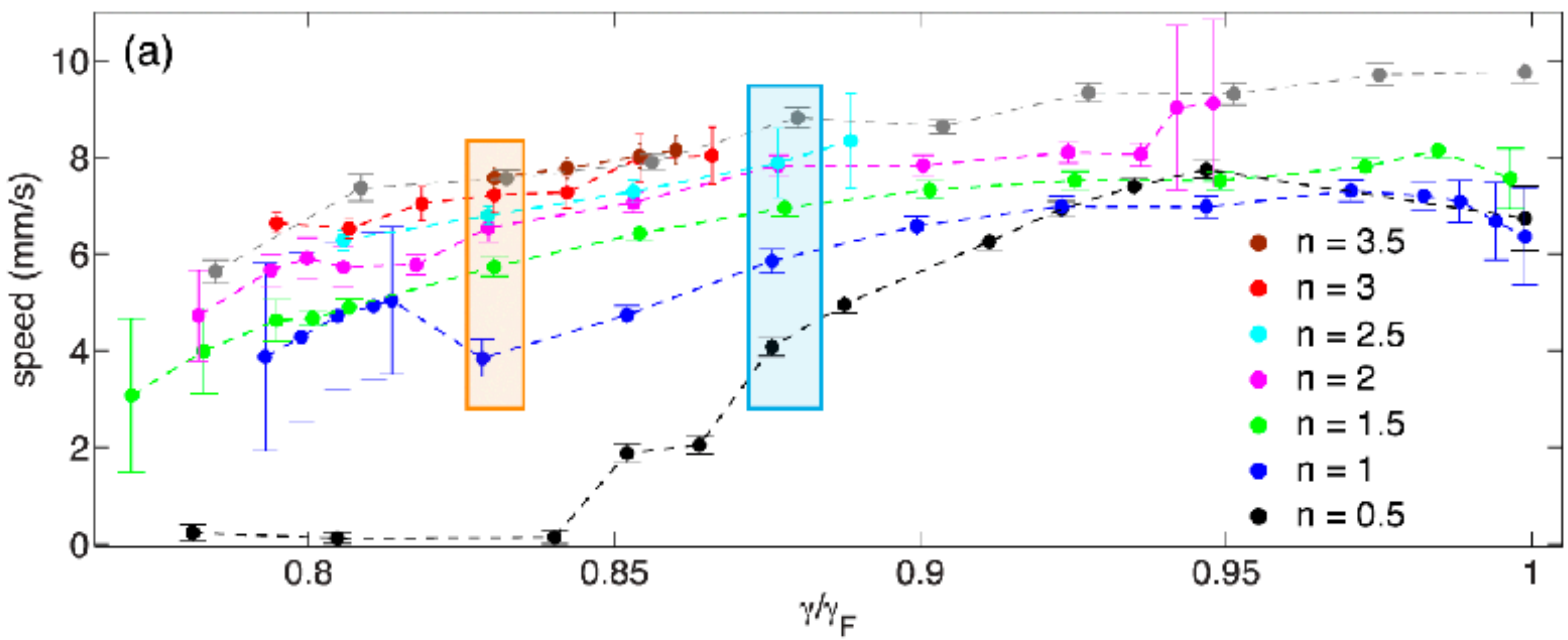
$\varsigma = \pm 1$ is the relative phase parameter

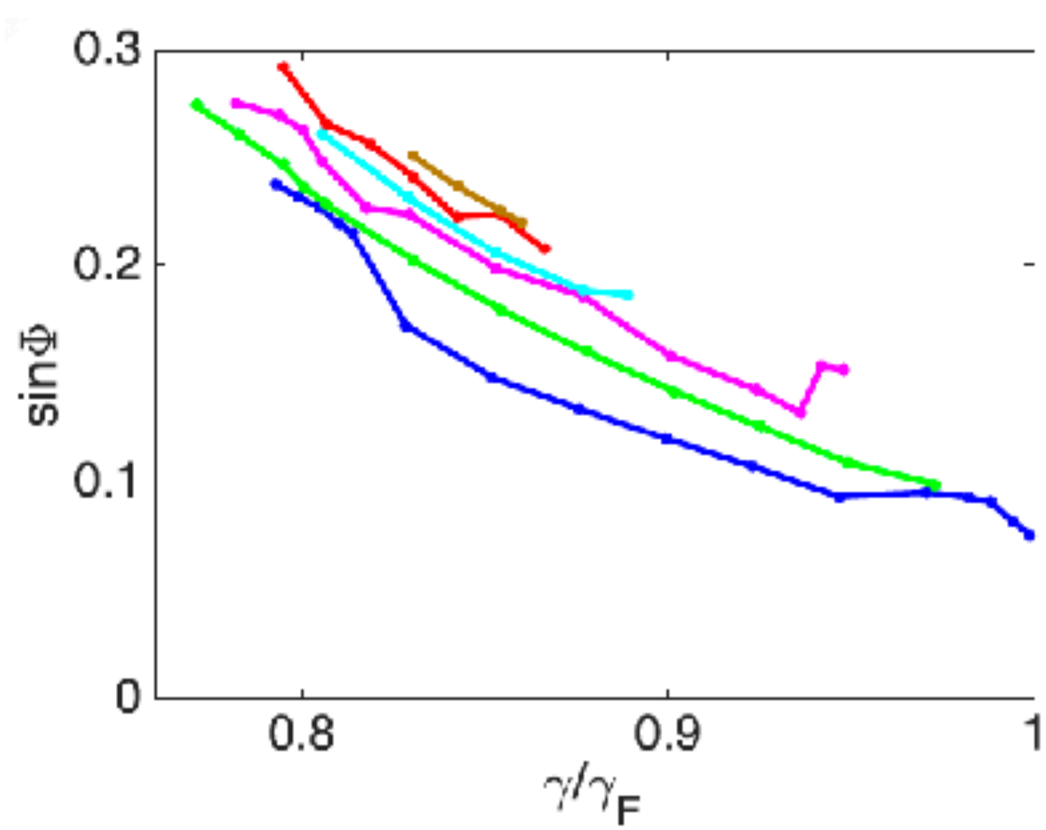
and $\mathcal{S} = \sin\Phi$ is the sine of the mean impact phase

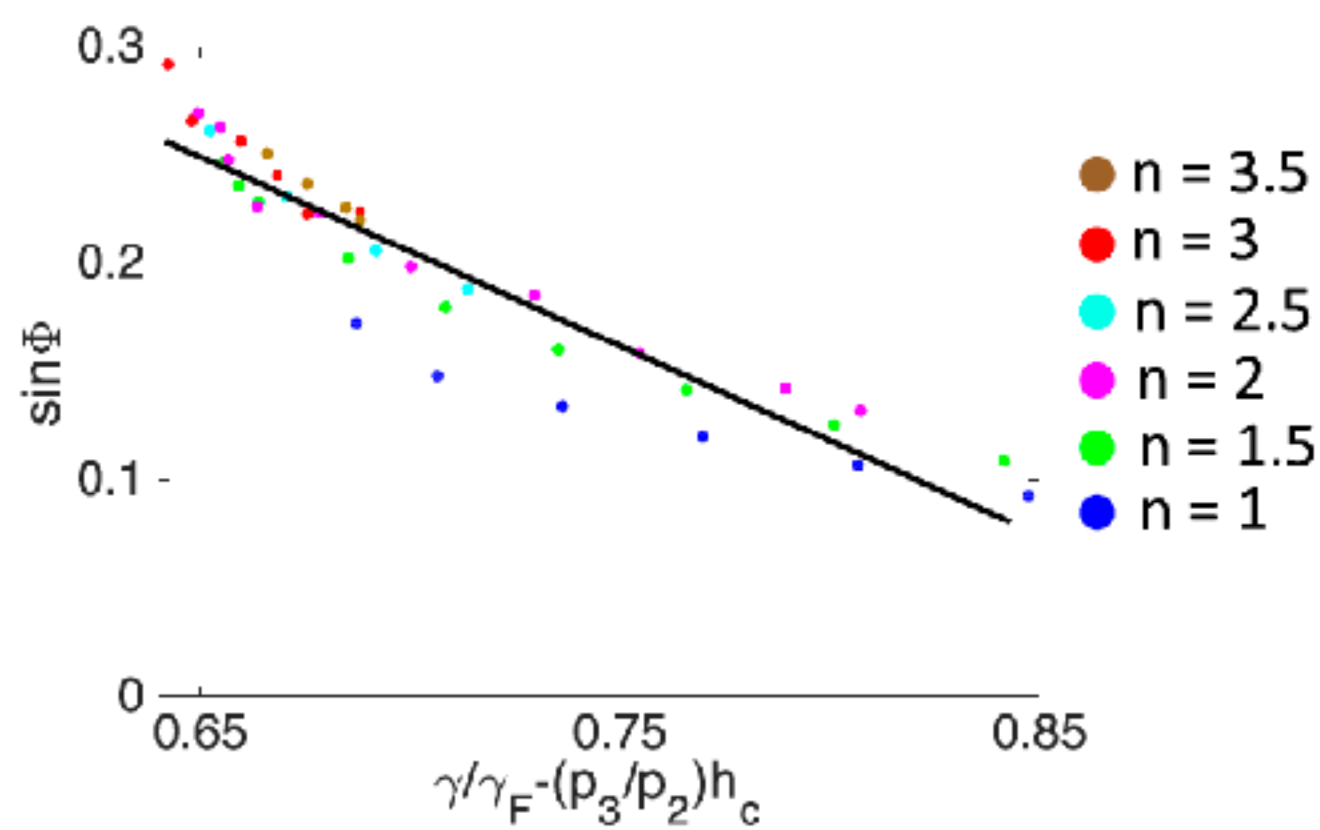
**Deduce S experimentally…**

**Note:** far-field form of wave field unimportant for many single-droplet settings, but must be considered here

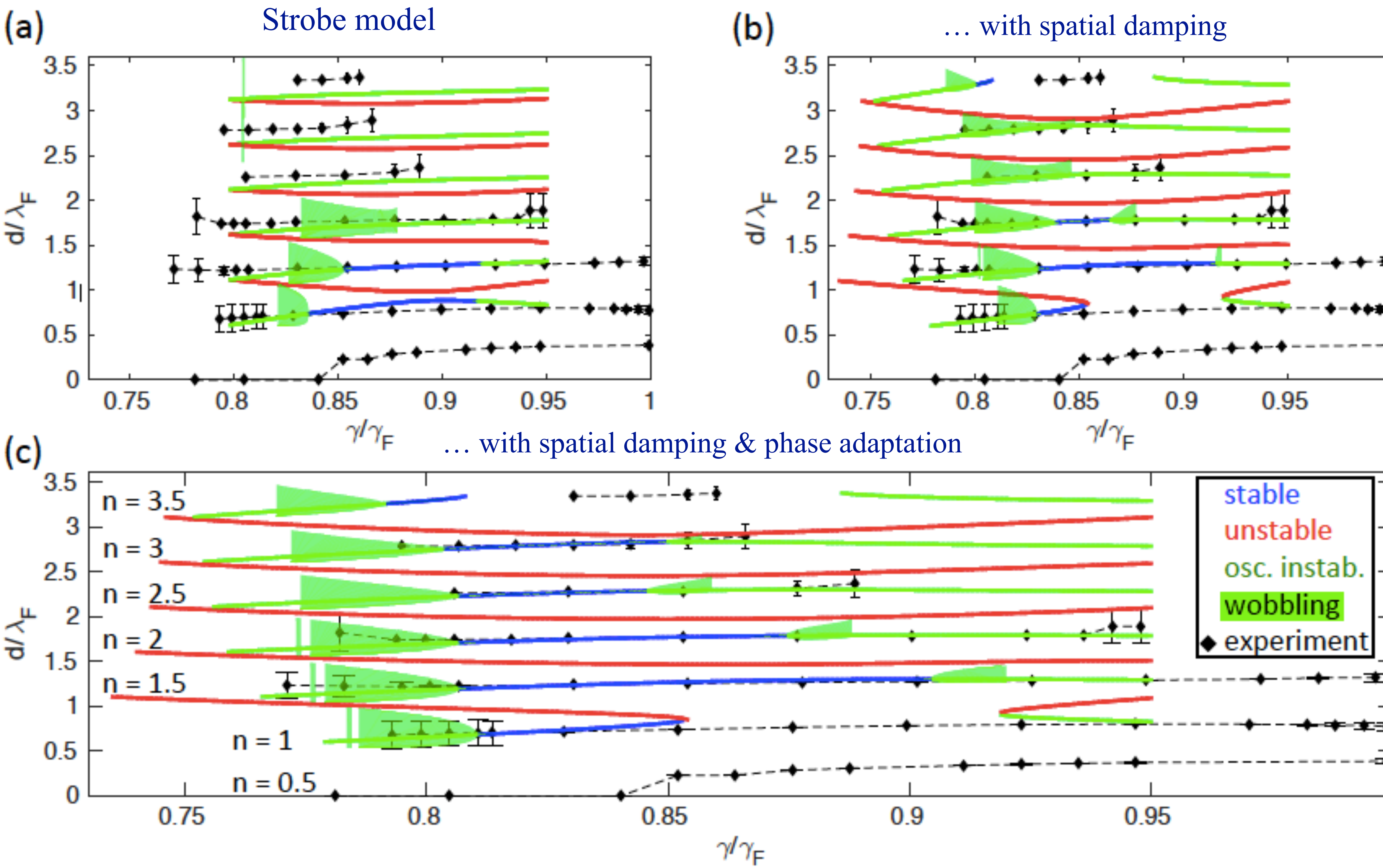
# Infer phase parameter by matching for orbital speed



# Orbiting identical pairs: stability *(Oza et al. 2017)*

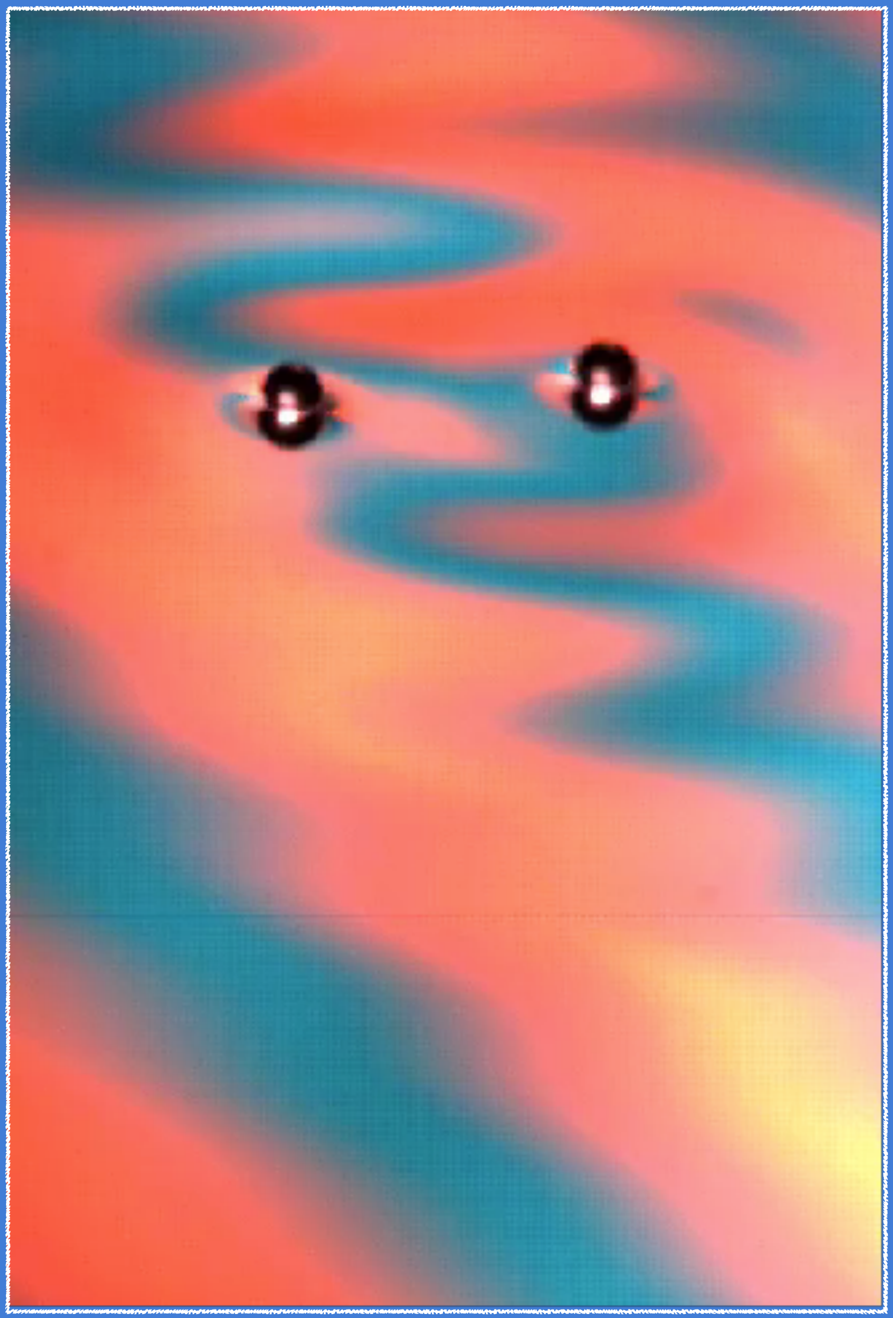



- indicates importance of spatial damping and **phase adaptation** on stability
- required deduction of dependence of the impact phase on memory, local wave amplitude

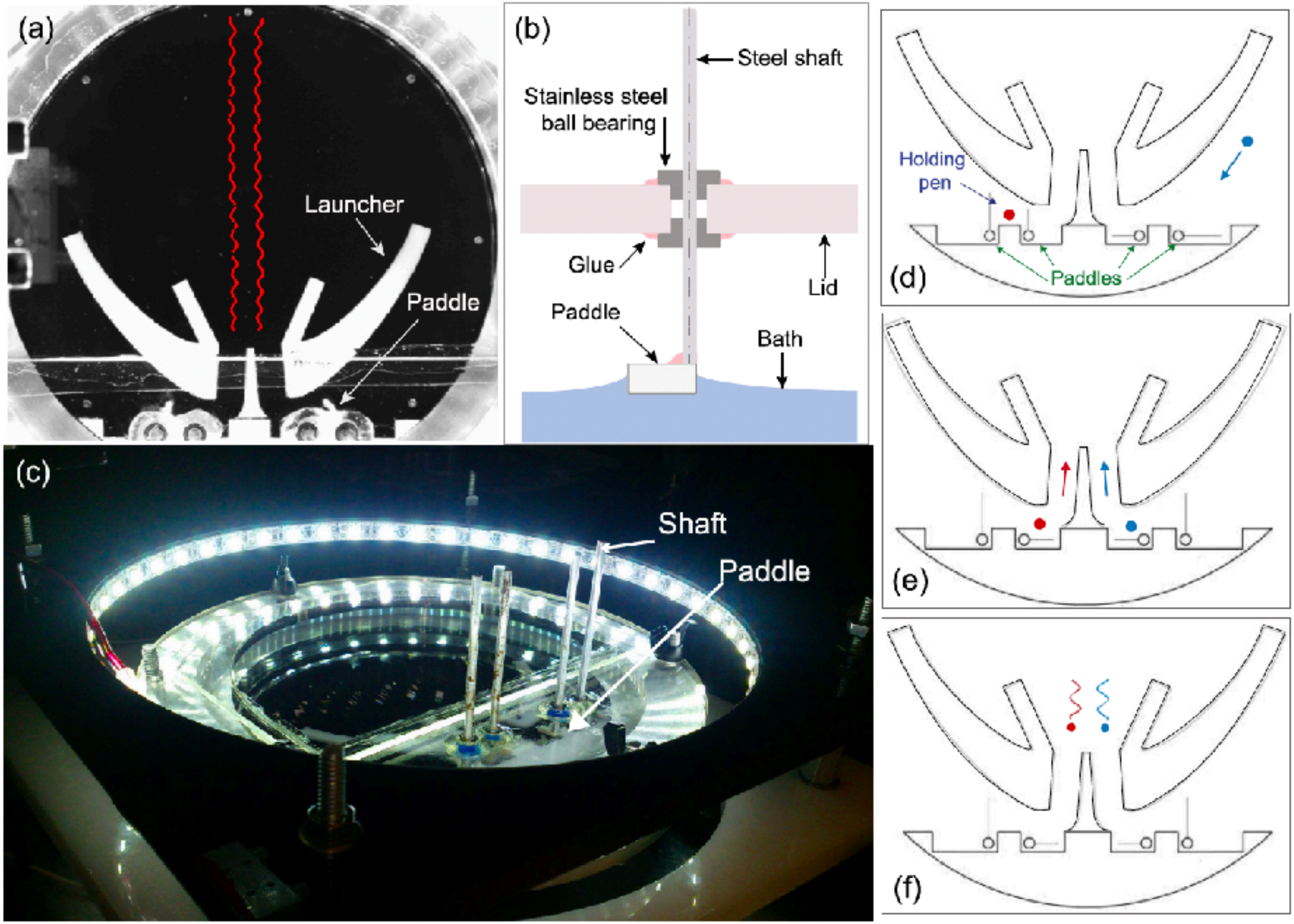
# Promenading pairs

*Borghesi et al. (2015)*
*Arbelaiz et al. (2017)*

- a pair of walking drops bound by their wave fields

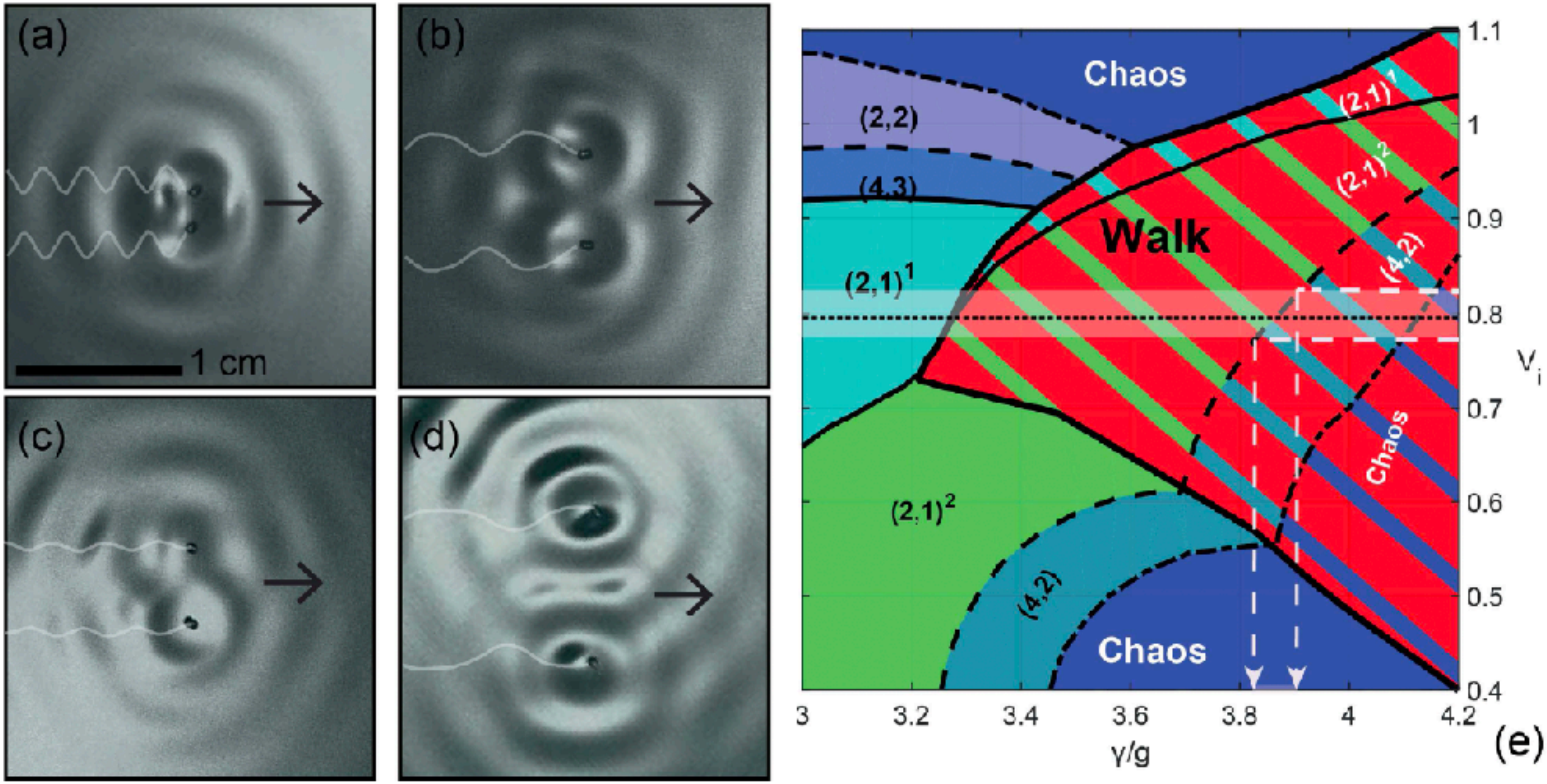


# Promenading pairs: Experiments



# Promenading pairs:  Experiments

$N = 1$ $N = 2$




$N = 1.5$ $N = 2.5$

- different modes accessible according to relative bouncing phase of walkers

# Promenading pairs:  Experiments

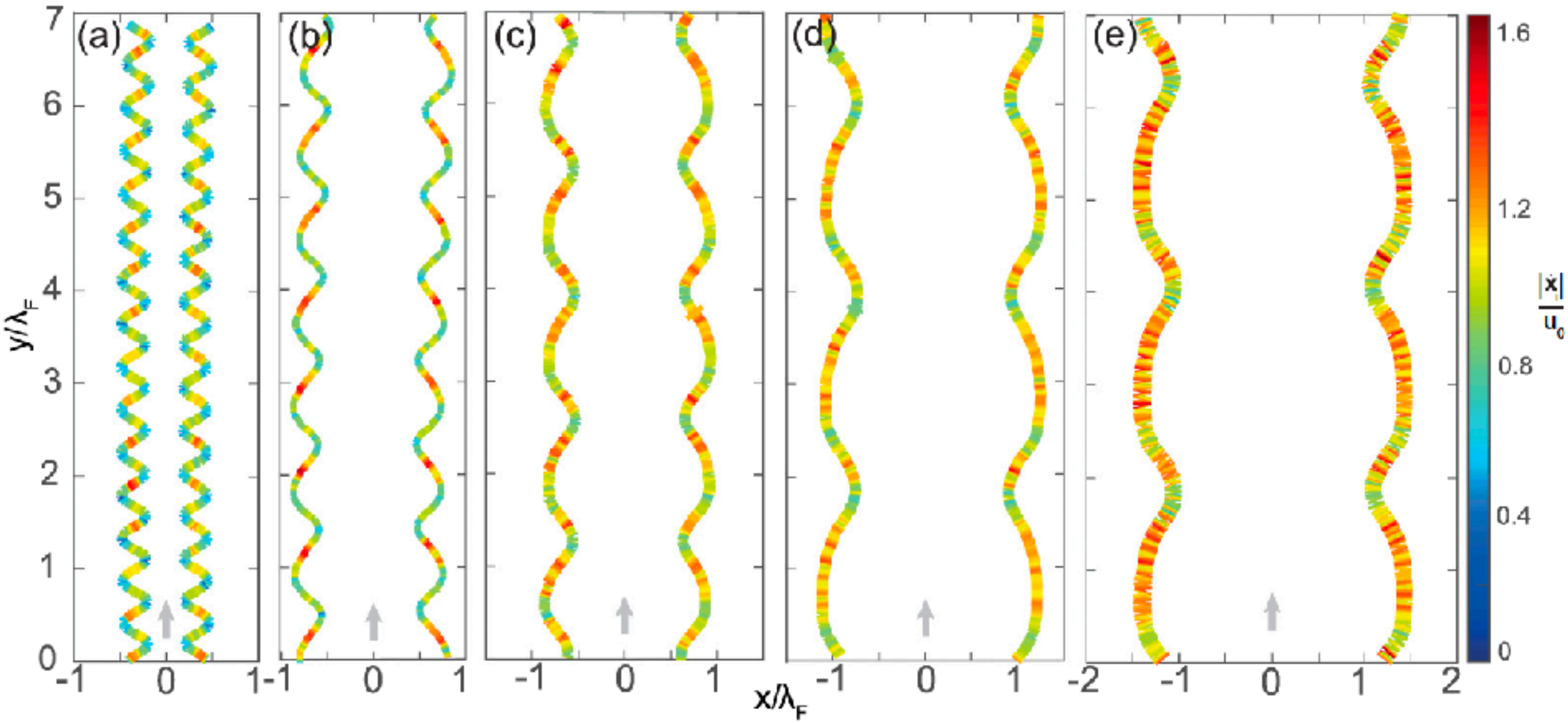
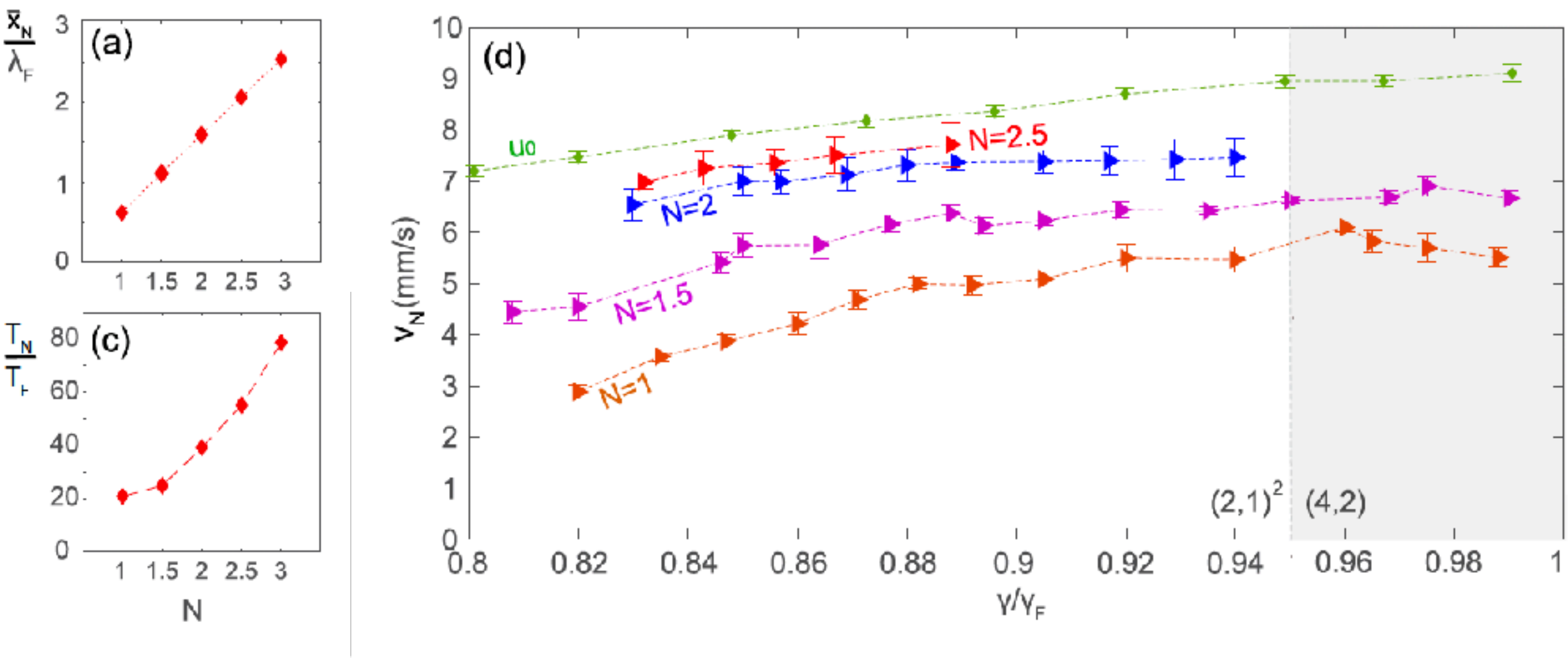



FIG. 4. Different stable promenading trajectories found in experiments at $\gamma/\gamma_F = 0.85 \pm 0.01$: (a) $N = 1$, (b) $N = 1.5$, (c) $N = 2$, (d) $N = 2.5$, and (e) $N = 3$. Trajectories are colored according to instantaneous speed $|\dot{x}_i|$, nondimensionalized by the free walking speed $u_0$. All the promenaders were generated with a drop size of $R = 0.38$ mm, except the $N = 3$ mode, for which $R = 0.35$ mm.

# Promenading pairs: Diagnostics



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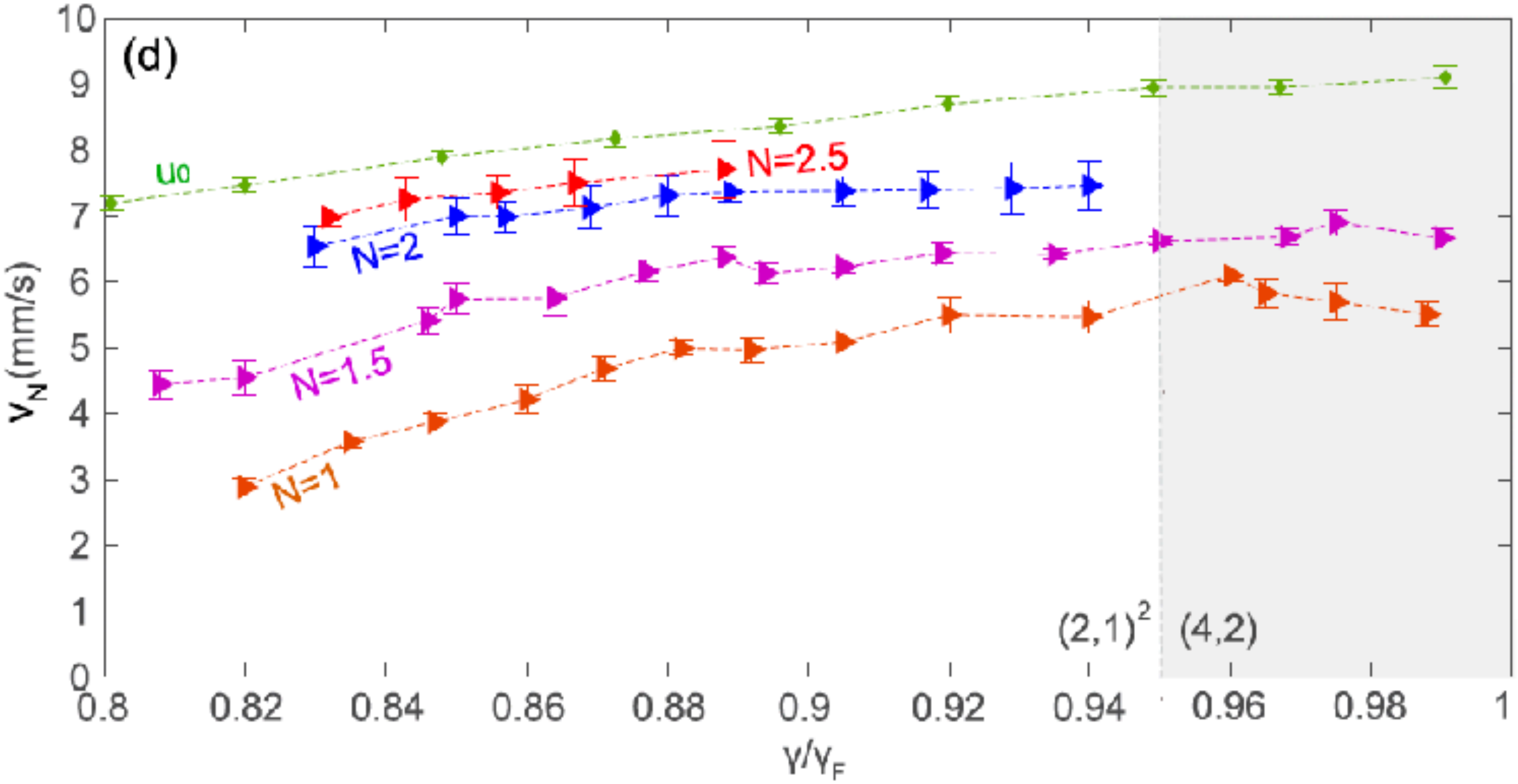
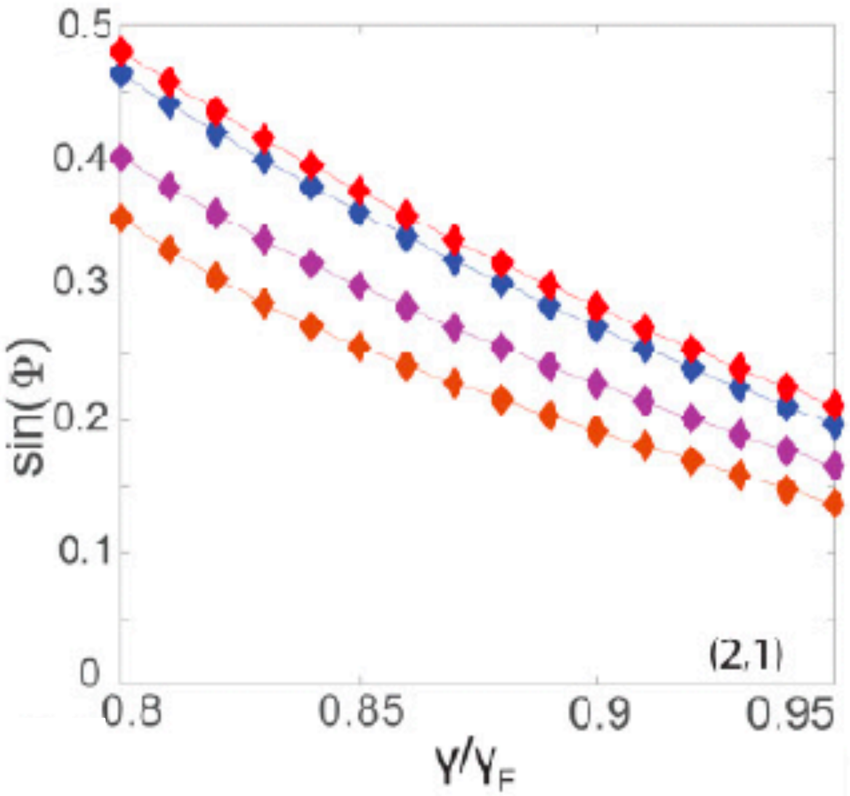
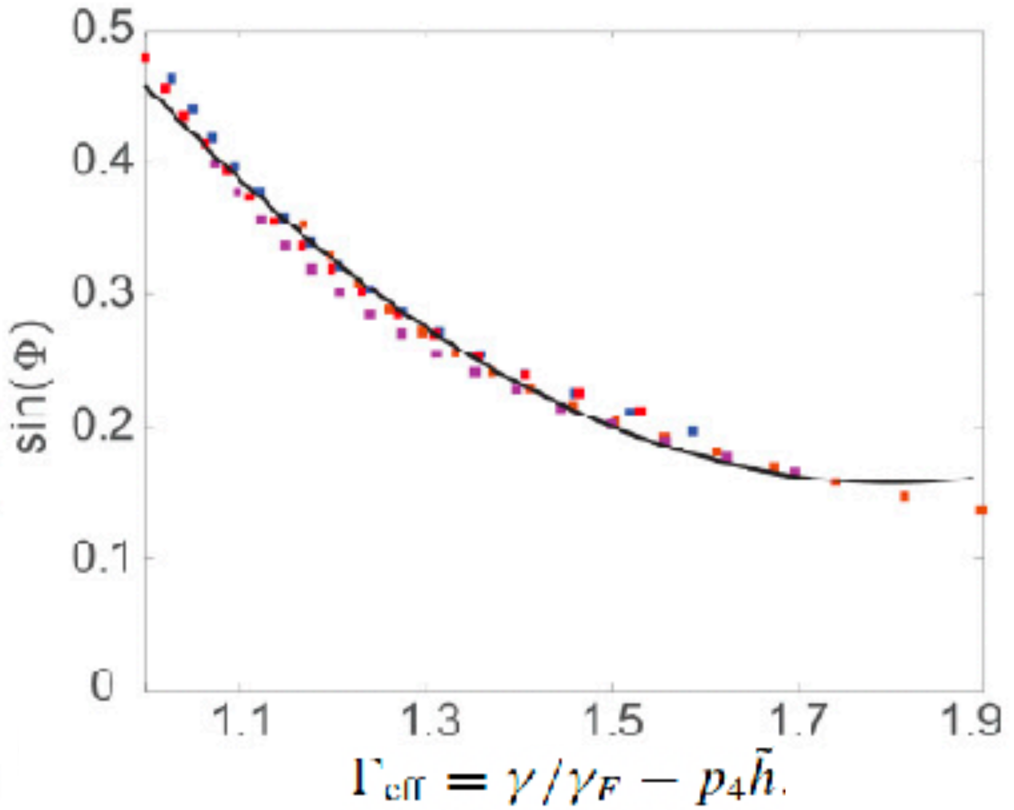
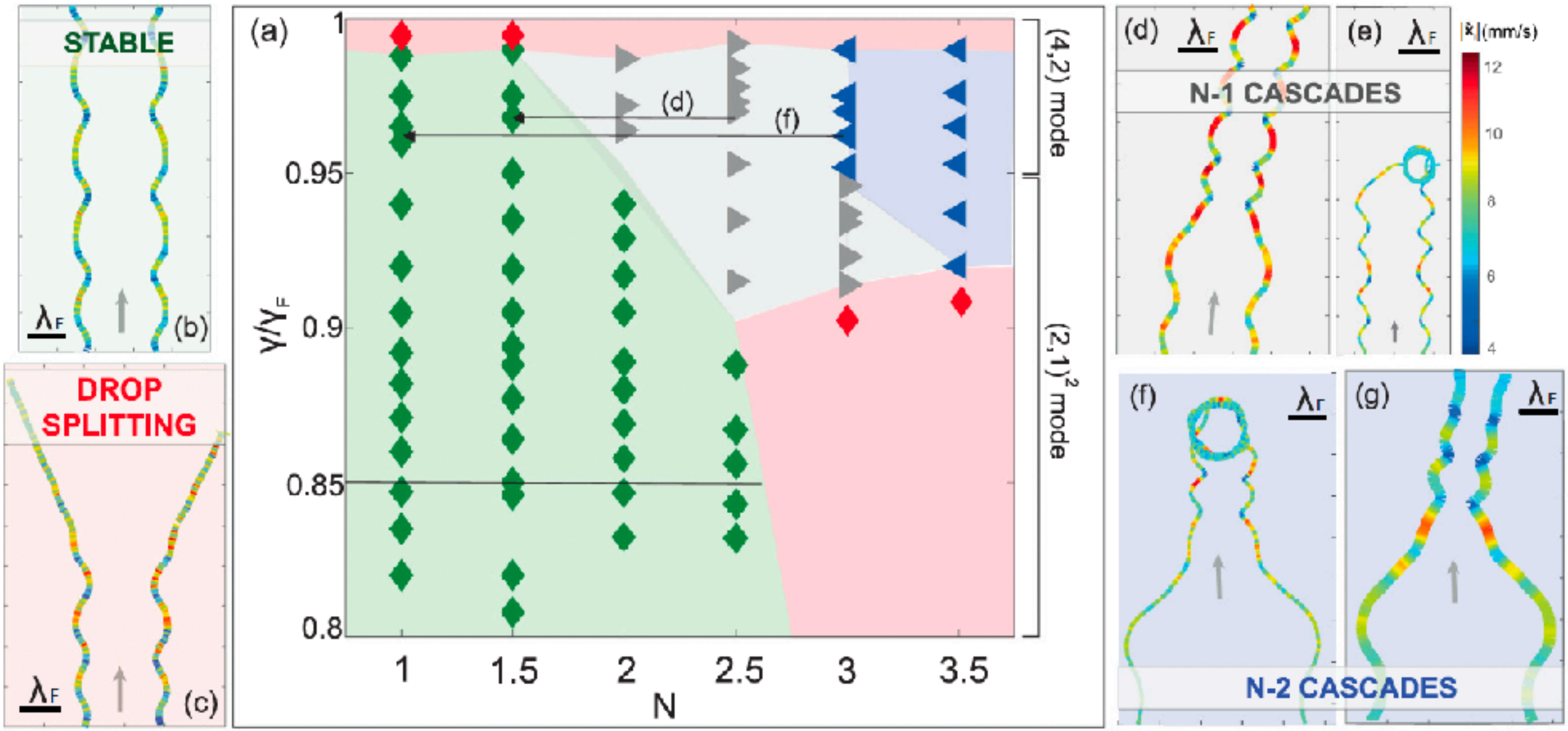


# Promenading pairs: Experiments

- promenade modes may destabilize at high memory
- shift from one dynamic bound state to another




## Energetic rationale?

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TEMPORAL DAMPING

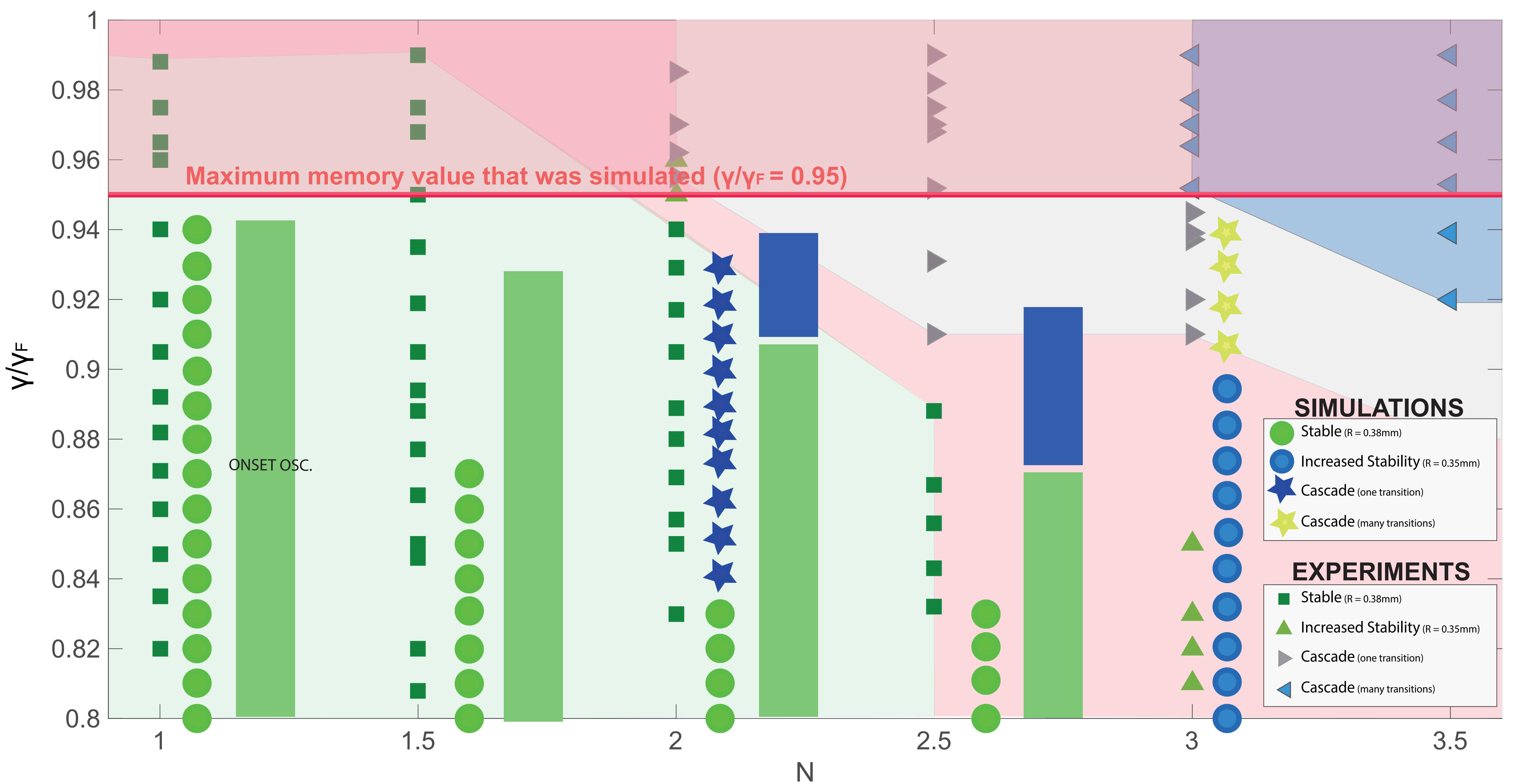
where $\varsigma = \pm 1$ is the relative phase parameter

and $\mathcal{S} = \sin\Phi$ is the sine of the mean impact phase

- stroboscopic model again has limitations in capturing observed behavior
- quantitative agreement requires consideration of **phase adaptation**

# Promenade mode: stability (Arbelaiz et al. 2016)

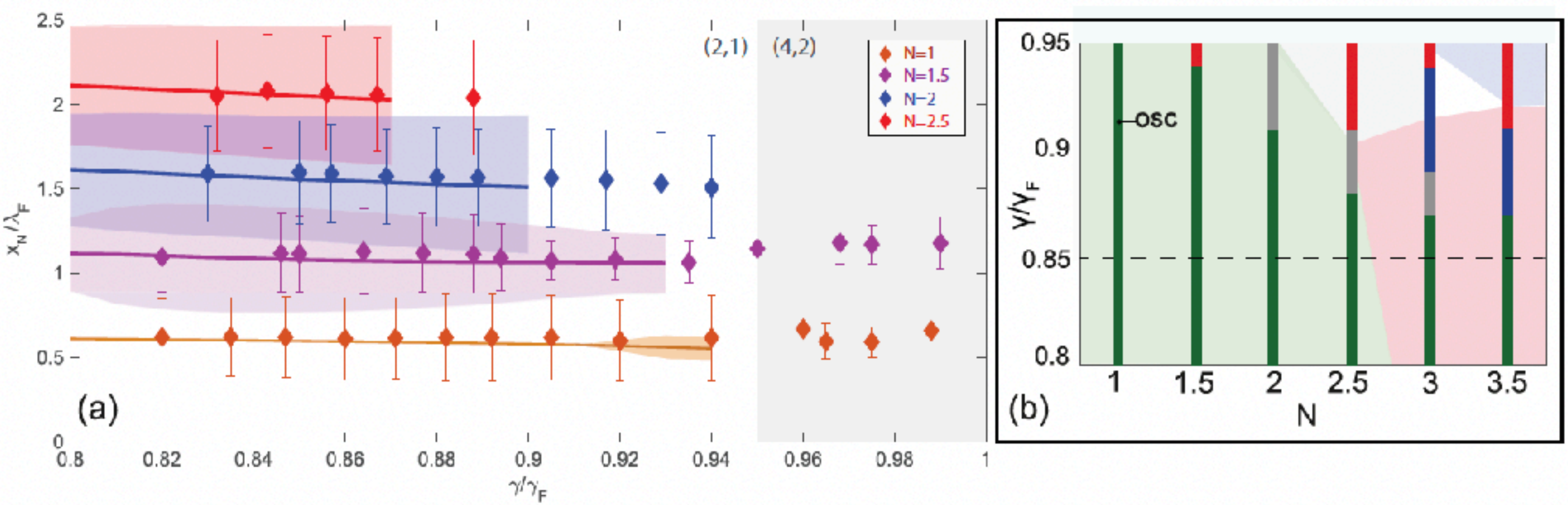
- assess stability of promenading states using the stroboscopic model




- indicates importance of **phase adaptation** on stability

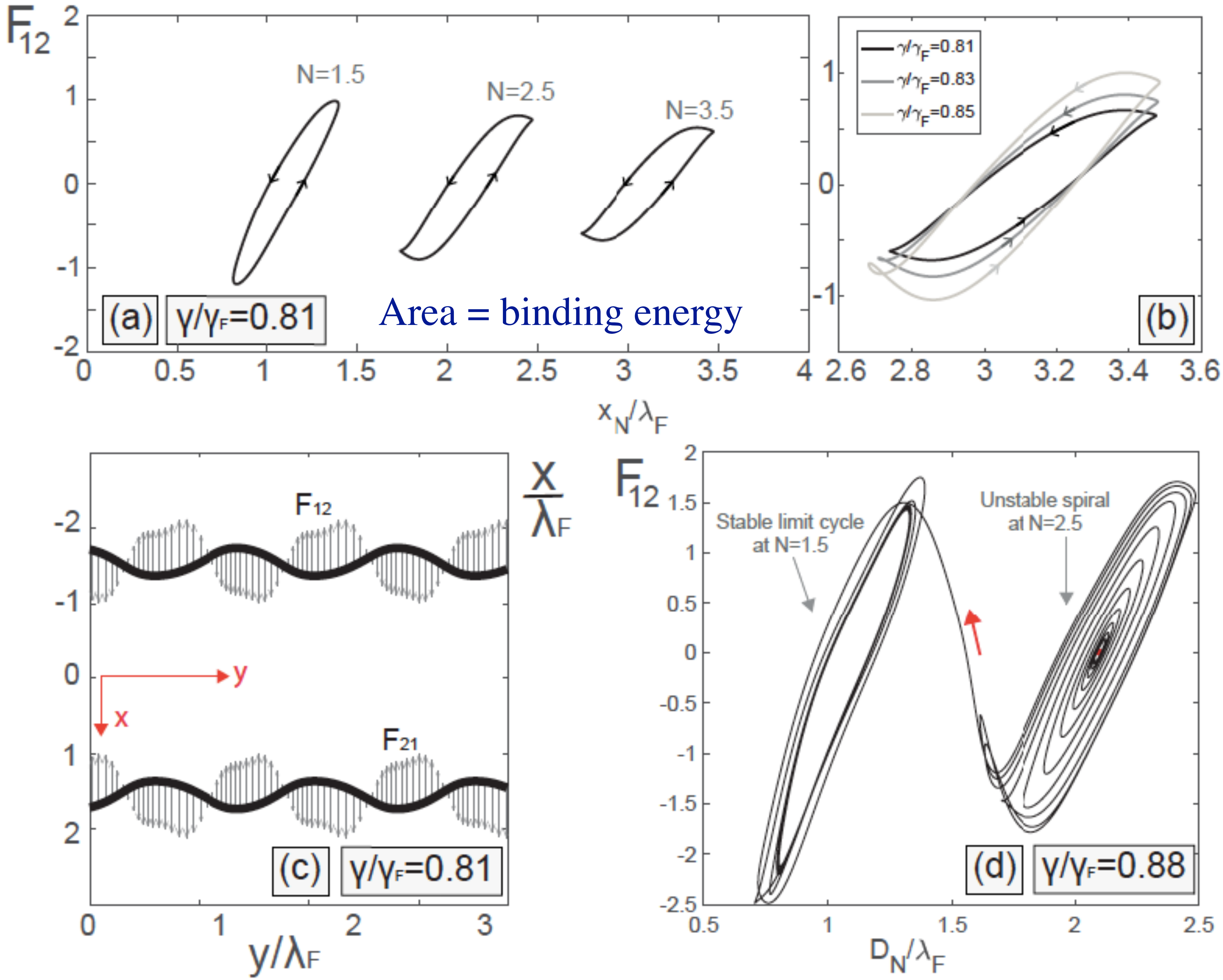
# Promenade mode: stability (Arbelaiz et al. 2016)

- compare observed and predicted onsets of instability

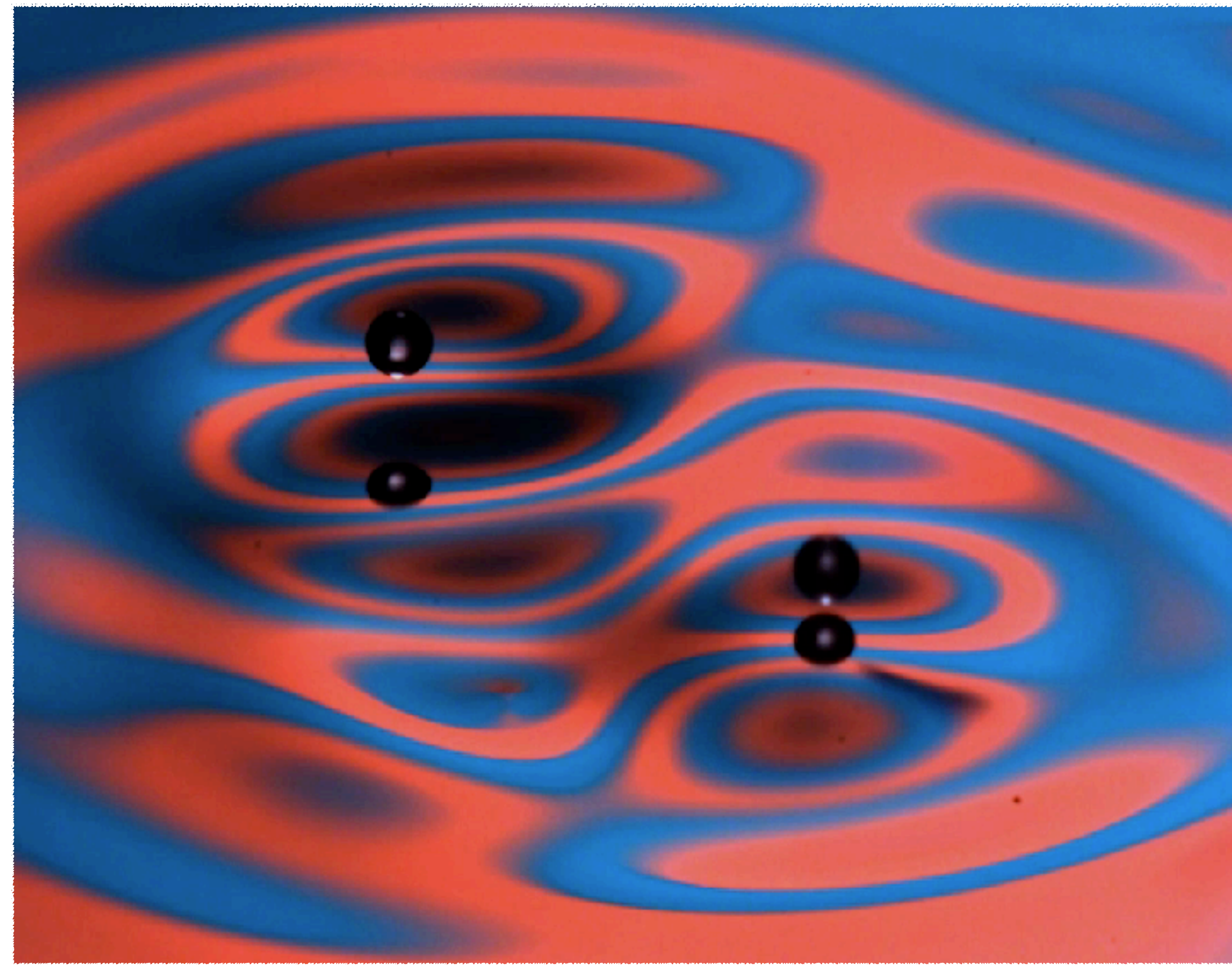



- stroboscopic model again has limitations in capturing observed behavior
- quantitative agreement requires consideration of **phase adaptation**

# Promenade mode: binding energy (Arbelaiz et al. 2016)




- binding energies of different promenade modes does not determine their relative stabilities: modulation in vertical dynamics energetically dominant



## Summary of dynamic bound states

- strobe model captures behavior of orbiting, promenading pairs *qualitatively*
- shortcomings suggest importance of phase variability, non-resonant effects

## Left on the table

- revisit, consider system energetics: which states are the most energetic?

# The wave-induced added mass of walking droplets

## Walking drops

- exhibit several features reminiscent of quantum systems (Couder & coworkers)
- bears a strong resemblance to de Broglie's relativistic pilot-wave theory

  - an attempt to reconcile relativity and quantum mechanics

## So, what is the field in QM?

- workers in Stochastic Electrodynamics suggest an EM pilot wave
- others have suggested the Higgs field, and gravitational waves

## A feature of particle motion in SED

*(Rueda & Haisch, 2005)*

- inertial mass is viewed a place holder for electromagnetic energy
- particle mass is altered through interaction with the zero-point field

**Is this true in the hydrodynamic pilot-wave system?**

# Perspective

- the walking droplet system is damped and driven
- there are steady and periodic states in which the driving and dissipation cancel

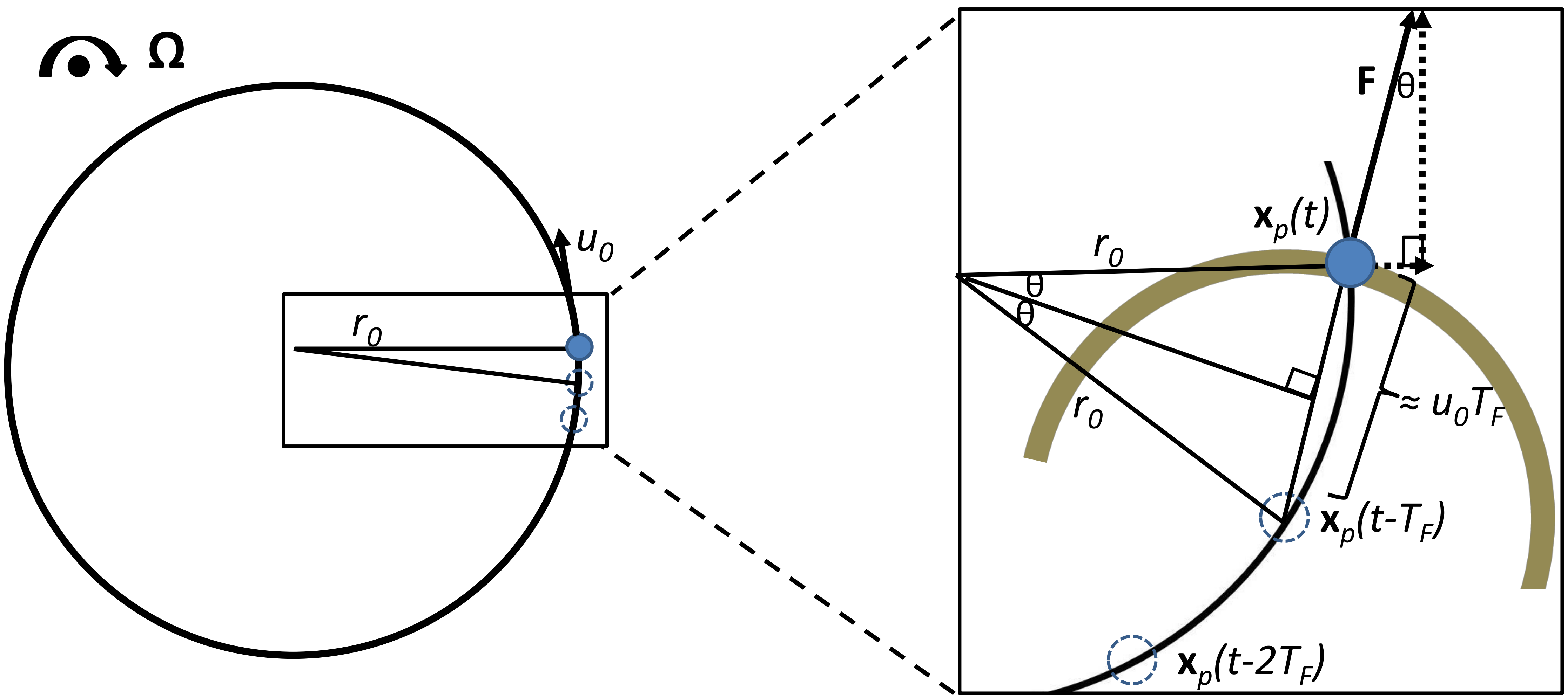
**Three interesting questions…**

**Is there an inviscid description of these states?**

**What is the mass of a walker?**

**What would the dynamics look like if we denied the presence of the guiding wave?**

# Origins of the boost factor idea




- the wave field is dominated by that from most recent impact
- owing to curvature of path, the resulting wave force has a radial component
- drop appears to be heavier than it is, owing to its pilot wave field

## Integro-differential trajectory equation

$$m\ddot{\mathbf{x}} + D\dot{\mathbf{x}} = \frac{F}{T_F}\int_{-\infty}^{t} \frac{J_1\left(k_F\,|\mathbf{x}(t)-\mathbf{x}(s)|\right)}{|\mathbf{x}(t)-\mathbf{x}(s)|}\left(\mathbf{x}(t)-\mathbf{x}(s)\right)\mathrm{e}^{-(t-s)/(T_F M_e)}\,\mathrm{d}s + \mathbf{f}$$

## Expand wave force in weak acceleration limit

Assume that $\mathbf{F} = \mathbf{F}(\epsilon t/T_M)$ (slowly varying force)

$|\ddot{\mathbf{x}}| \ll |\dot{\mathbf{x}}|/(T_F M_e)$ (weak acceleration)

- walker accelerates over a time long relative to the memory time

## Trajectory equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\gamma_B\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right) + \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\left[D - \frac{mgA}{T_F\left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|^2}\left(1 - \frac{1}{\sqrt{1+(k_F T_F M_e)^2\left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|^2}}\right)\right] = \mathbf{f}$$

## Boost Factor

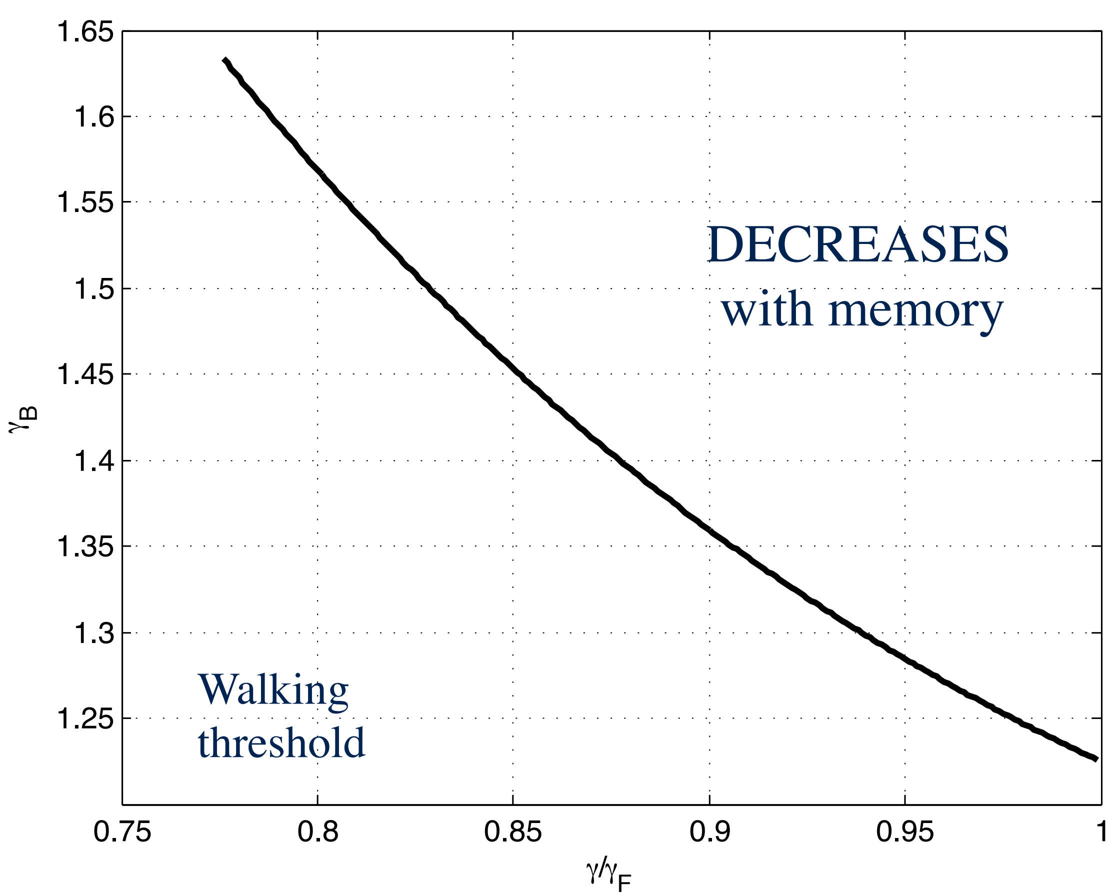
$$\gamma_B = 1 + \frac{mgA\;k_F^2 T_F^2 M_e^3}{2\left(1 + k_F^2 T_F^2 M_e^2\;\left|\frac{d\mathbf{x}}{dt}\right|^2\right)^{3/2}}$$

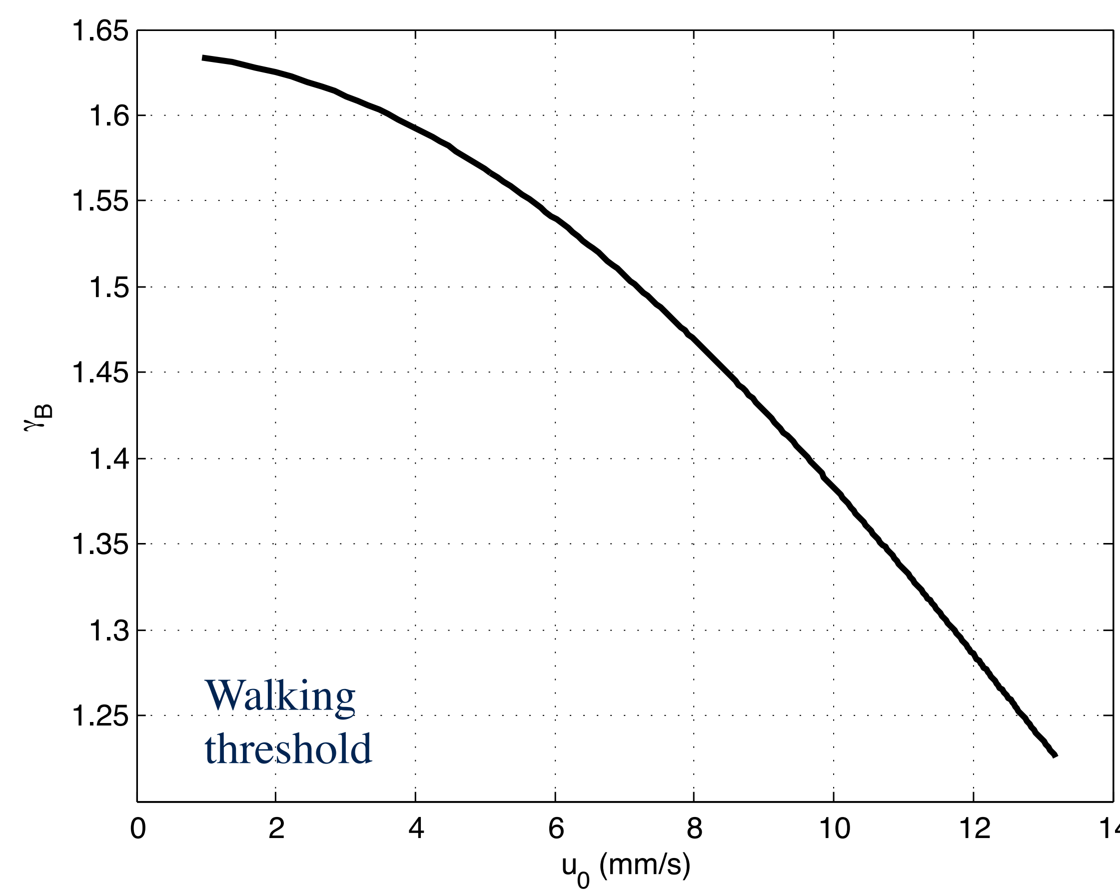
# The hydrodynamic boost factor

$$\boxed{\gamma_B = 1 + \frac{\beta}{2\kappa(1+v^2)^{3/2}}}$$

**Walker mass:** $$m_w = \gamma_B \, m_0$$

**Walker momentum:** $$p_w = \gamma_B \, m_0 \, v$$



## Trajectory equation

at free walking speed

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\gamma_B\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)+\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\left[D-\cancel{\frac{mgA}{T_F\left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|^2}\left(1-\frac{1}{\sqrt{1+(k_FT_FM_e)^2\left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|^2}}\right)}\right]=\mathbf{f}$$

## Boost Factor

$$\boxed{\gamma_B = 1+\frac{mgA\;k_F^2T_F^2M_e^3}{2\left(1+k_F^2T_F^2M_e^2\;\left|\frac{d\mathbf{x}}{dt}\right|^2\right)^{3/2}} = 1+\frac{\beta}{2\kappa(1+v^2)^{3/2}}}$$

## Walker mass

$$m_w = \gamma_B\; m_0$$

rest mass

## Walker momentum

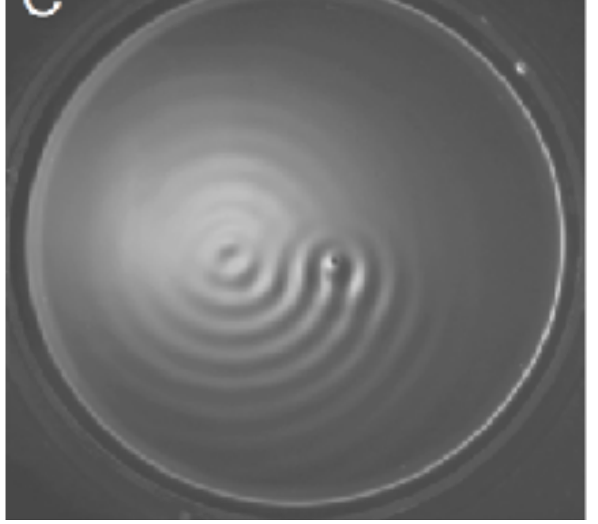
$$\mathbf{p} = m_w\;\mathbf{v} = \gamma_B\; m_0\;\mathbf{v}$$

## Trajectory equation

$$\boxed{\frac{d}{dt}\,\mathbf{p} = \mathbf{f}}$$

- inviscid dynamics of a particle with a speed-dependent mass

# Steady circular motion



$$\mathbf{x}(t) = ( r_0 \cos\omega t,\ r_0 \sin\omega t )$$

Tangential and normal components of trajectory equation:

**Tangential:**

$$r_0\omega\left[D - \frac{m_0 g A}{T_F (r_0\omega)^2}\left(1 - \frac{1}{\sqrt{1+(k_F T_F M_e r_0\omega)^2}}\right)\right] = \mathbf{f}\cdot\mathbf{t} = 0$$

Steady walking speed obtains: $\quad u_0 = \omega\, r_0$

**Radial:**

$$-m_0\gamma_B r_0\omega^2 = \mathbf{f}\cdot\mathbf{n}$$

## Two special cases

**Coriolis force**

$$\boxed{\mathbf{f} = 2m_0\, \mathbf{v}\times\boldsymbol{\Omega}}$$

Fort et al. (2010), Harris & Bush (2014)

**Central force**

$$\boxed{\mathbf{f} = -k\mathbf{x}}$$

Perrard et al. (2014)

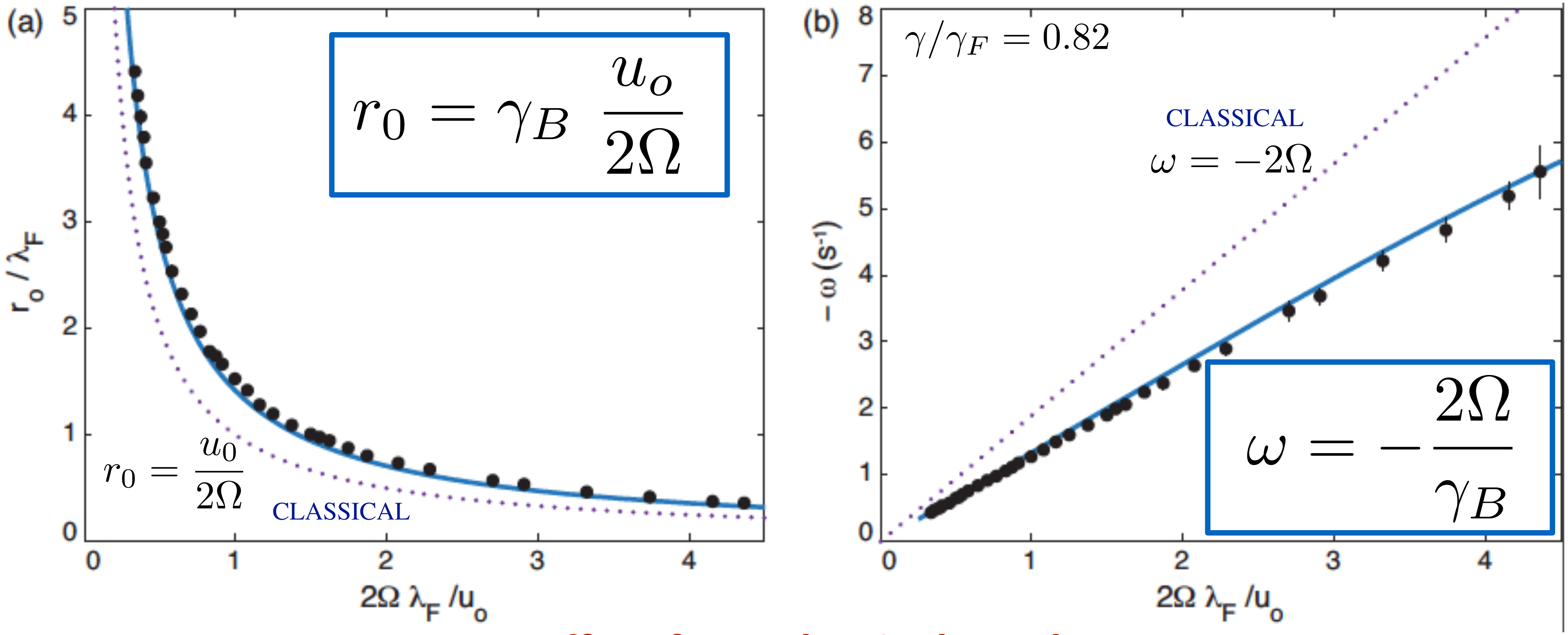
# Walkers in a rotating frame

Harris & Bush, JFM (2014)

**Coriolis force:** $$\mathbf{f} = 2m_0\,\mathbf{v}\times\boldsymbol{\Omega}$$

**Radial force balance in inertial orbits:**

$$-m_0\gamma_B r_0\omega^2 = \mathbf{f}\cdot\mathbf{n} = 2m_0 r_0\omega\Omega$$



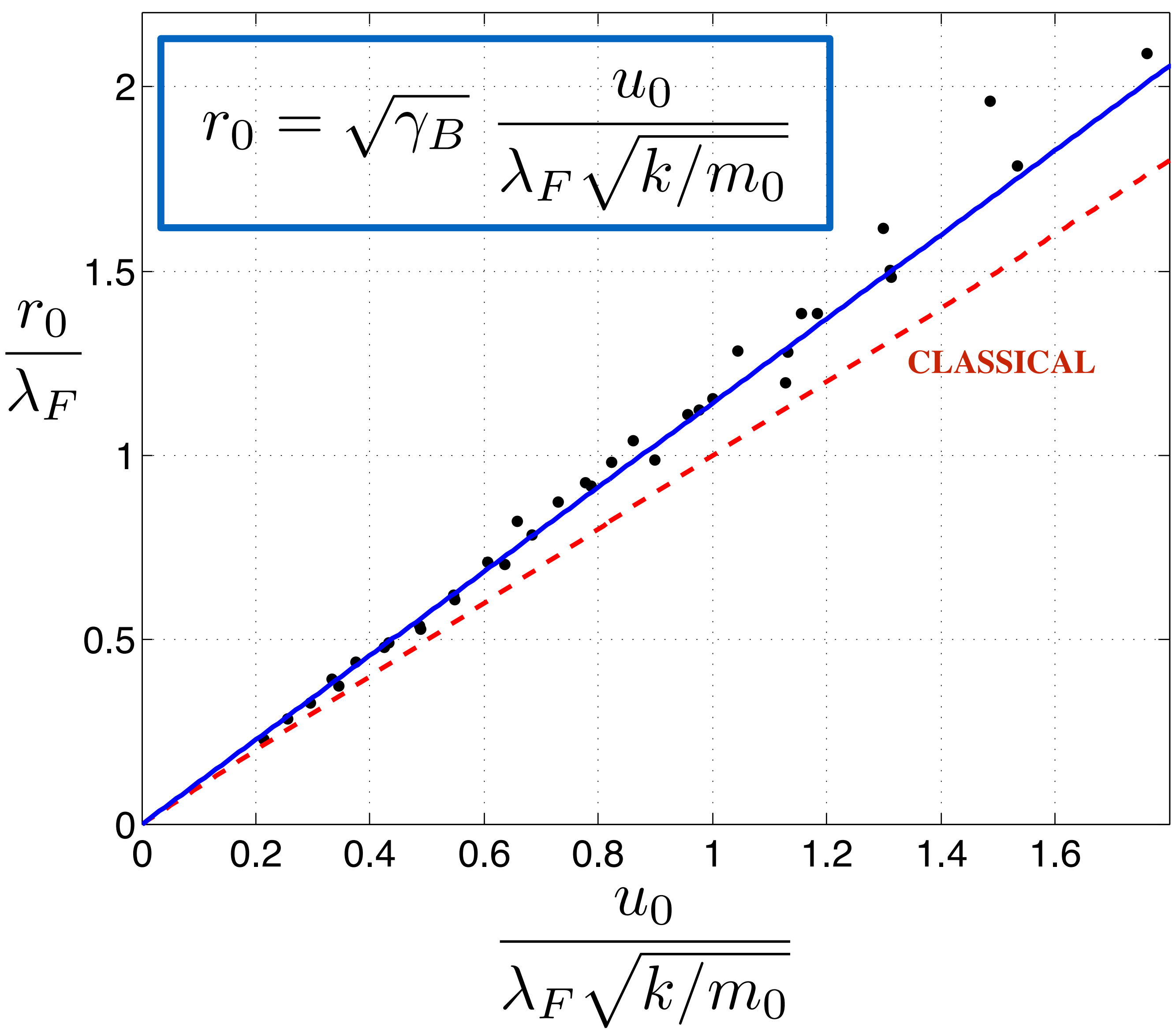

**offset from classical results**
**due to hydrodynamic boost factor**

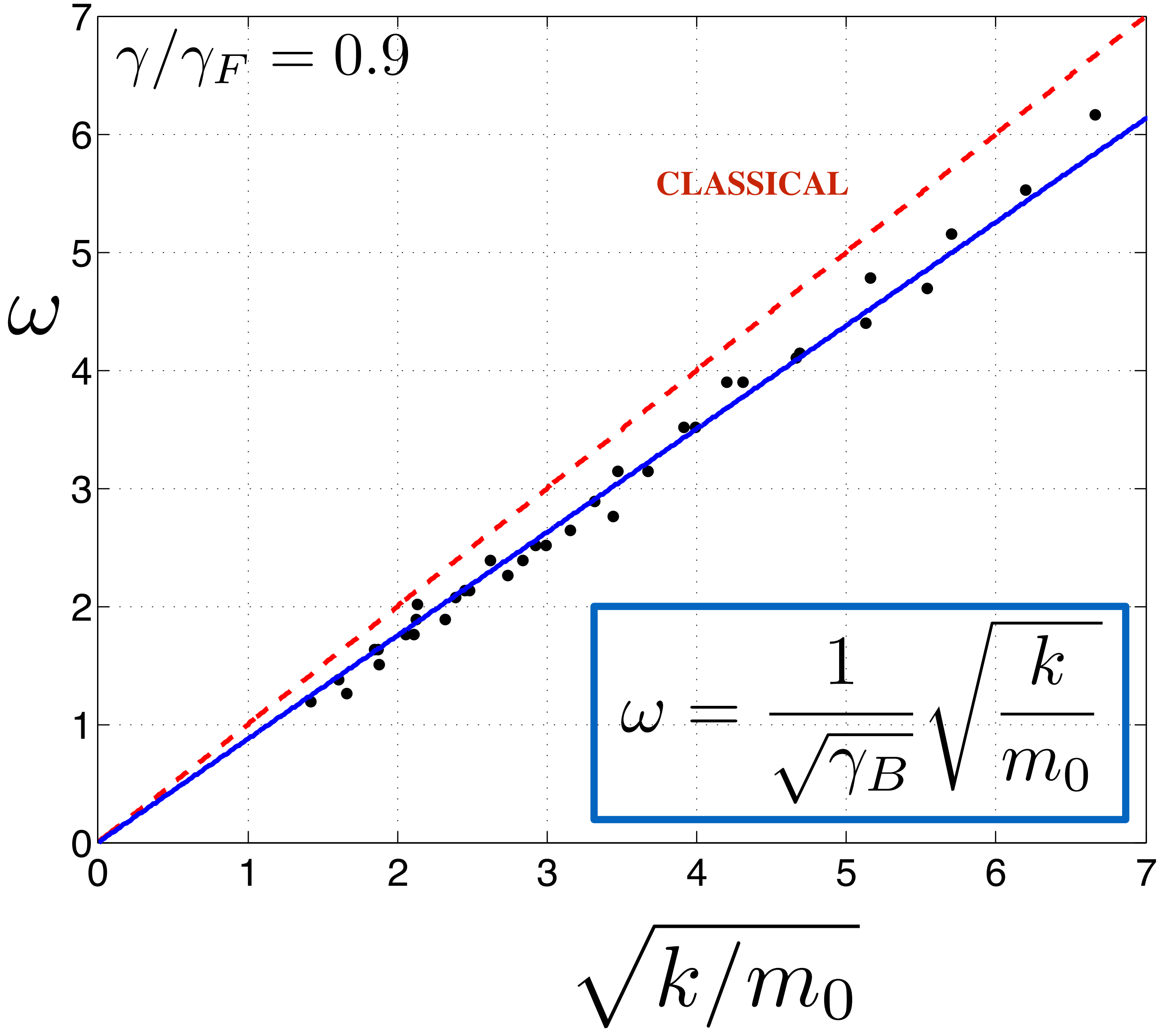
# Orbits in a central force

$$\mathbf{f} = -k\mathbf{x}$$

**Radial force balance:** $$-m_0\gamma_B r_0\omega^2 = \mathbf{f}\cdot\mathbf{n} = -kr_0$$

*Data: Perrard et al. (2013)*




**offset from classical results**
**due to hydrodynamic boost factor**

# The Boost equation

Bush et al. (JFM 2014)

In the weak-acceleration limit, the trajectory equation takes the form

$$\boxed{\frac{d}{dt}\,\mathbf{p_w} \;+\; D_w\,\mathbf{v} \;=\; \mathbf{F}}$$

where the walker mass $m_w = \gamma_B(v)\; m_0$ , momentum $\mathbf{p_w} = m_w\mathbf{v}$

depend on the ***hydrodynamic boost factor*****:** $\gamma_B \;=\; 1 \;+\; \dfrac{\beta}{2\kappa(1+v^2)^{3/2}}$

and a nonlinear drag $D_w \;=\; D_0\left(\dfrac{\mathbf{v}^2}{u_0^2} - 1\right)$ drives it to its free walking speed.

**For motion at the free walking speed:**

$$\boxed{\frac{d}{dt}\,\mathbf{p_w} \;=\; \mathbf{F}}$$

- the inviscid dynamics of a particle with a speed-dependent mass

Answers to our interesting questions…

## What is the mass of a walker?

$$\boxed{m_w \;=\; \gamma_B(v)\; m_0}$$

**BOOST FACTOR**

## What would the dynamics look like if we denied the presence of the guiding wave?

***One would observe the dynamics of a particle with a speed-dependent mass and a nonlinear drag that drives it towards a constant speed.***

## Is there an inviscid description of these states?

$$\boxed{\frac{d}{dt}\,\mathbf{p_w} \;=\; \mathbf{f}} \quad \text{where} \quad \mathbf{p_w} = m_0 \gamma_B(v)\;\mathbf{v}$$

- the inviscid dynamics of a particle with a speed-dependent mass

# Compare generalized boost model and LAD equation

**Sam Turton's thesis**

$$\boxed{\frac{d}{dt}(\kappa\gamma_B v) + D(v)v = B_1(v)\ddot{v} + B_2(v)\dot{v}^2 v,}$$

$$B_1(v) = \frac{\beta}{2}\left(\frac{1-4v^2}{(1+v^2)^{7/2}}\right),$$

$$\gamma_B(\mathbf{v}) = 1 + \frac{\beta}{2\kappa(1+|\mathbf{v}|^2)^{3/2}}, \quad D(\mathbf{v}) = 1 - \frac{\beta}{|\mathbf{v}|^2}\left(1 - \frac{1}{\sqrt{1+|\mathbf{v}|^2}}\right) \quad B_2(v) = \frac{15\beta}{8}\left(\frac{4v^2-3}{(1+v^2)^{9/2}}\right).$$

**LAD equation:**

$$\boxed{\frac{d\mathbf{v}}{dt} = \frac{e}{mc}\mathbf{F}\cdot\mathbf{v} + \tau_0\left(\ddot{\mathbf{v}} + \frac{\mathbf{v}}{c^2}|\dot{\mathbf{v}}|^2\right),} \tag{3.57}$$

used in electrodynamics to describe the trajectory of a particle with mass $m$, charge $e$, and velocity $\mathbf{v}$, moving in response to a background field $\mathbf{F}$. The last two terms are the 'self force', arising from the particle interacting with its own electromagnetic field, specifically the *Schott* force, $\mathbf{F}_S \sim \ddot{\mathbf{v}}$, and the *radiation reaction* force, $\mathbf{F}_R \sim \mathbf{v}|\dot{\mathbf{v}}|^2$. Notoriously, equation (3.57) admits unphysical, runaway solutions, a problem that remains unresolved in classical electrodynamics. The LAD equation is based on an approximation to the equations of motion required due to uncertainty of the precise form of the electromagnetic field in the vicinity of the particle. The unphysical runaway solutions, which also exist within the higher-order boost equation (3.15), might conceivably be understood as being a consequence of analogous approximation errors.

## Boost summary

- walking drops exhibit features of SED

**If one were unaware of the underlying wave field, one would observe the inviscid dynamics of a particle with a speed-dependent mass.**

- hydrodynamic boost factor indicates the dependence of walker mass on speed
- provides rationale for anomalous radii of walkers in orbital states

## Caveats

- analysis holds only in the low-acceleration regime
- analysis, based on strobed model, neglects variations in bouncing phase

# Wave field measurements

Exp Fluids (2009) 46:1021–1036
DOI 10.1007/s00348-008-0608-z

RESEARCH ARTICLE

## A synthetic Schlieren method for the measurement of the topography of a liquid interface

Frédéric Moisy · Marc Rabaud · Kévin Salsac

Exp Fluids (2016) 57:163
DOI 10.1007/s00348-016-2251-4

CrossMark

RESEARCH ARTICLE

## Surface topography measurements of the bouncing droplet experiment

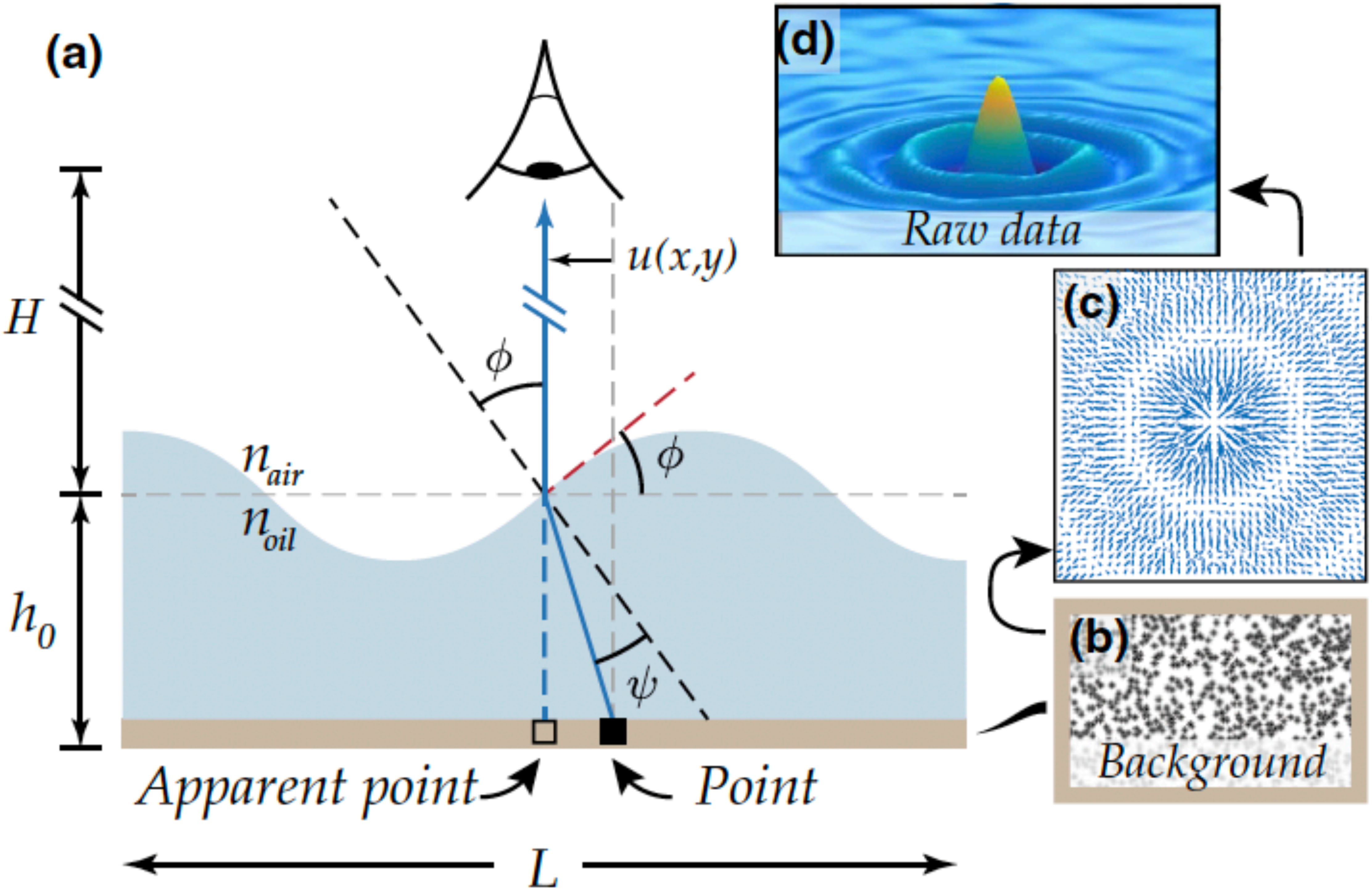
Adam P. Damiano[1] · P.-T. Brun[1] · Daniel M. Harris[1] · Carlos A. Galeano-Rios[2,3] · John W. M. Bush[1]

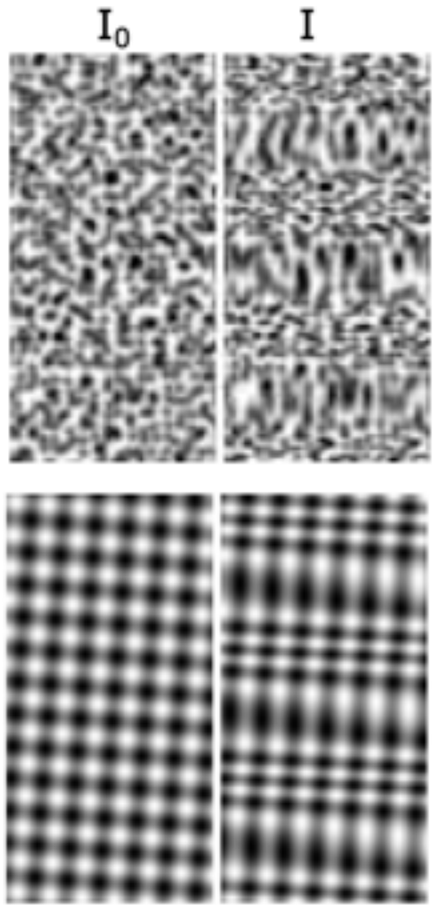
**Exp. Fluids manuscript No.**
(will be inserted by the editor)

## Real-time quantitative Schlieren imaging by fast Fourier demodulation of a checkered backdrop

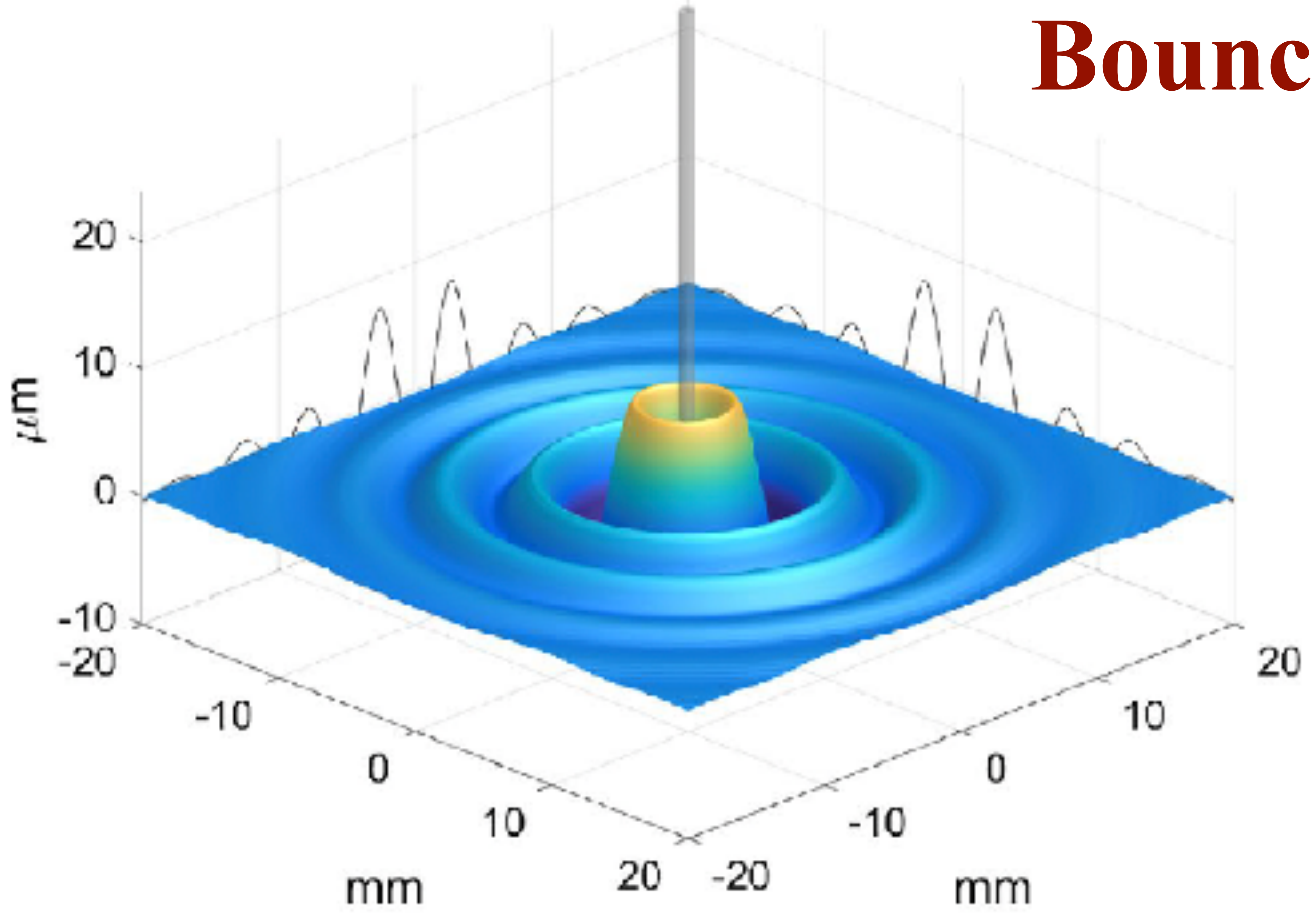
Sander Wildeman

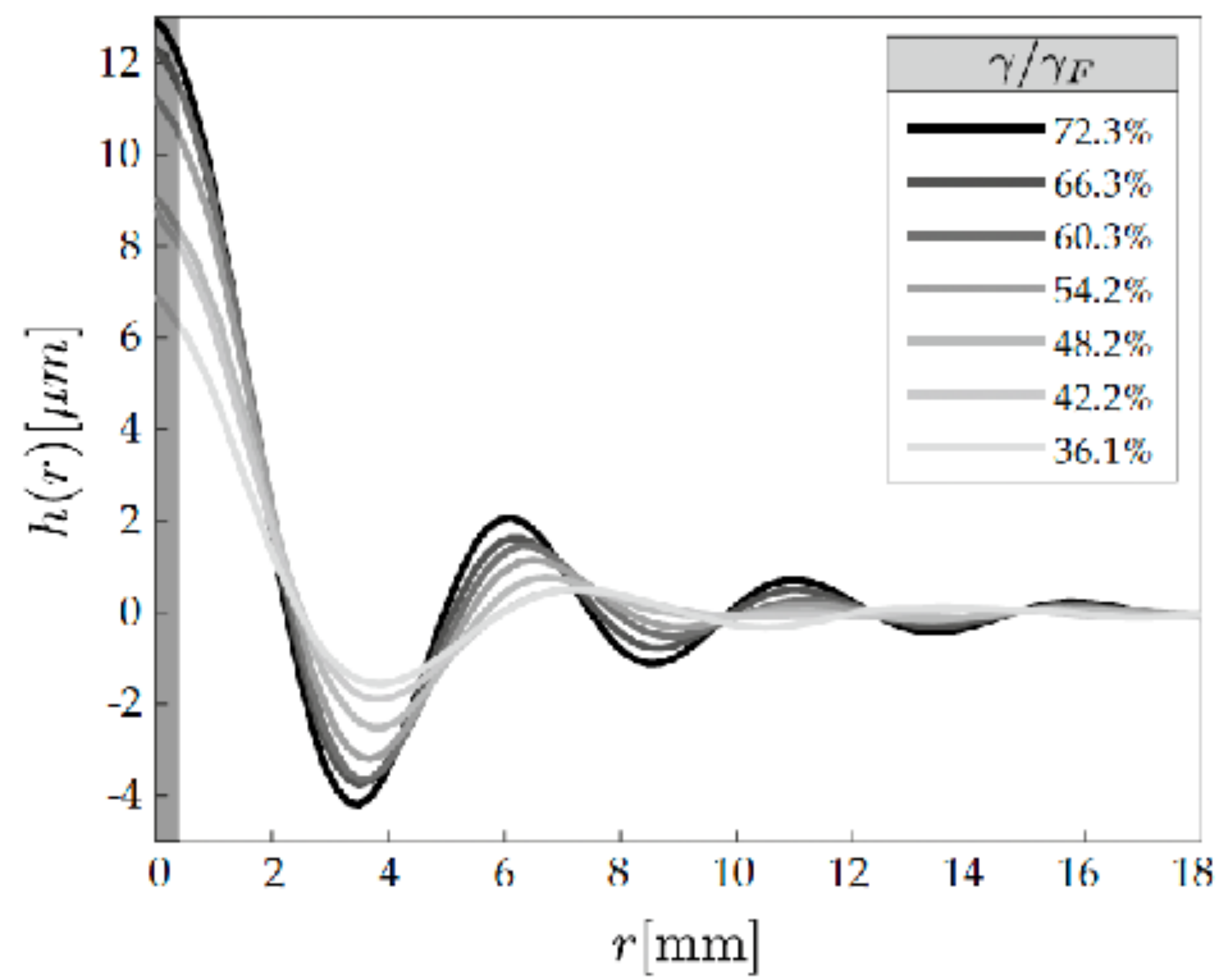
# Free-surface Schlieren

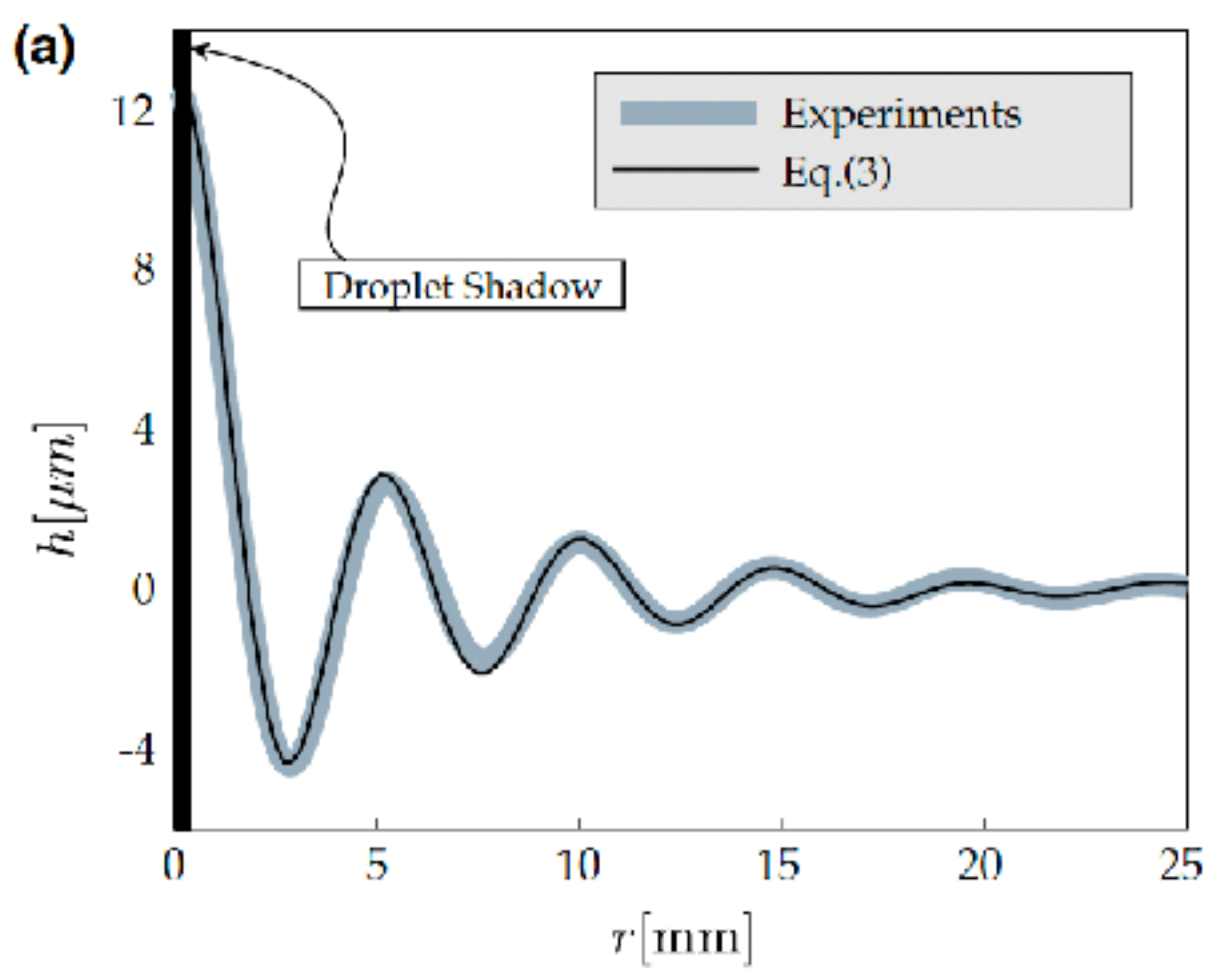


# Bouncer wave field



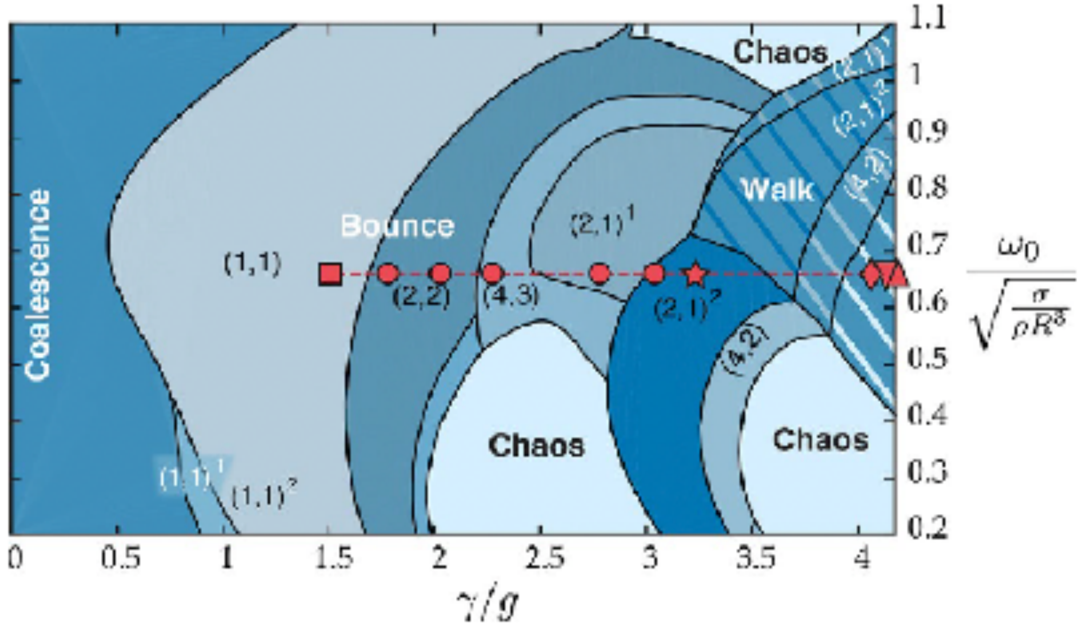




Well fit by

$$h(r) = 2A\sqrt{b}\mathbf{J}_0(k_F r) r \mathbf{K}_1(2\sqrt{br})$$

where A = 12.6 μm and $b = 0.0053\ \mathrm{mm}^{-2}$

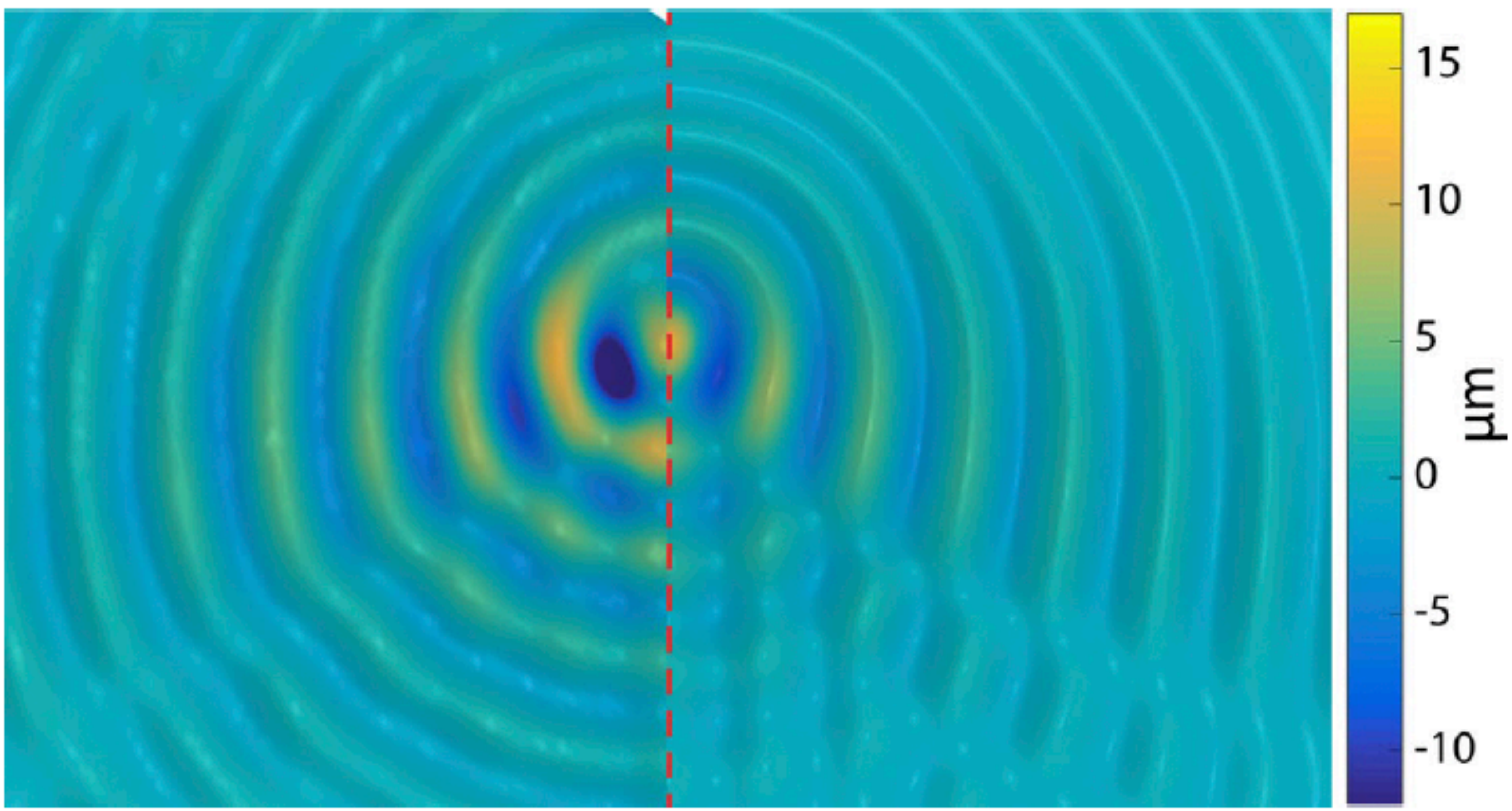
Eq. (A47) in Molacek & Bush (2013a)




# Walker wave field

Experiment

Theory



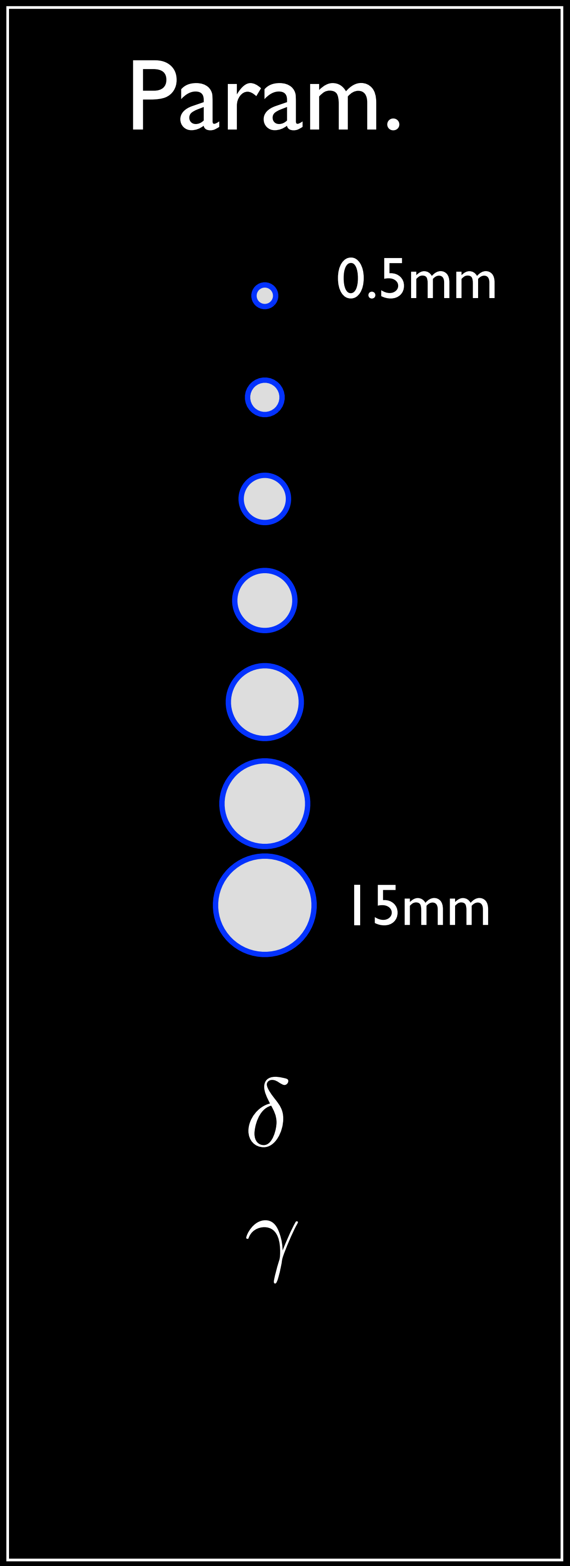
# Scattering off a submerged pillar

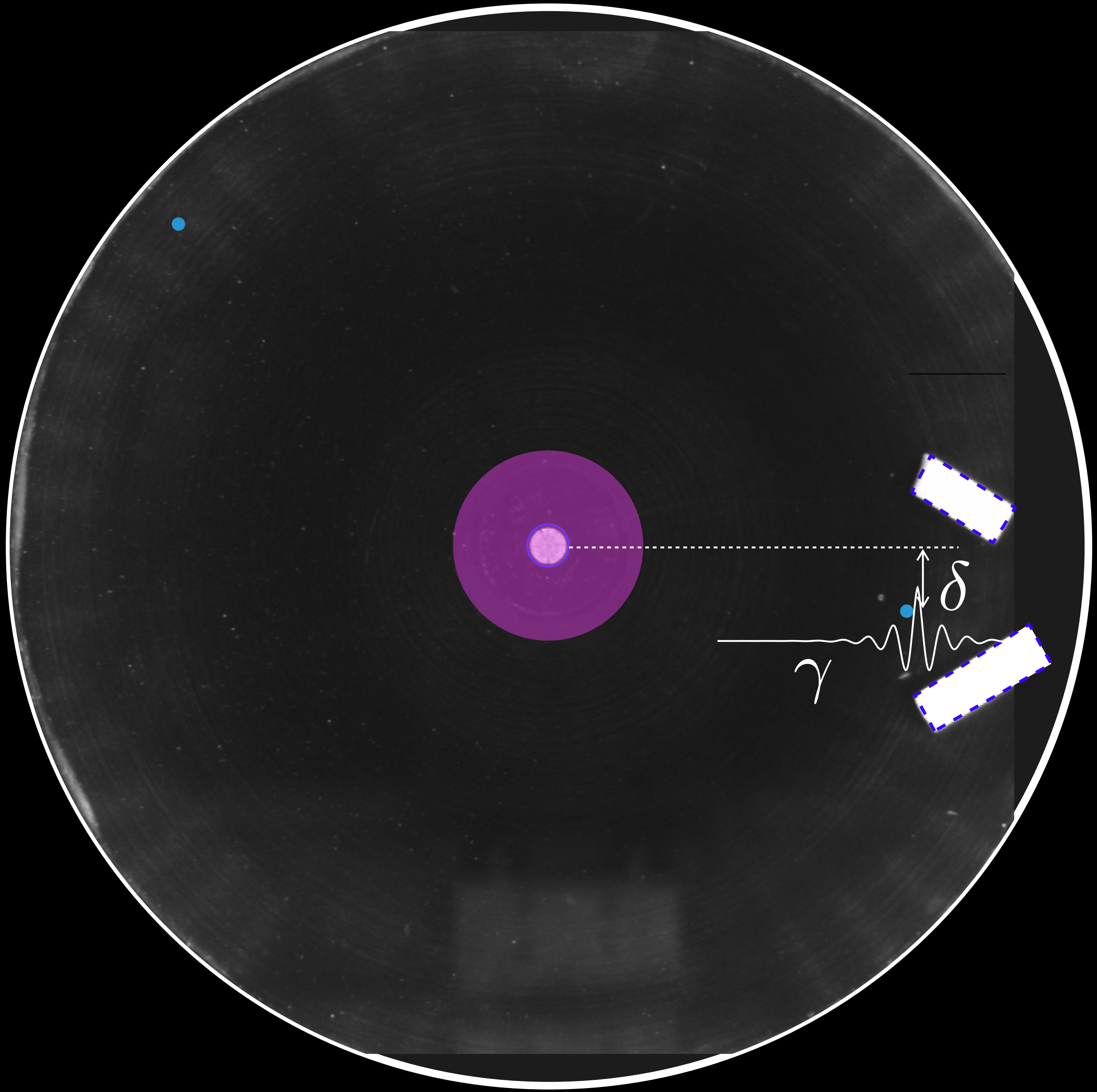
CHAOS 28, 096105 (2018)

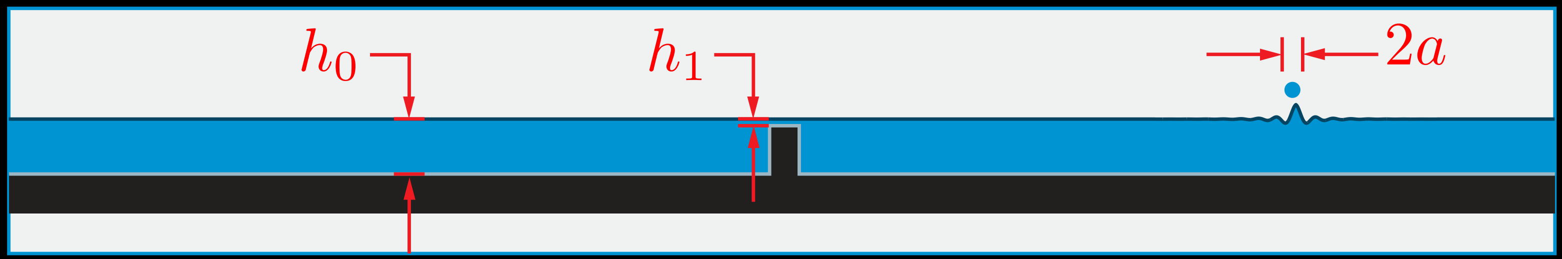
## The interaction of a walking droplet and a submerged pillar: From scattering to the logarithmic spiral

Daniel M. Harris,[1,2] P.-T. Brun,[1,3] Adam Damiano,[1] Luiz M. Faria,[1] and John W. M. Bush[1]
[1] *Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[2] *School of Engineering, Brown University, Providence, Rhode Island 02912, USA*
[3] *Department of Chemical and Biological Engineering, Princeton University, Princeton, New Jersey 08540, USA*

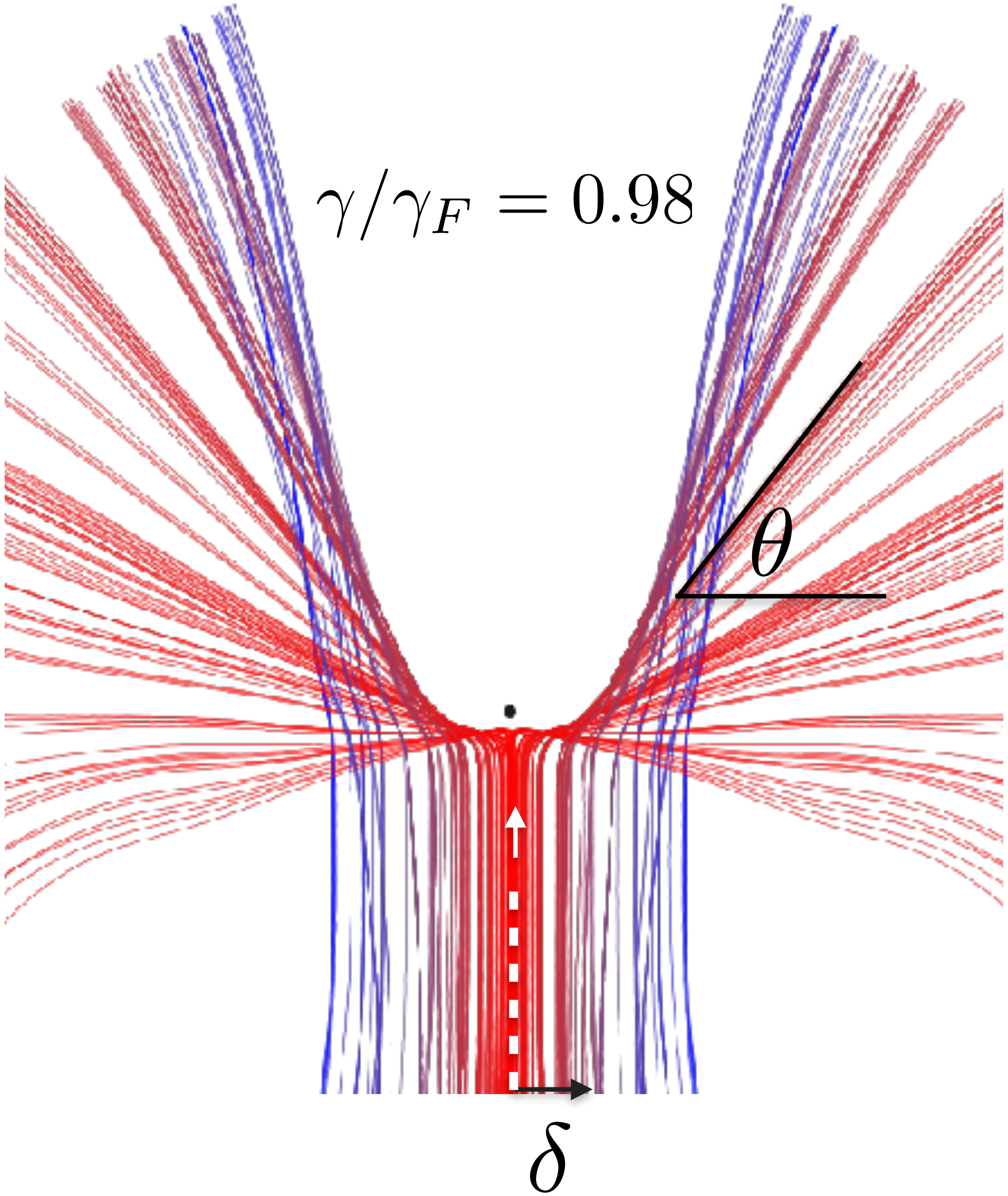
# The experiment

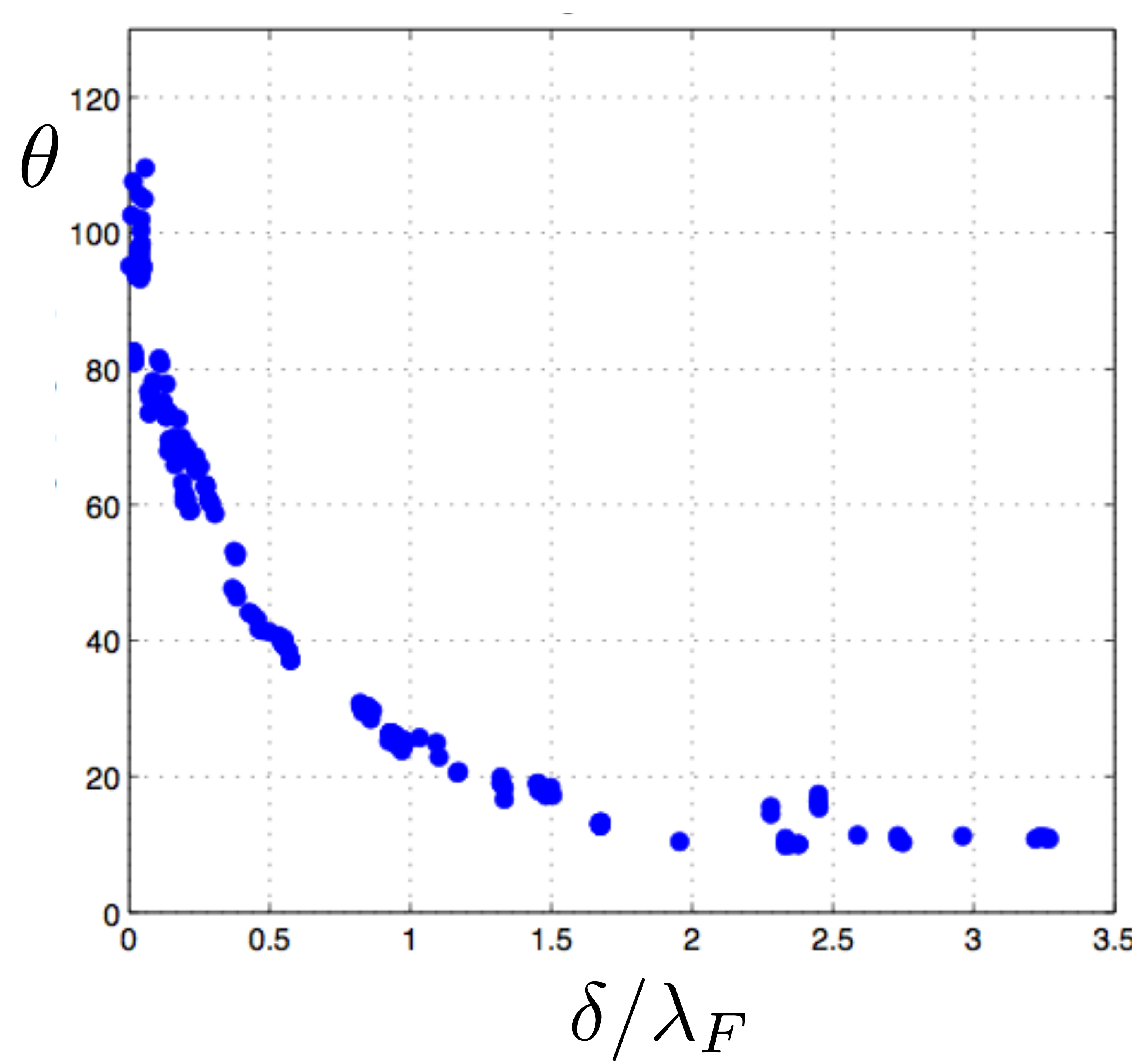


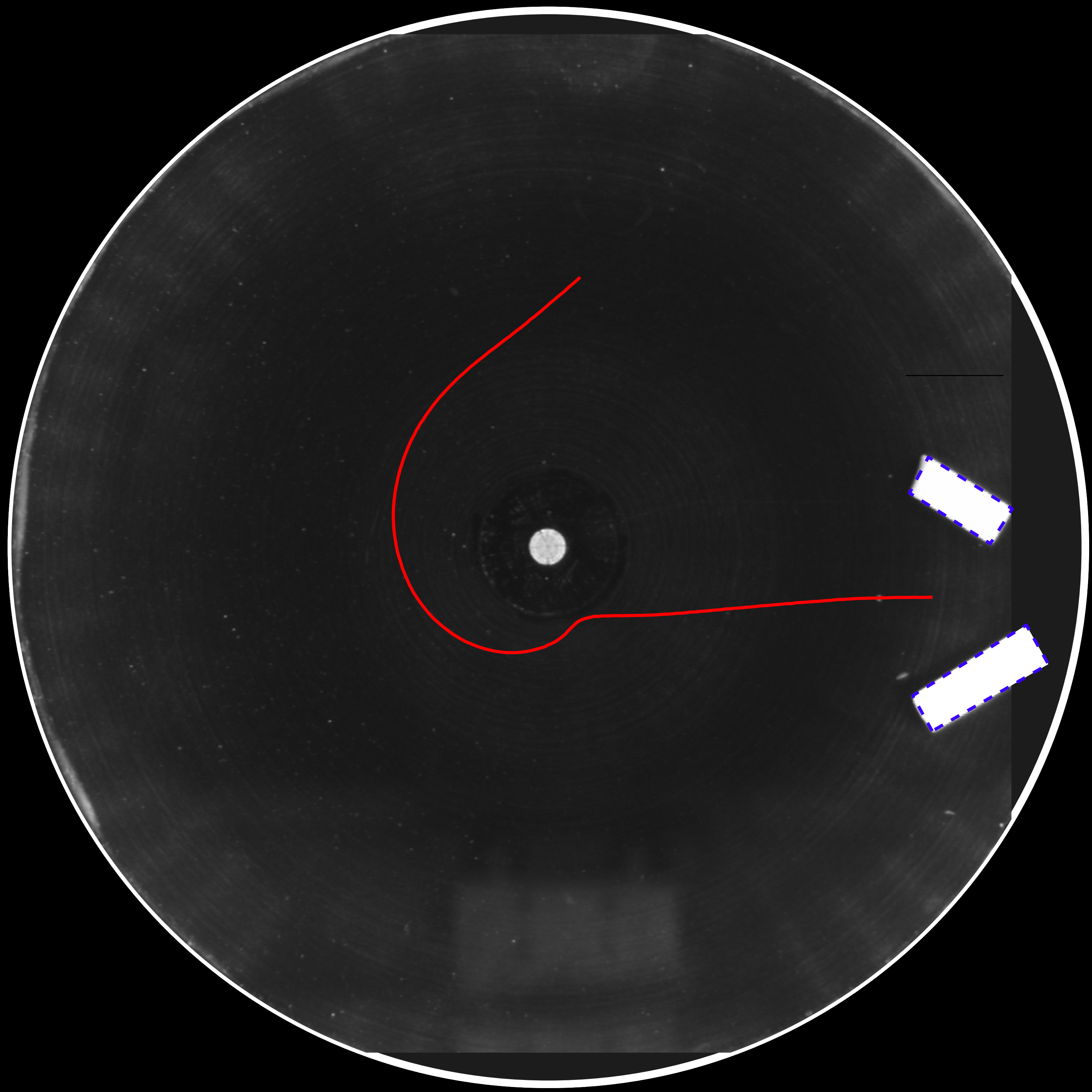
# Small Pillars (R = 0.5 mm)



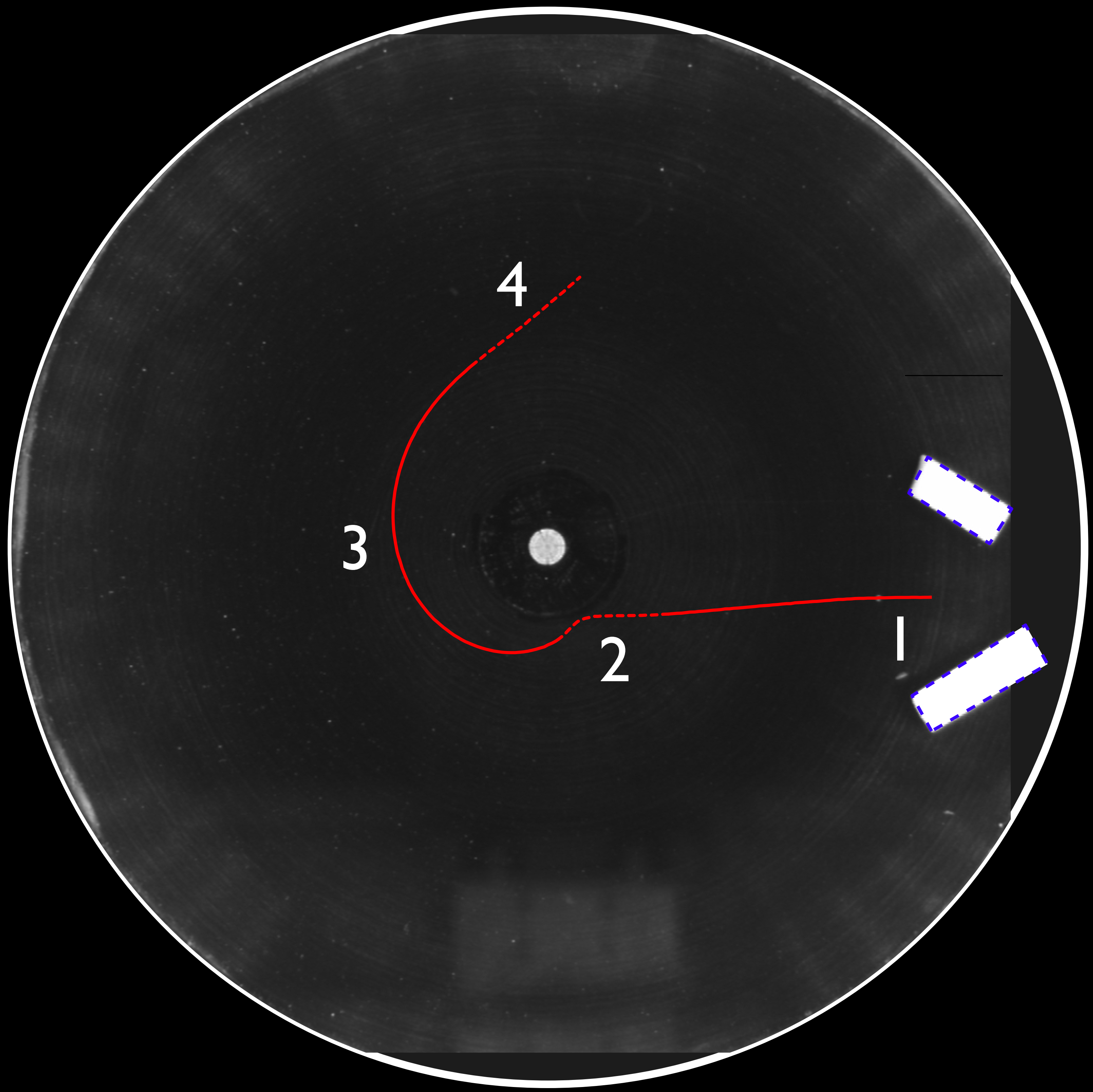

- walker simply scattered by small pillars
- continuous evolution of scattering angle $\theta$ with impact parameter $\delta$

# Larger pillars:  R = 2.5mm



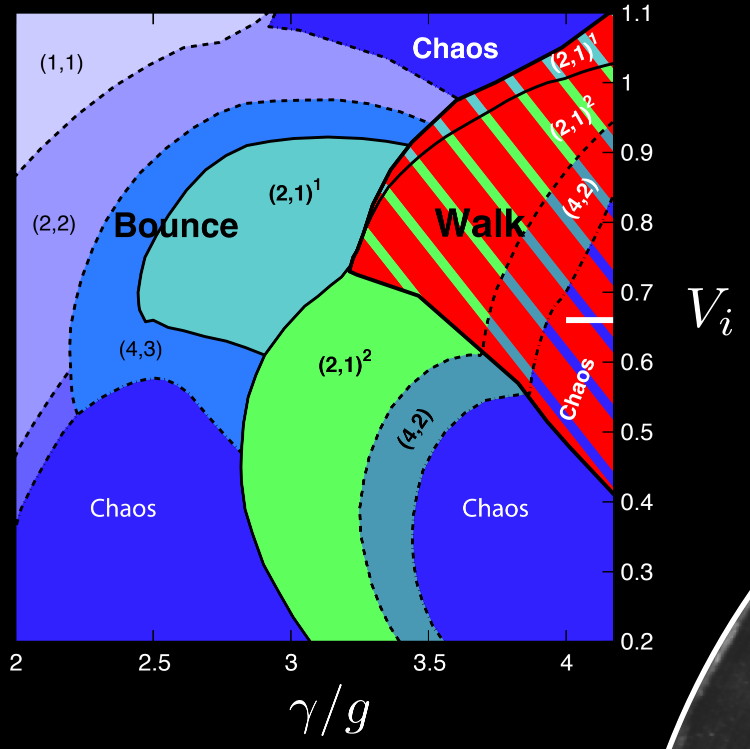
# A typical trajectory

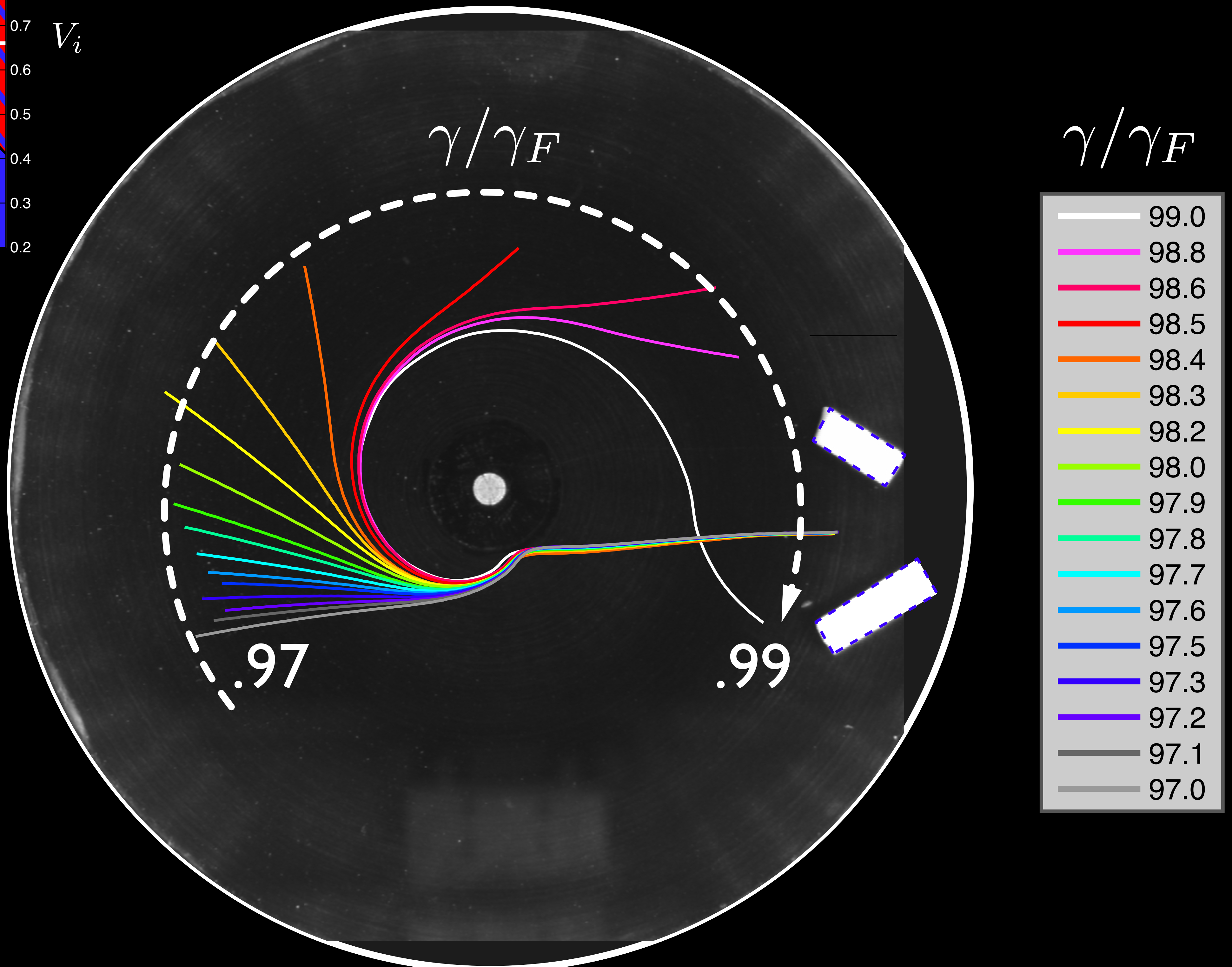
1. straight
2. bump
3. curved
4. straight




- walker bends towards the pillar

# Effect of memory (R = 2.5mm)

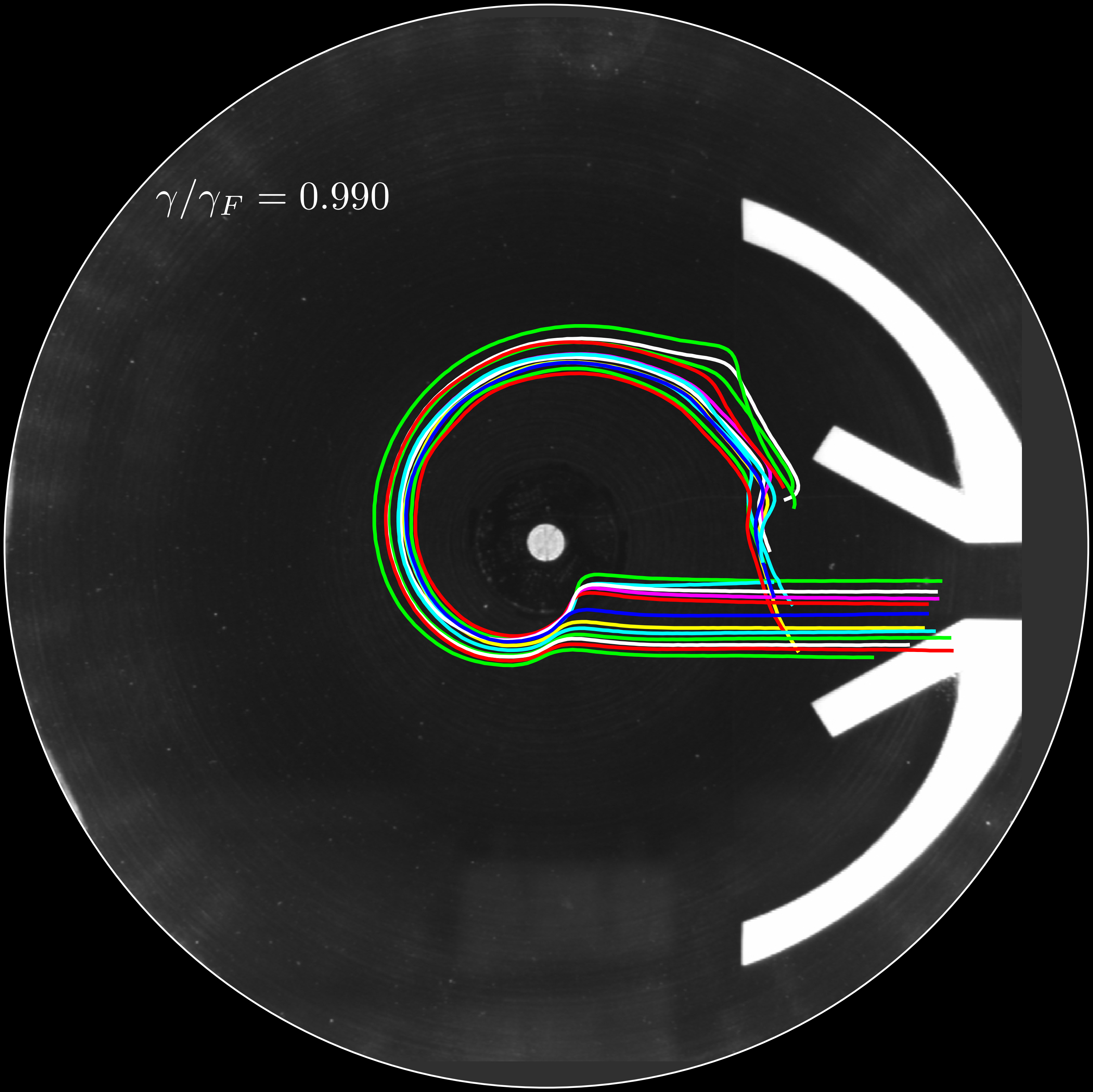



- walker locks onto a spiral trajectory at high memory

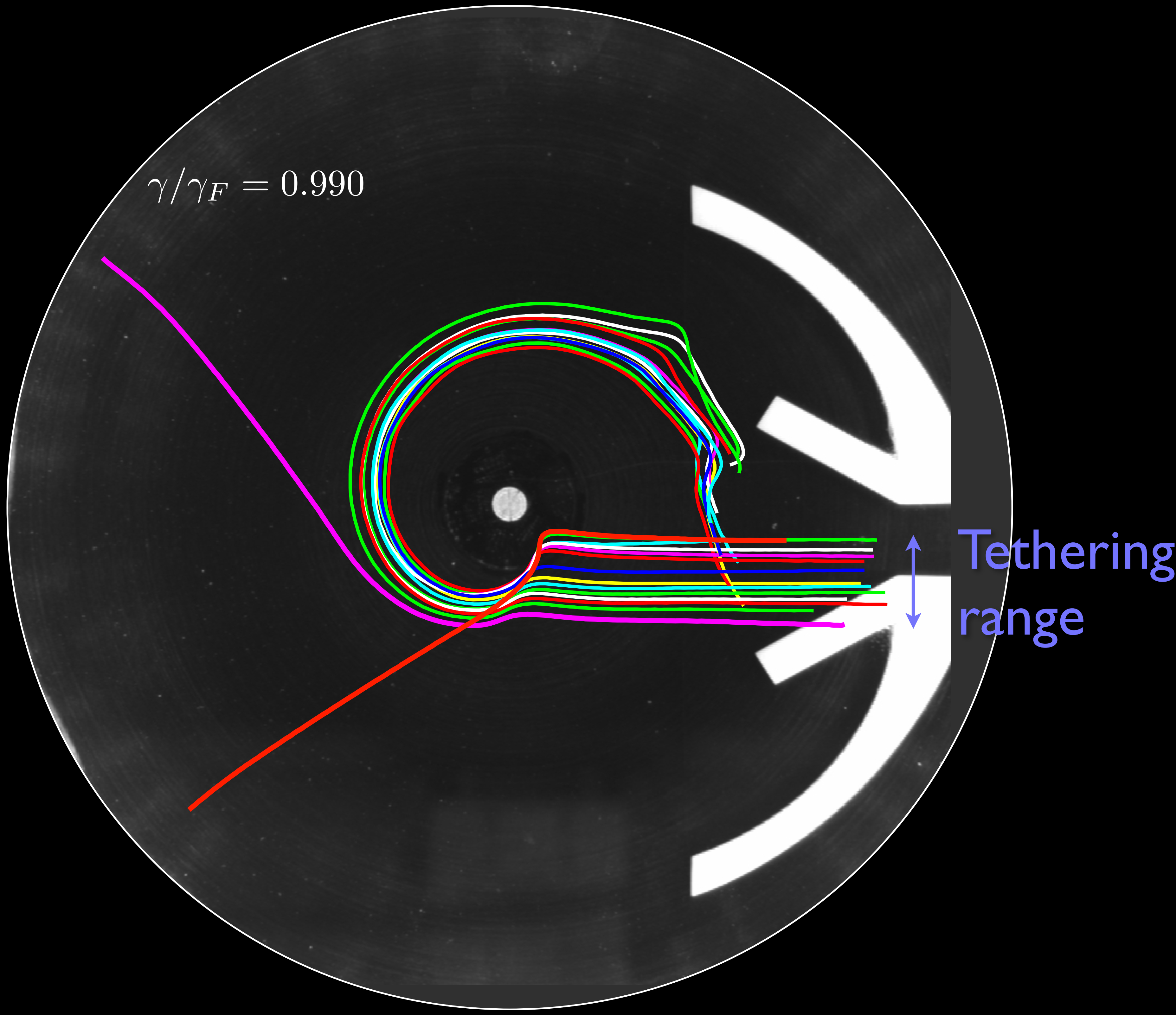
# Effect of impact parameter

# Effect of impact parameter




- preferred angle replaced by a preferred spiral

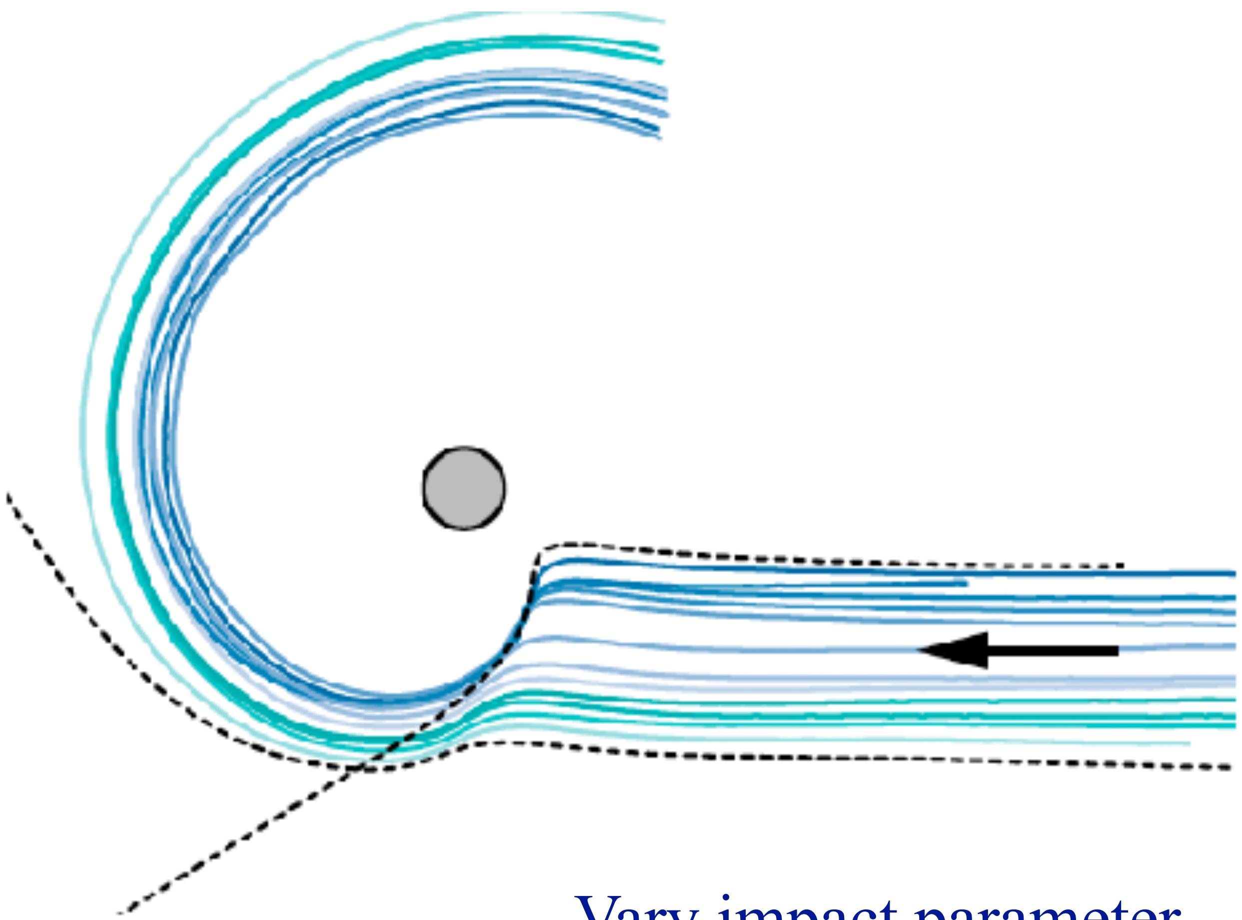
# Effect of impact parameter




- walker locks onto the same spiral for most impact parameters

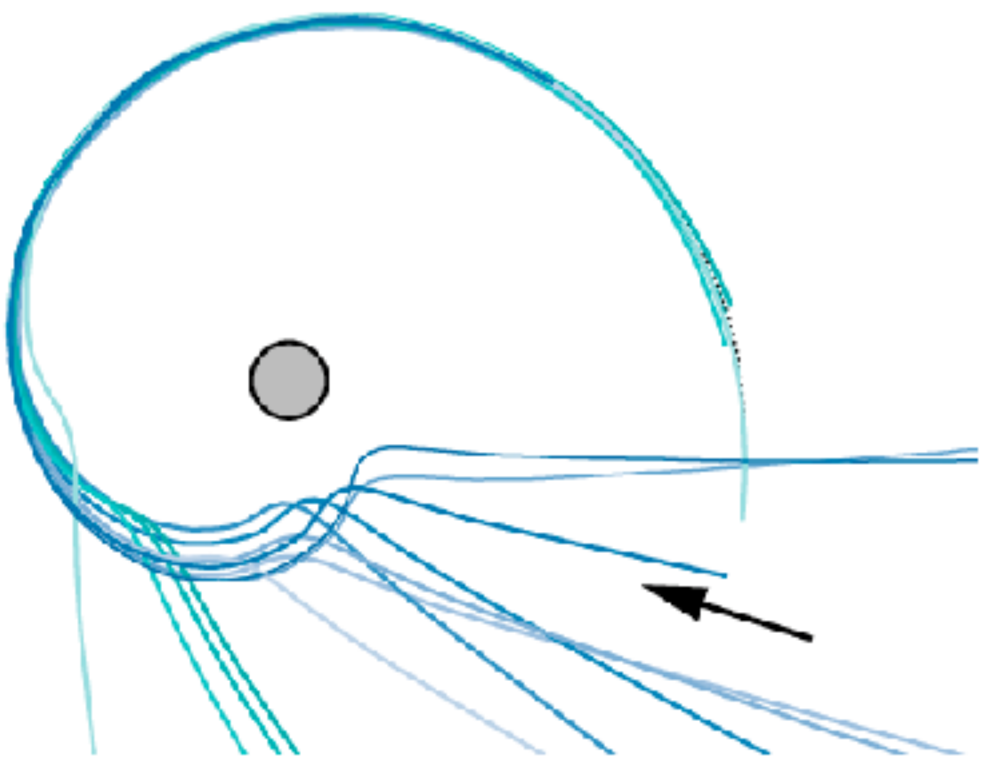
# Universal spiral for given pillar size, memory

**Align incoming trajectories**

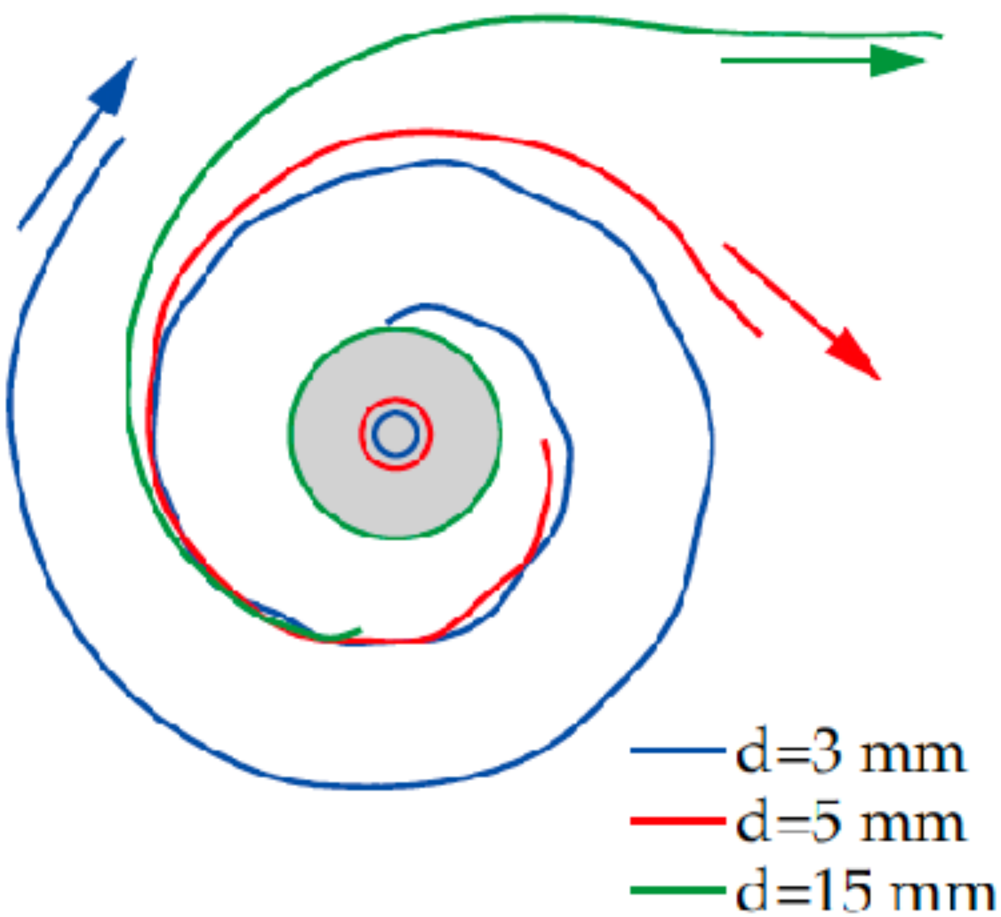


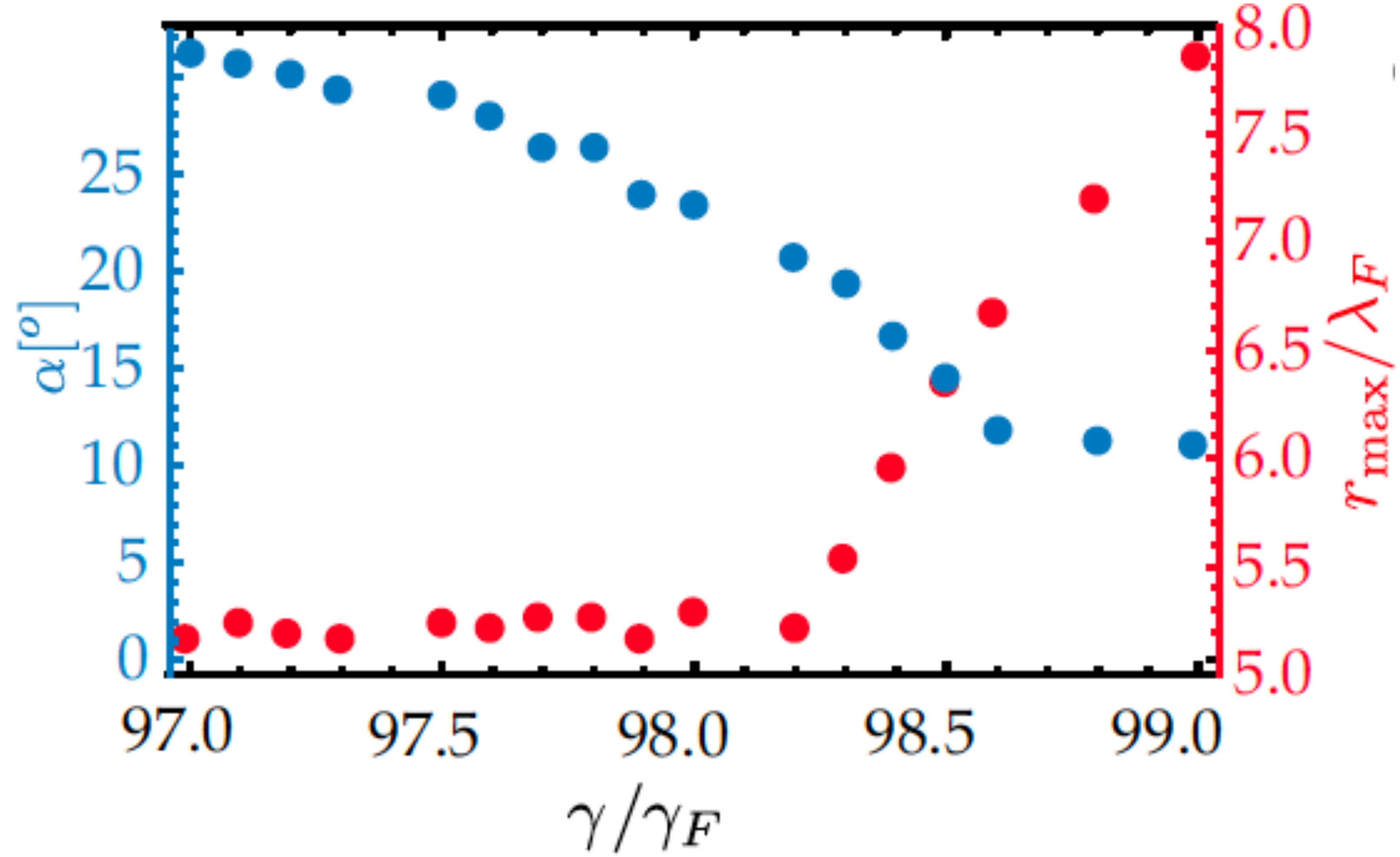
Vary impact parameter

**Twist**



# Summary: effect of pillar size and memory on spiral

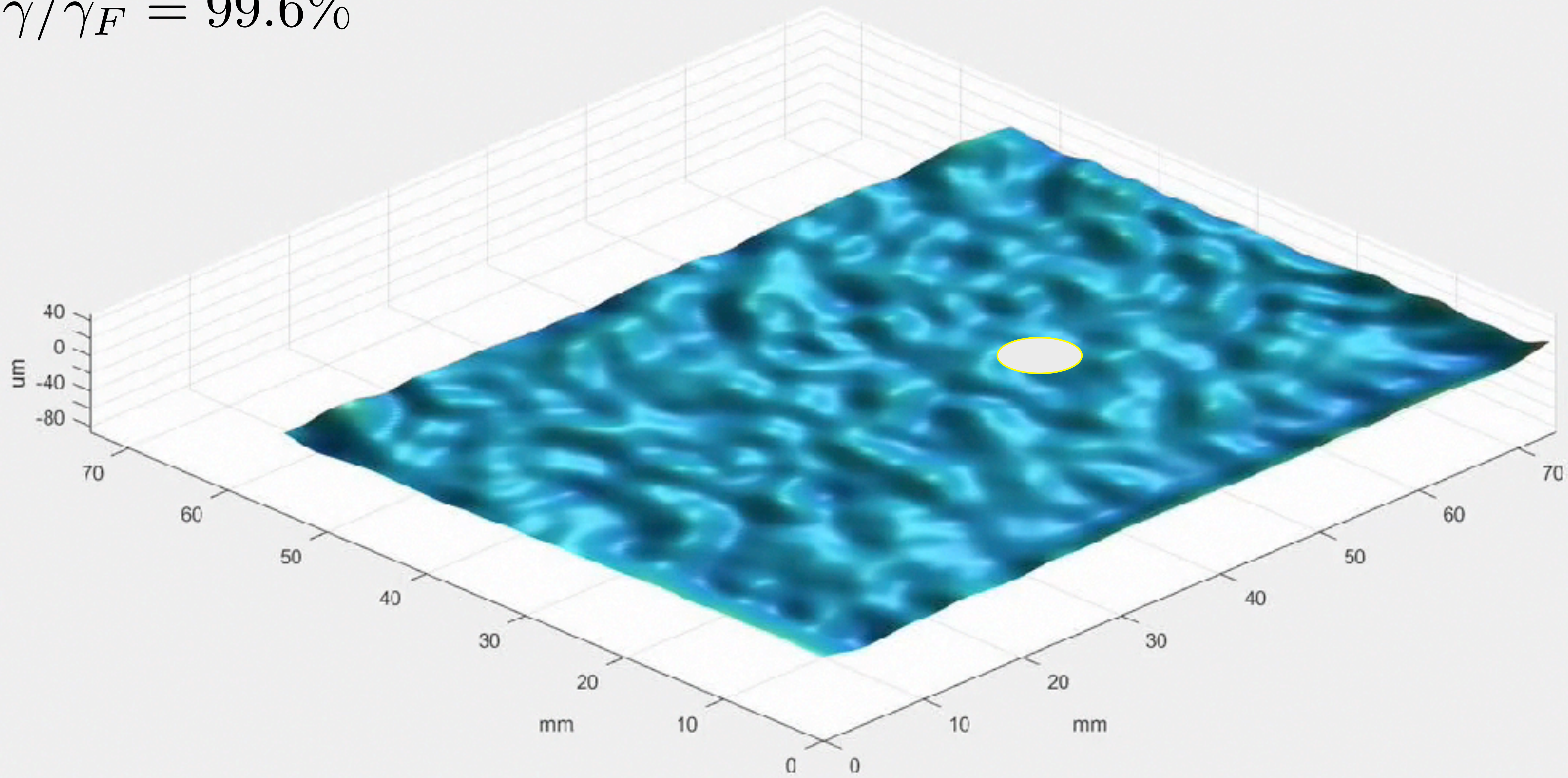


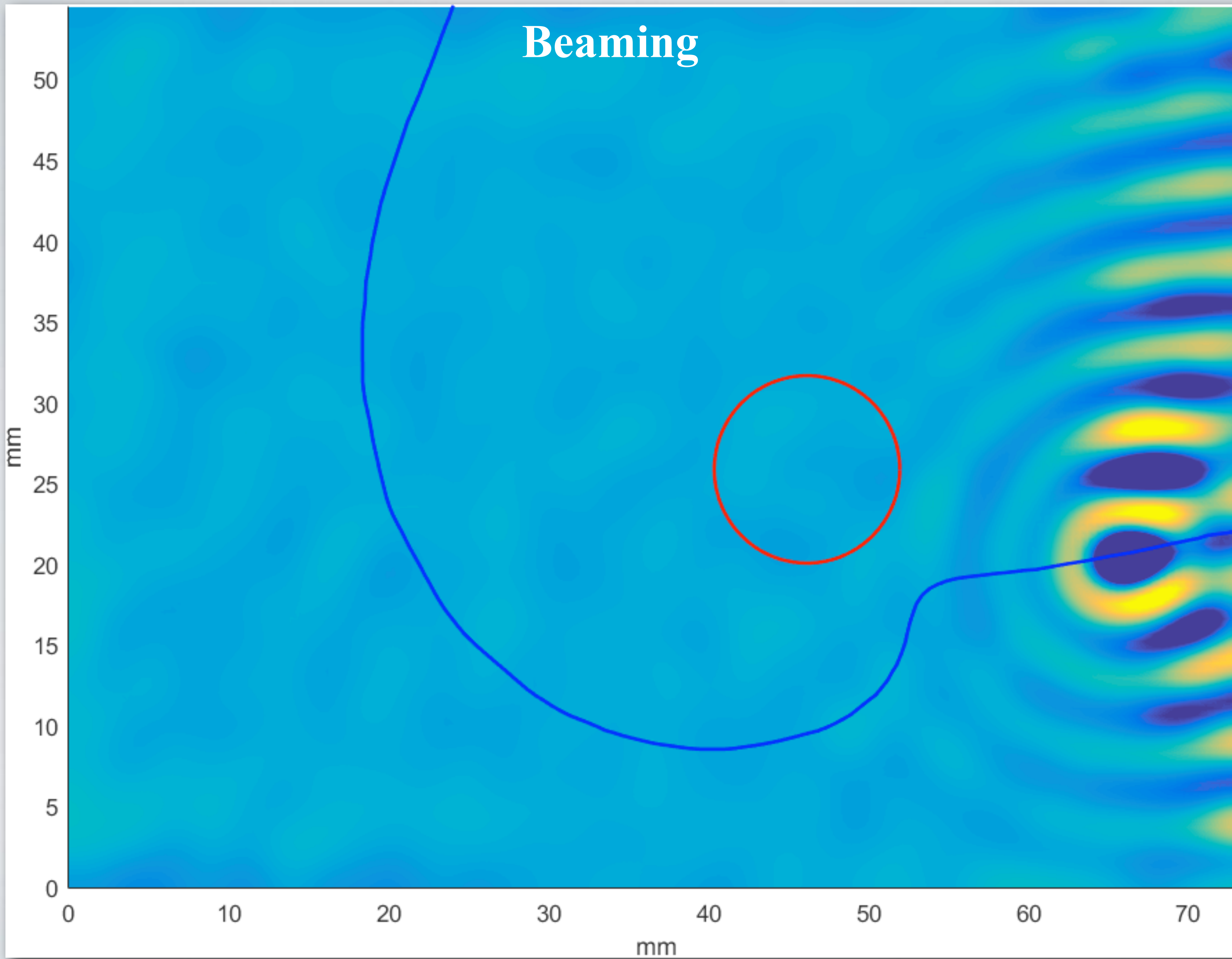

- pitch angle increased with pillar size: smallest pillars have tightest spirals
- pitch angle decreases with system memory: tightest spirals at high Me
- extent of spiral, tethering range increase with system memory

# Surface Schlieren imaging: a walker interacts with a pillar

$\gamma / \gamma_F = 99.6\%$




- pillar acts to locally suppress the walker-induced wave field


Beaming
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60
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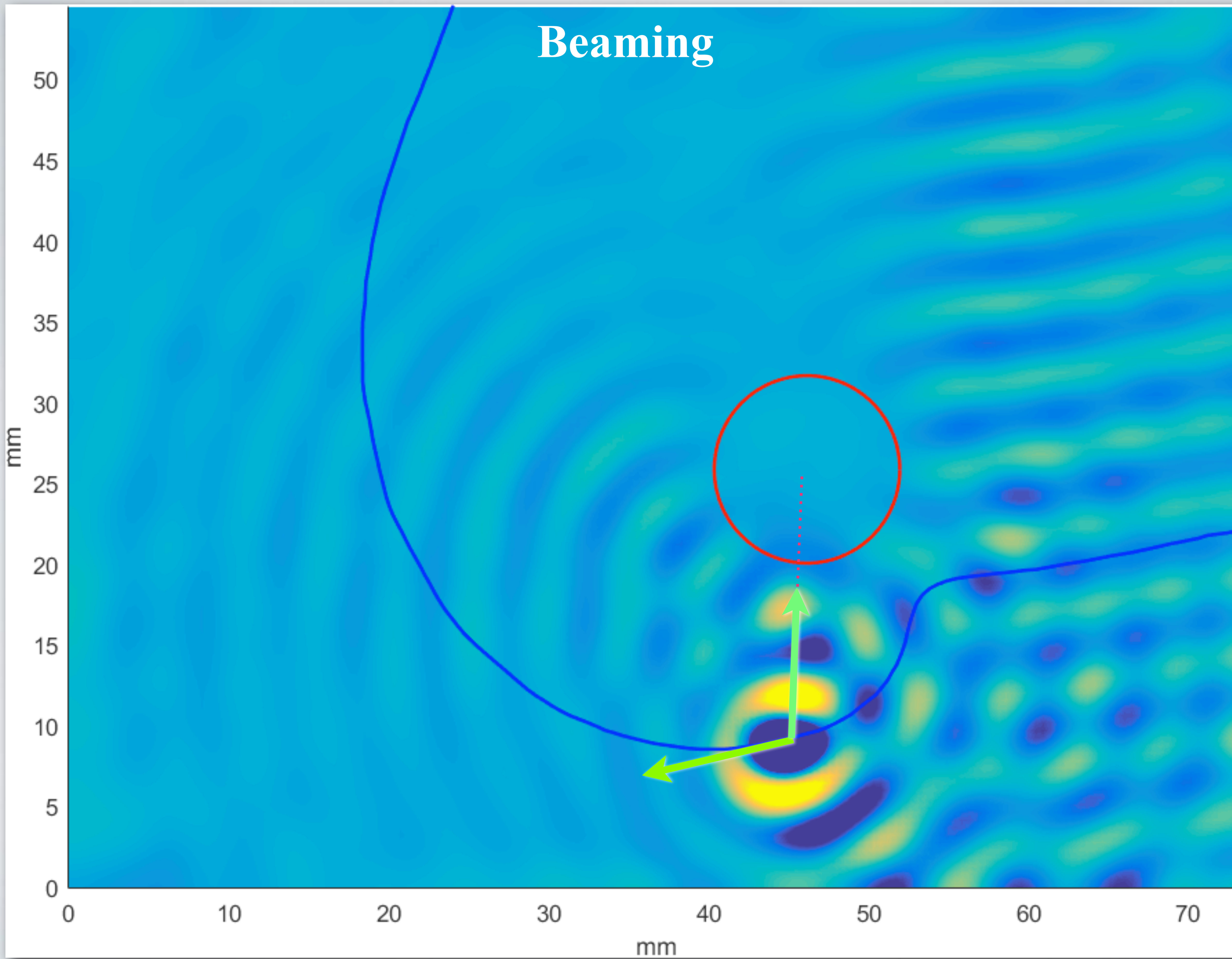
mm



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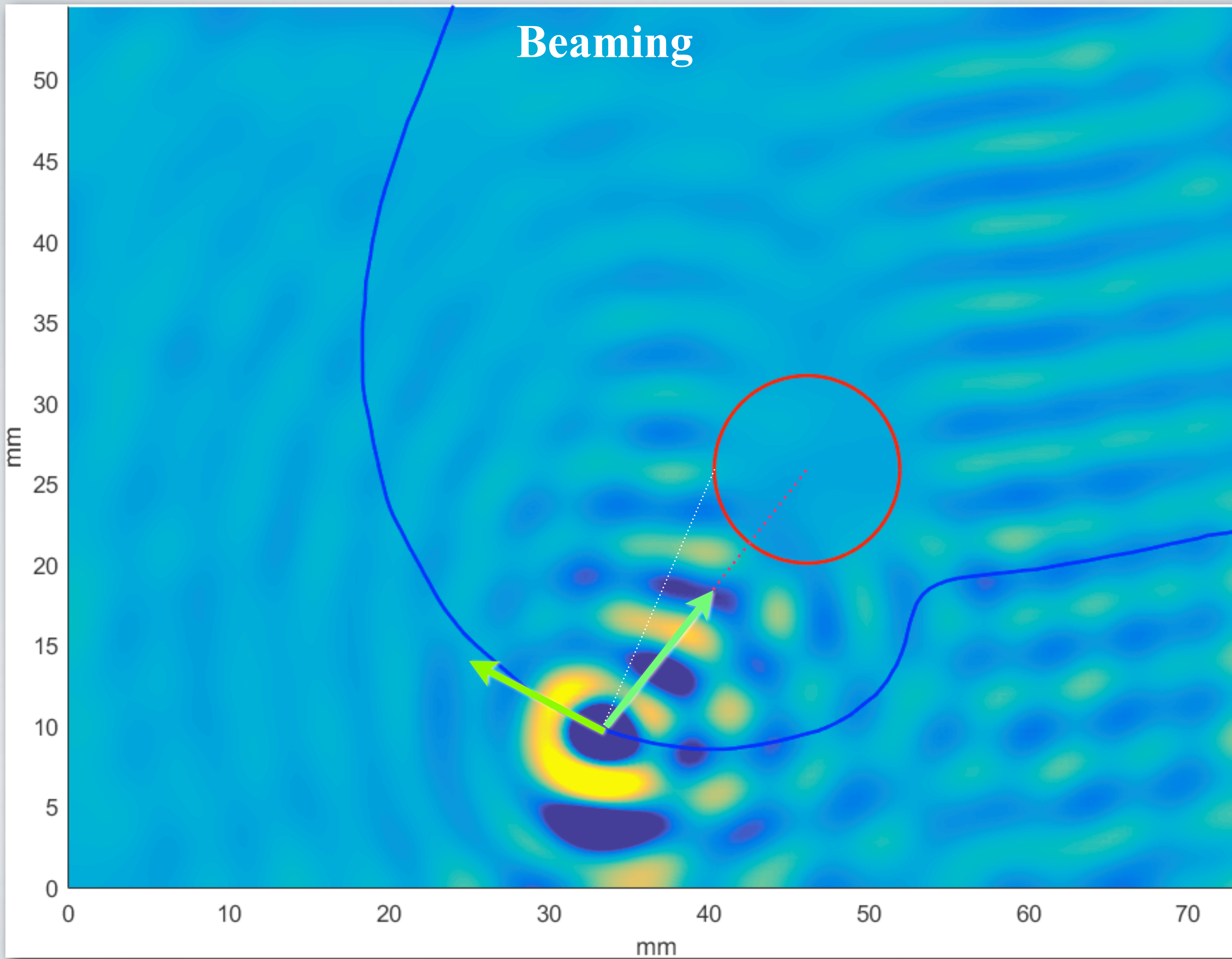
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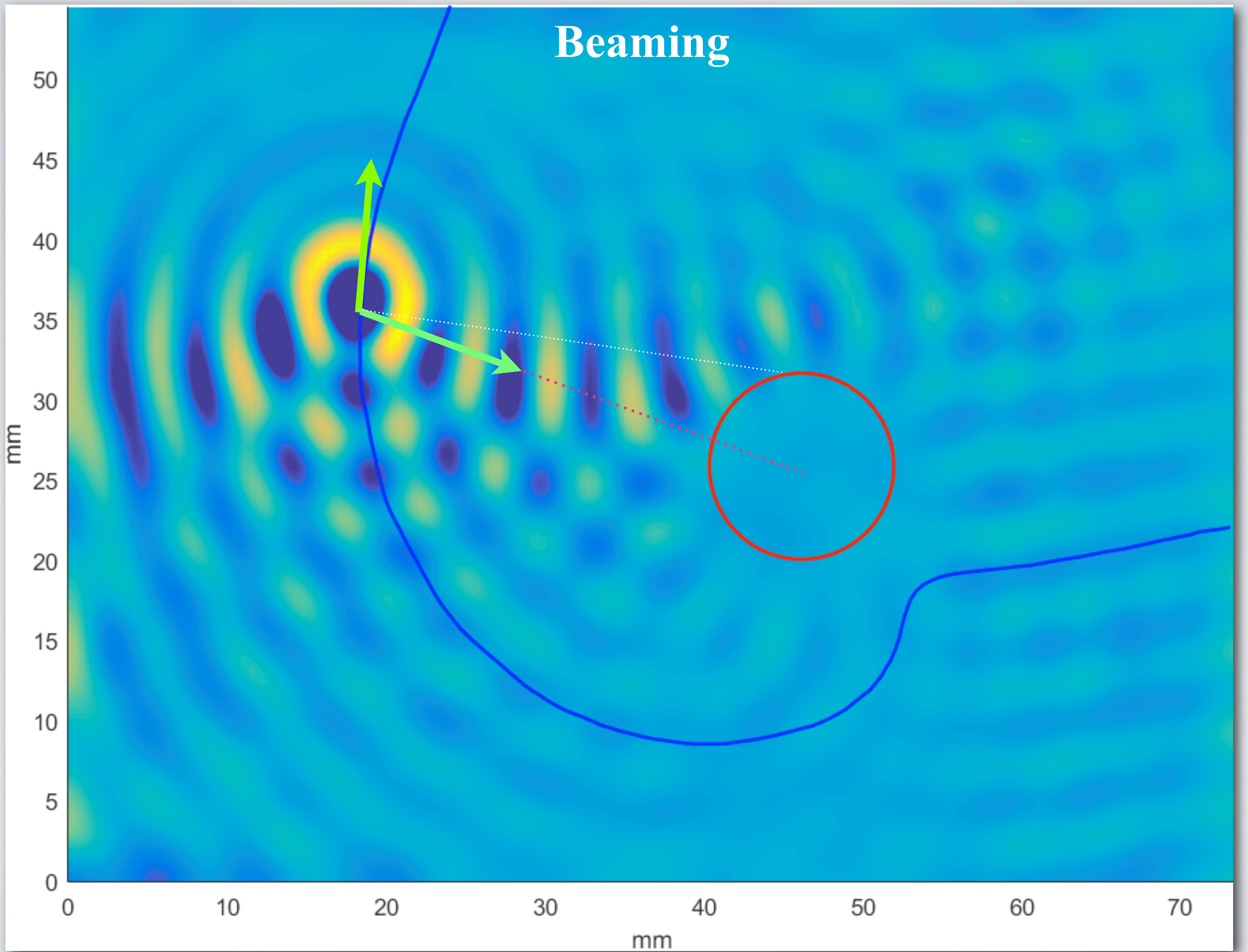
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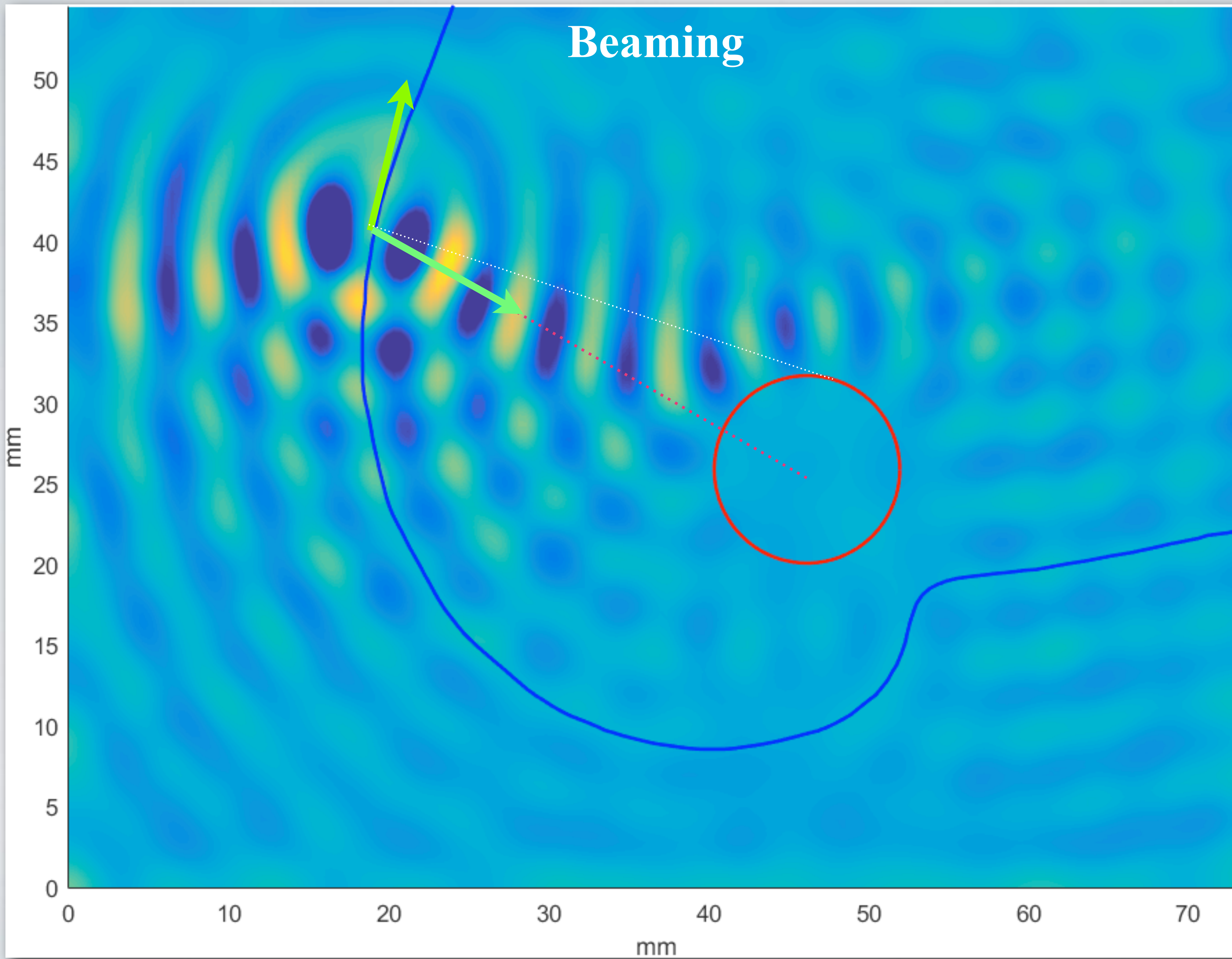

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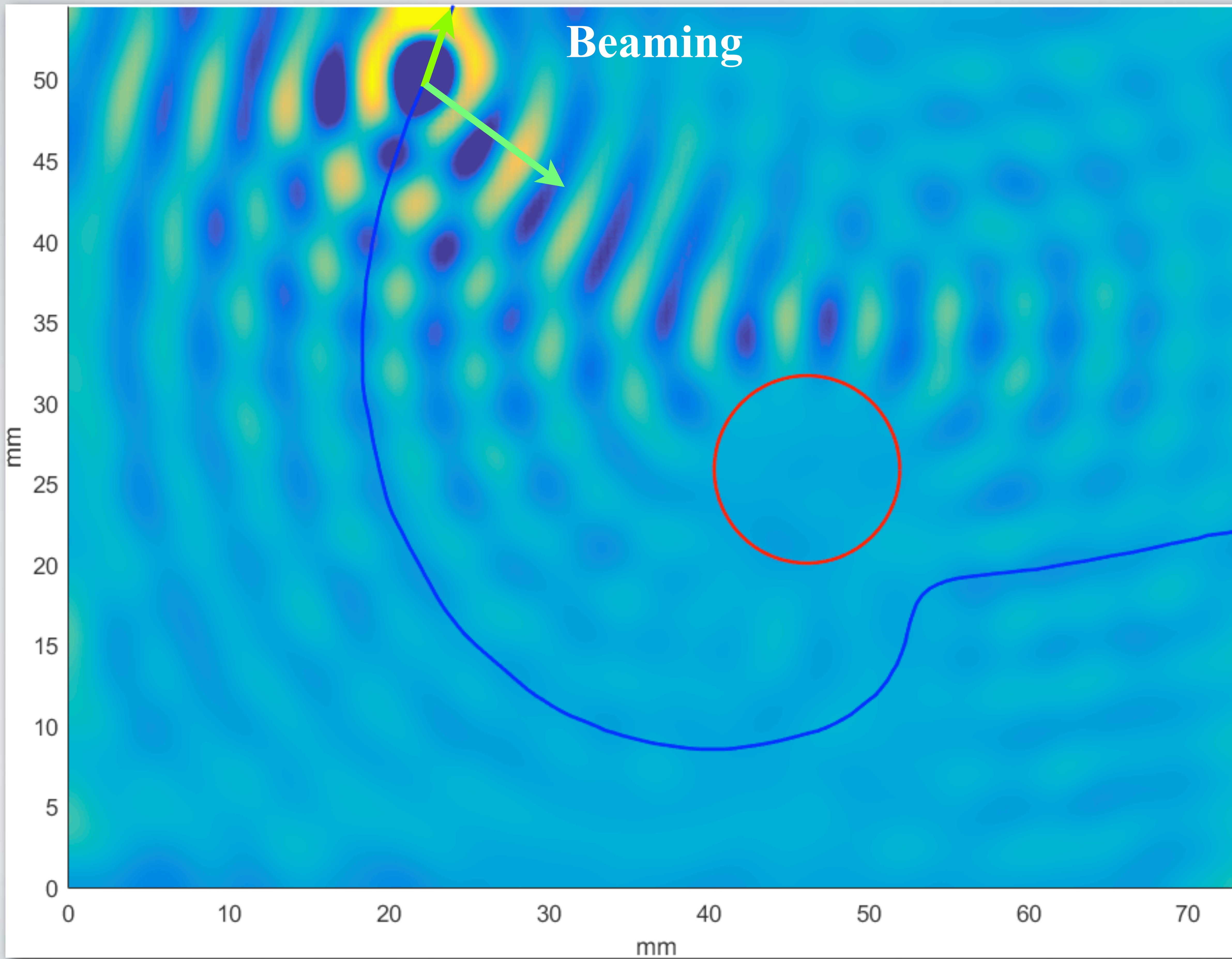

Beaming
mm
mm
0
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15
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45
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0
10
20
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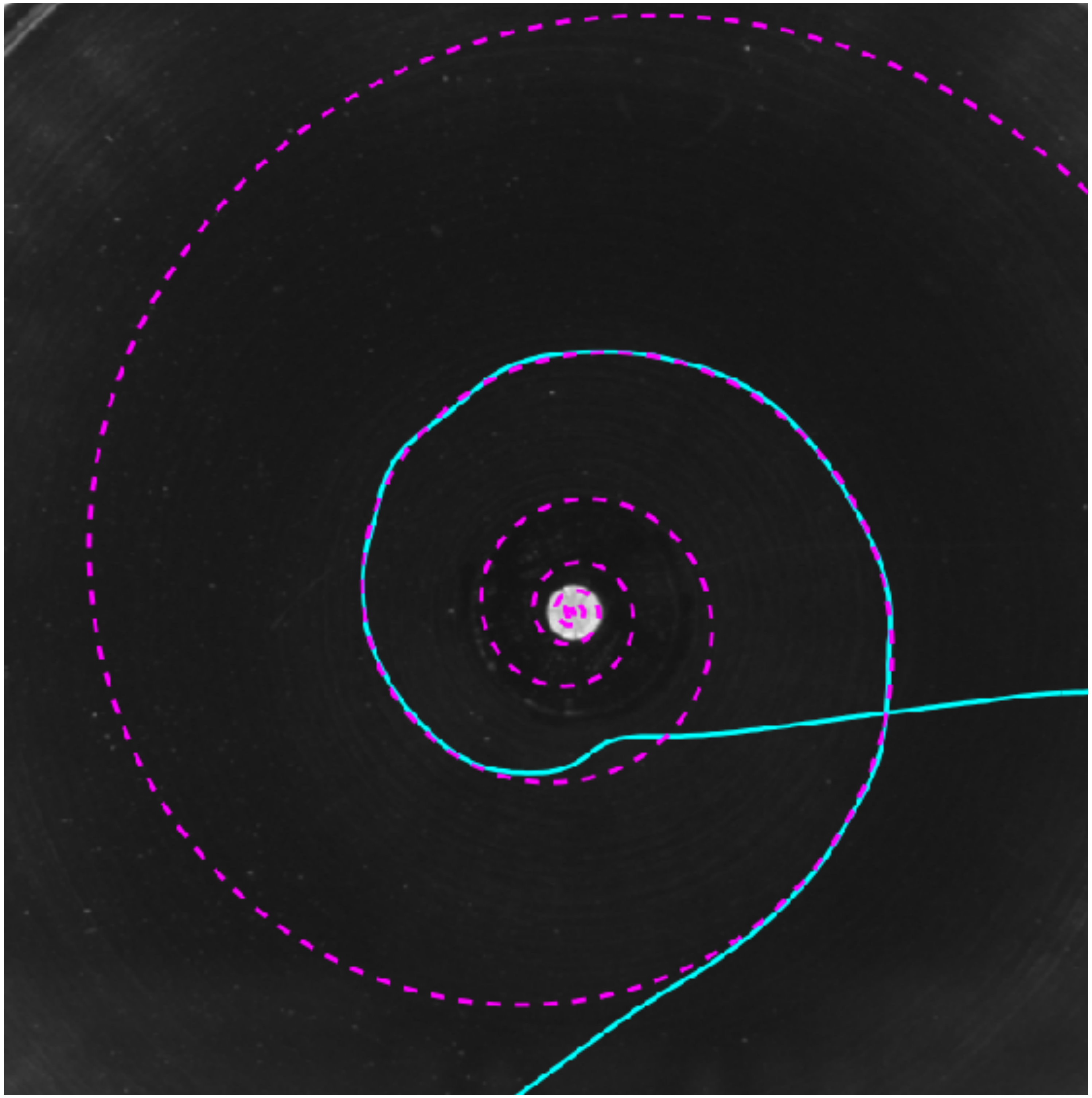
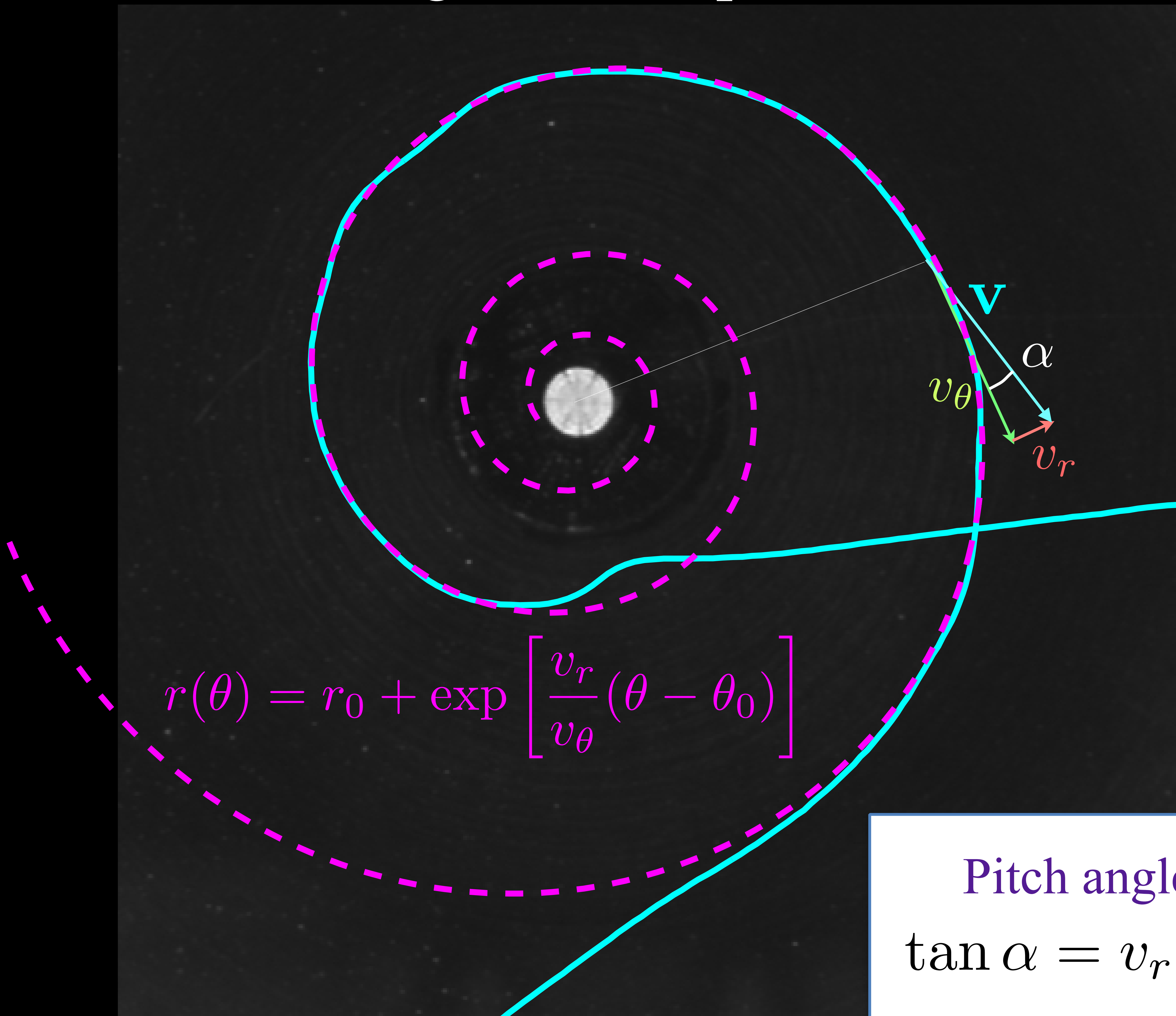
40
50
60
70

# What type of spiral?



- evident for all pillars above a critical size (R~ 0.5mm)

# Logarithmic spiral

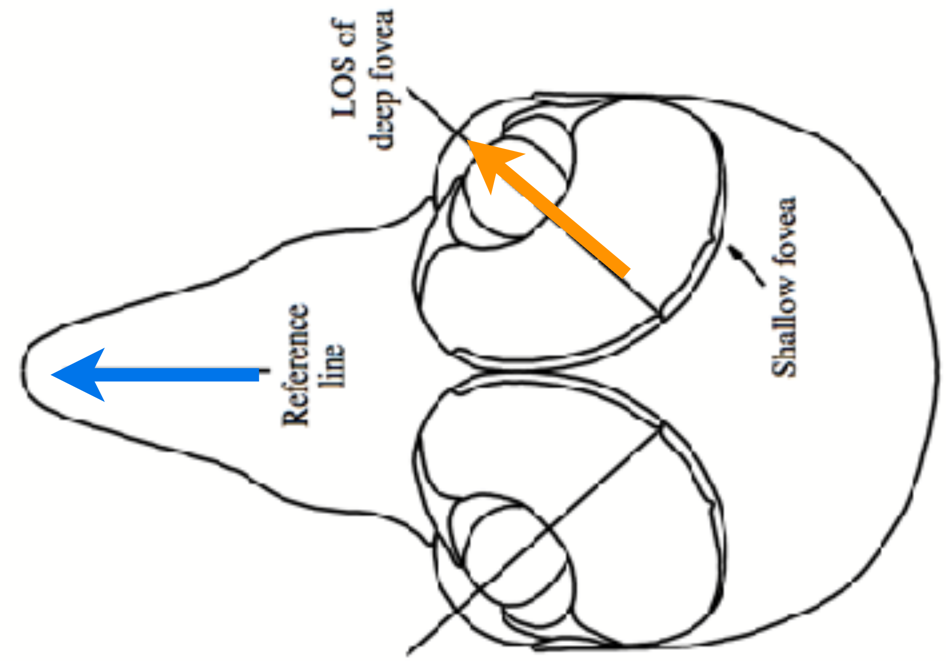



$$r(\theta) = r_0 + \exp\left[\frac{v_r}{v_\theta}(\theta - \theta_0)\right]$$

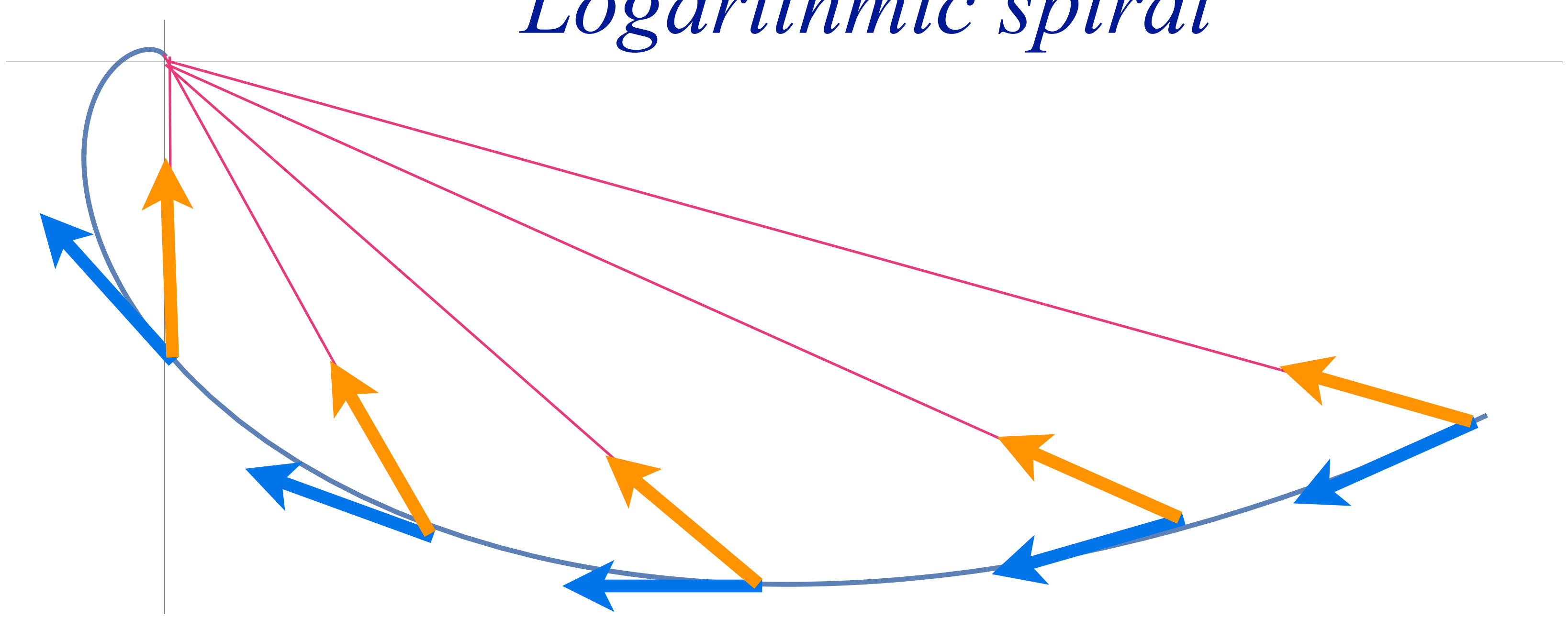
Pitch angle

$$\tan\alpha = v_r / v_\theta$$

# Spirals: the blueprint of Nature

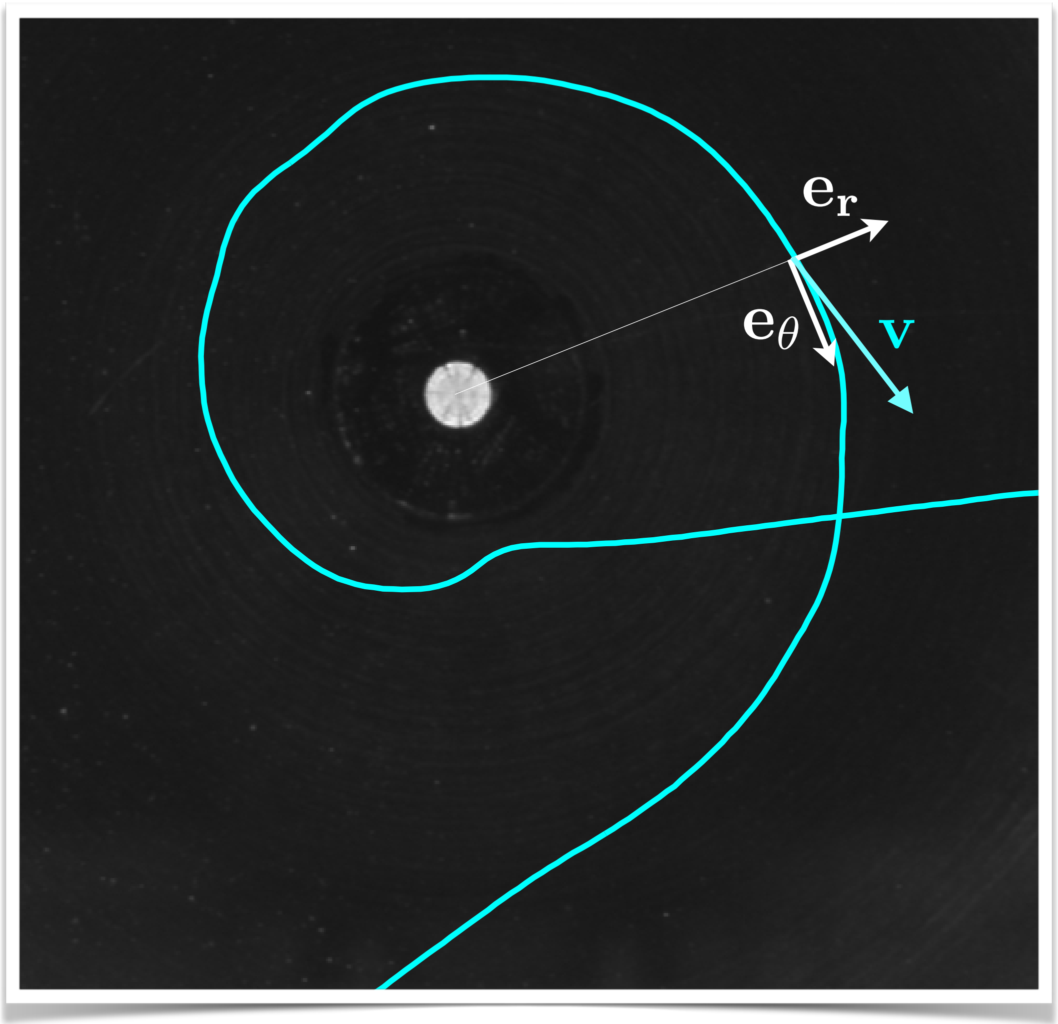


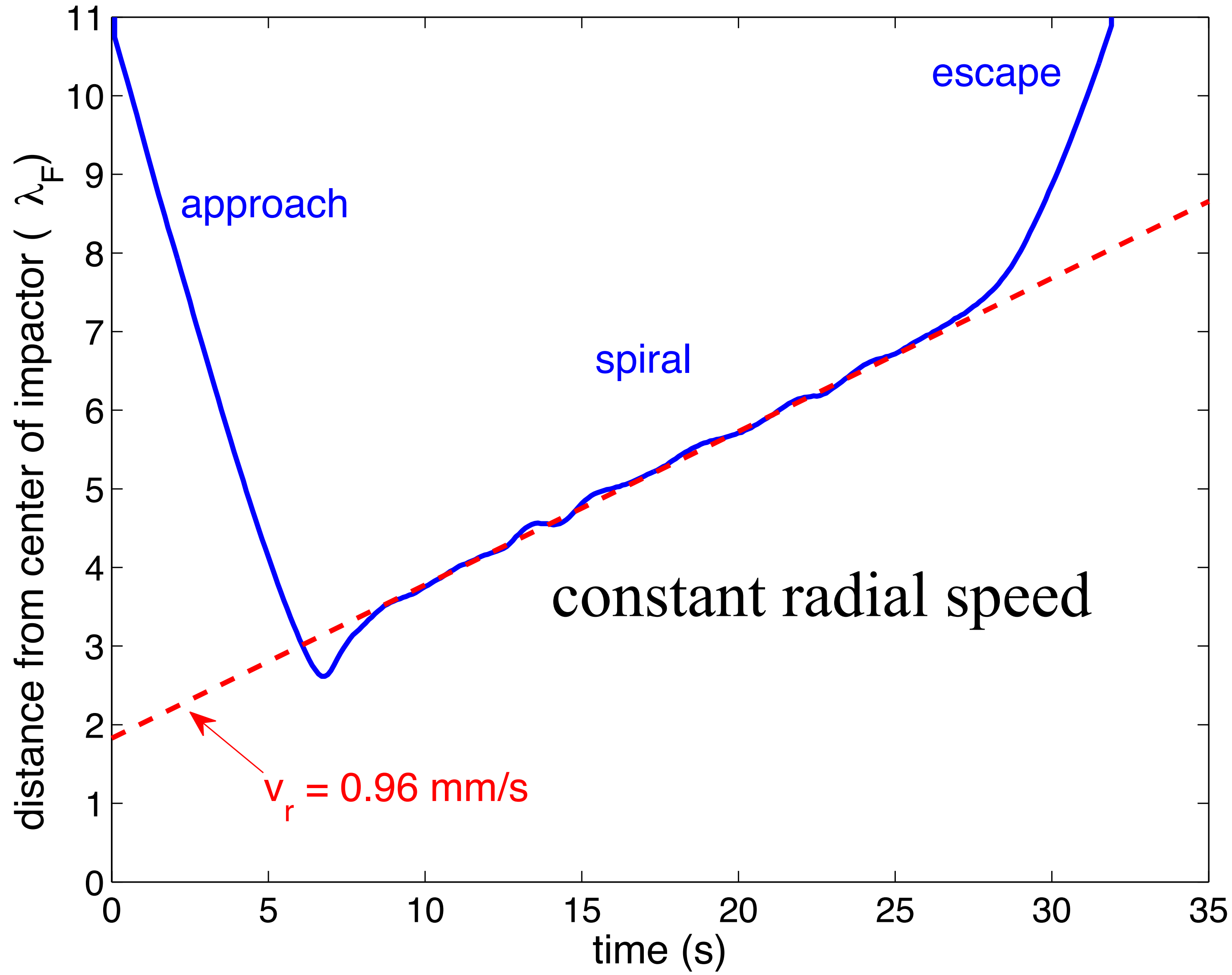

*Logarithmic spiral*



Tucker et al. Jour. Exp. Bio 2010

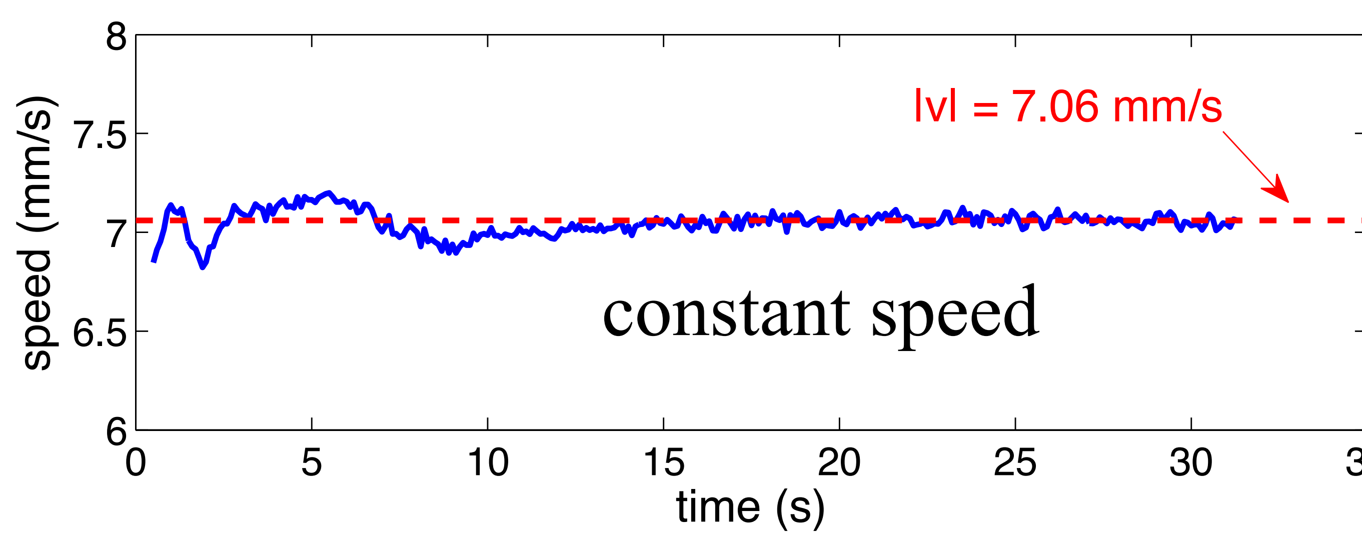
# The logarithmic spiral: kinematics



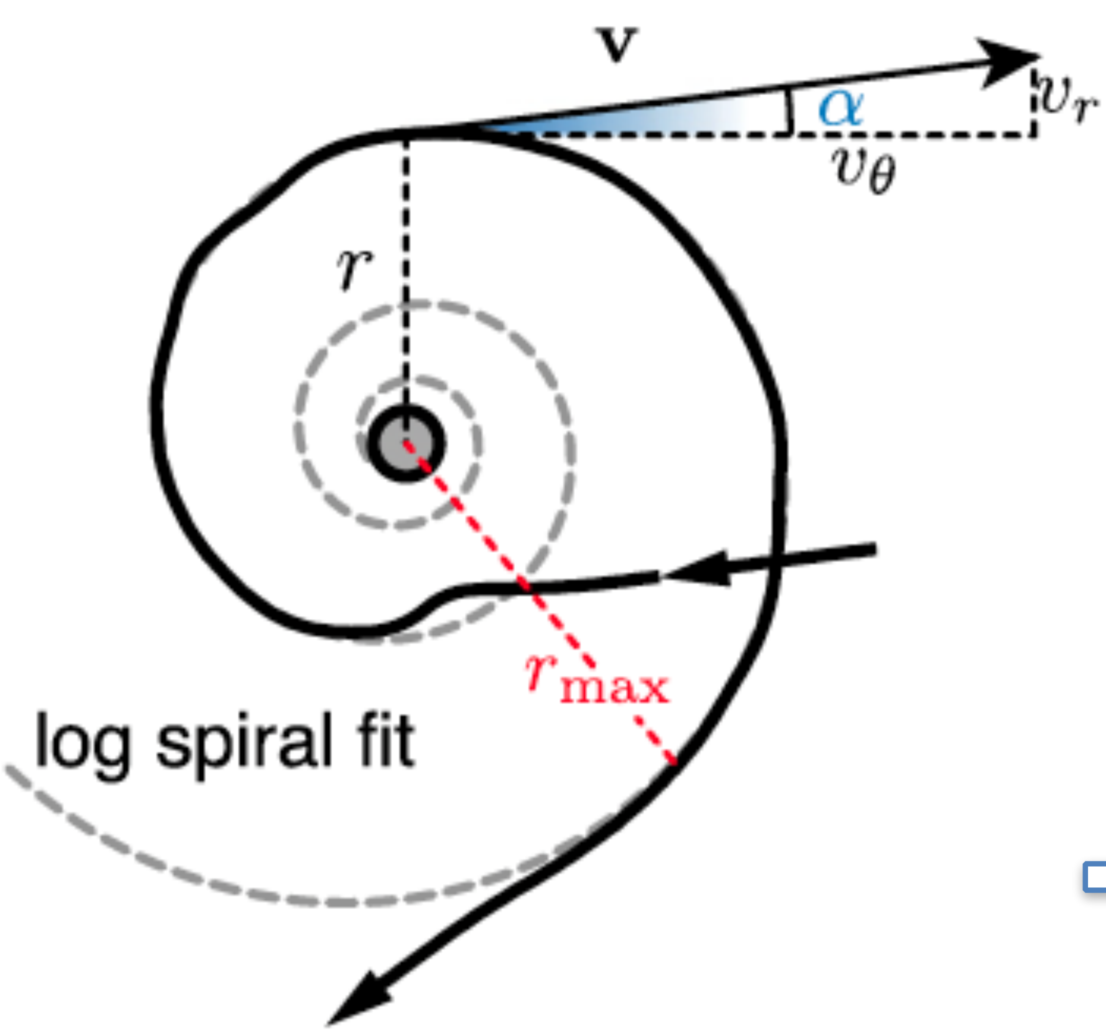

**Constant azimuthal speed**

$$v_\theta = \sqrt{\mathbf{v}^2 - v_r^2}$$




**walker maintains a constant speed**

# The logarithmic spiral




## Radial and azimuthal displacements:

$$r(t) = v_r t + r_o$$

$$\dot{\theta}(t) = \frac{v_\theta}{r} = \frac{v_\theta}{v_r t + r_o}$$

$$\Rightarrow \quad \theta(t) = \frac{v_\theta \ln(v_r t + r_o)}{v_r} + \theta_o$$

Rearrange and exponentiate:

$$v_r t + r_o = e^{\frac{v_r}{v_\theta}(\theta - \theta_o)}$$

**Parametric equation:**

$$\boxed{r(\theta) = r_{\min} e^{\frac{v_r}{v_\theta}(\theta - \theta_0)}}$$

# The Boost equation

Bush et al. (JFM 2014)

In the weak-acceleration limit, the trajectory equation takes the form

$$\boxed{\frac{d}{dt}\,\mathbf{p_w} + D_w\,\mathbf{v} = \mathbf{F}}$$

where the walker mass $m_w = \gamma_B(v)\; m_0$ , momentum $\mathbf{p_w} = m_w\mathbf{v}$

depend on the ***hydrodynamic boost factor***: $\gamma_B = 1 + \dfrac{\beta}{2\kappa(1+v^2)^{3/2}}$

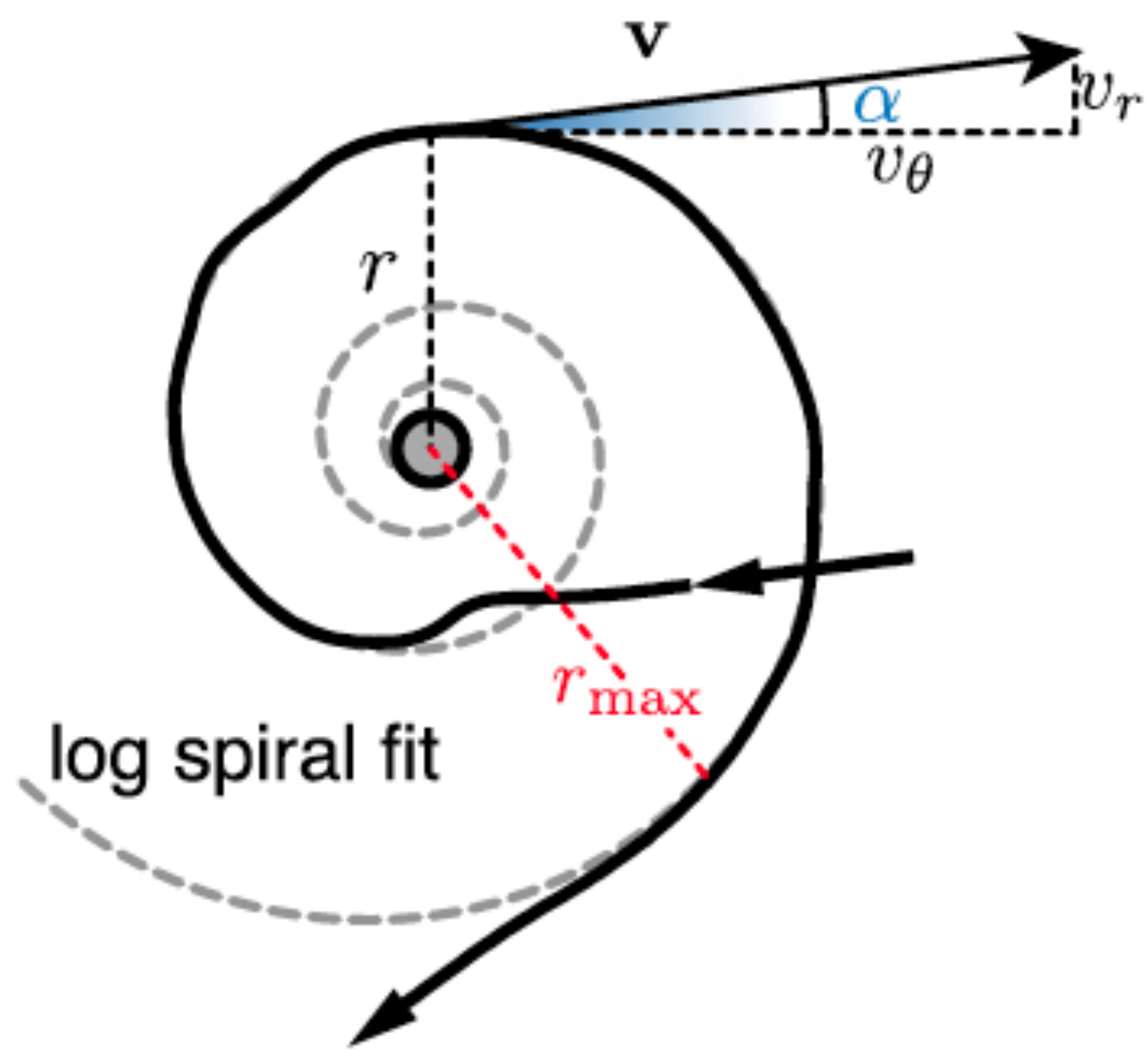
and a nonlinear drag $D_w = D_0\left(\dfrac{\mathbf{v}^2}{u_0^2} - 1\right)$ drives it to its free walking speed.

**For motion at the free walking speed:**

$$\boxed{\frac{d}{dt}\,\mathbf{p_w} = \mathbf{F}}$$

- the inviscid dynamics of a particle with a speed-dependent mass

## Force balance:




$$m\gamma_B\ddot{\mathbf{x}} = \mathbf{F}$$

$$\widehat{e}_r: \quad m\gamma_B(\ddot{r} - r\dot{\theta}^2) = F_r$$

$$\widehat{e}_\theta: \quad m\gamma_B(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F_\theta$$

Along the logarithmic spiral

$$r(t) = v_r t + r_o, \quad \dot{r}(t) = v_r, \quad \ddot{r}(t) = 0.$$

$$\theta(t) = \frac{v_\theta \ln(v_r t + r_o)}{v_r} + \theta_o, \quad \dot{\theta}(t) = \frac{v_\theta}{v_r t + r_o}, \quad \ddot{\theta}(t) = -\frac{v_\theta v_r}{(v_r t + r_o)^2}$$

## Inferred force

$$\Rightarrow \quad \mathbf{F} = \frac{m\gamma_B}{r}\left(-v_\theta^2\widehat{e}_r + v_\theta v_r \widehat{e}_\theta\right) = \frac{m\gamma_B v_\theta}{r}\left(\widehat{e}_z \times \mathbf{v}\right)$$

# Use Boost equation to infer pillar-induced force

$$\frac{d}{dt}\,\mathbf{p_w} = \mathbf{F_p}$$

**Force required for a logarithmic spiral:** $\boxed{\mathbf{F}_p = 2\pi\,\gamma_B m\,\mathbf{v}\times\boldsymbol{\Omega}}$

where $\boldsymbol{\Omega} = \dfrac{v_\theta}{2\pi r}\,\hat{\mathbf{k}}$ is the walker's instantaneous angular velocity

- identical forms of Coriolis force acting on a mass $\mathbf{F_C} = 2m(\mathbf{v}\wedge\boldsymbol{\Omega})$ and the Lorentz force acting on a charge $\mathbf{F_B} = q(\mathbf{v}\wedge\mathbf{B})$ was the basis for the analogy between inertial orbits and Landau levels *(Fort et al. 2010)*

- here, it indicates that the walker is analogous to a charge moving in the magnetic field associated with its own motion

→ **hydrodynamic self-induction**

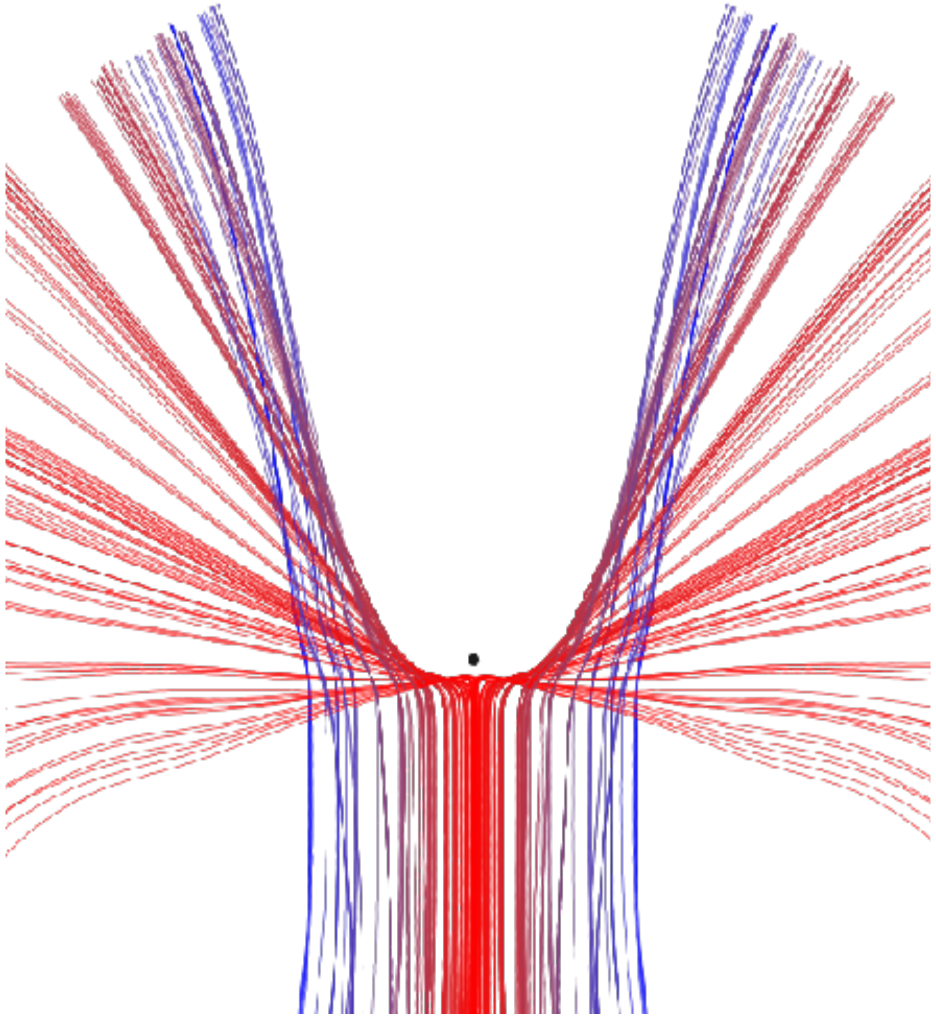
# Gravitoelectromagnetism

- in limit of weak spacetime curvature (weak gravitational fields)

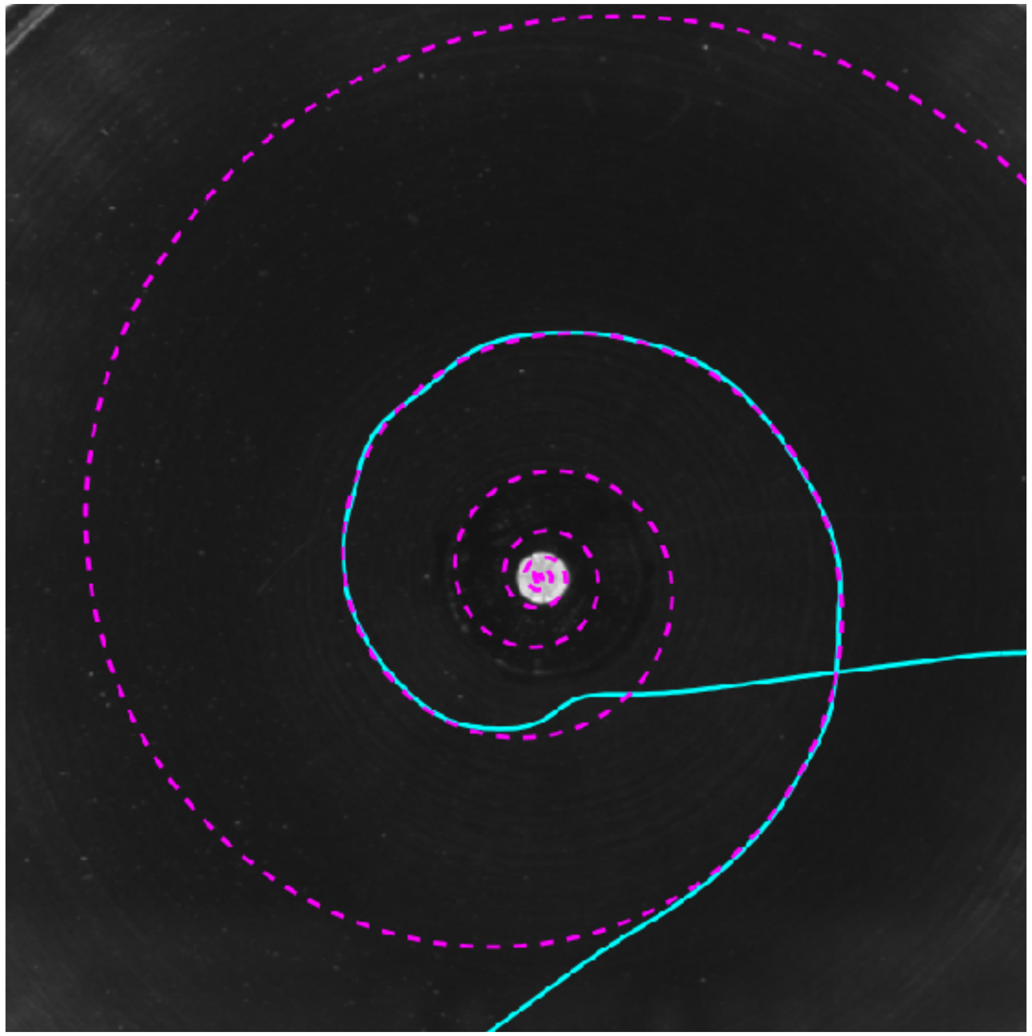
| GEM equations | Maxwell's equations |
|---|---|
| $\nabla \cdot \mathbf{E}_g = -4\pi G \rho_g$ | $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$ |
| $\nabla \cdot \mathbf{B}_g = 0$ | $\nabla \cdot \mathbf{B} = 0$ |
| $\nabla \times \mathbf{E}_g = -\dfrac{\partial \mathbf{B}_g}{\partial t}$ | $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ |
| $\nabla \times \mathbf{B}_g = -\dfrac{4\pi G}{c^2}\mathbf{J}_g + \dfrac{1}{c^2}\dfrac{\partial \mathbf{E}_g}{\partial t}$ | $\nabla \times \mathbf{B} = \dfrac{1}{\epsilon_0 c^2}\mathbf{J} + \dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t}$ |

## Lorentz force

| GEM equation | EM equation |
|---|---|
| $\mathbf{F}_g = m\left(\mathbf{E}_g + \mathbf{v} \times 4\mathbf{B}_g\right)$ | $\mathbf{F}_e = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$ |



# Conclusions



- walkers scatter from small submerged pillars as if from a repulsive force
- walkers lock into a logarithmic spiral when they encounter larger pillars

➡ EM analogy indicates ``***hydrodynamic self-induction***''

➡ gravitoEM indicates analogy with gravitational self-induction

- motivates models capable of accounting for boundary interactions