18.S996 Hydrodynamic quantum analogs

Lecture 12: The stroboscopic model

The stroboscopic model

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A trajectory equation for walking droplets: hydrodynamic pilot-wave theory

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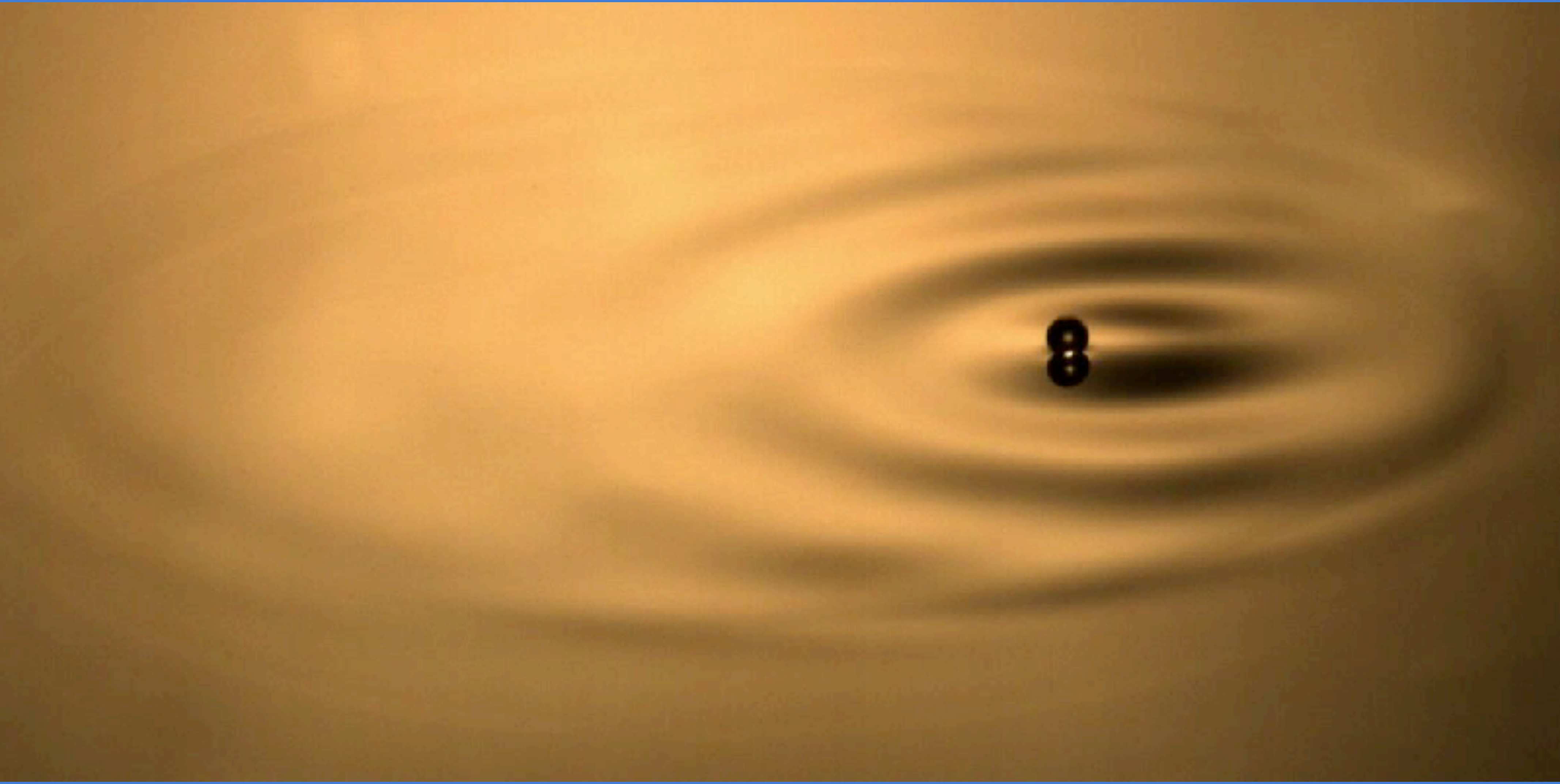
A droplet walking in a circular corral



- the droplet generates and explores its wave field

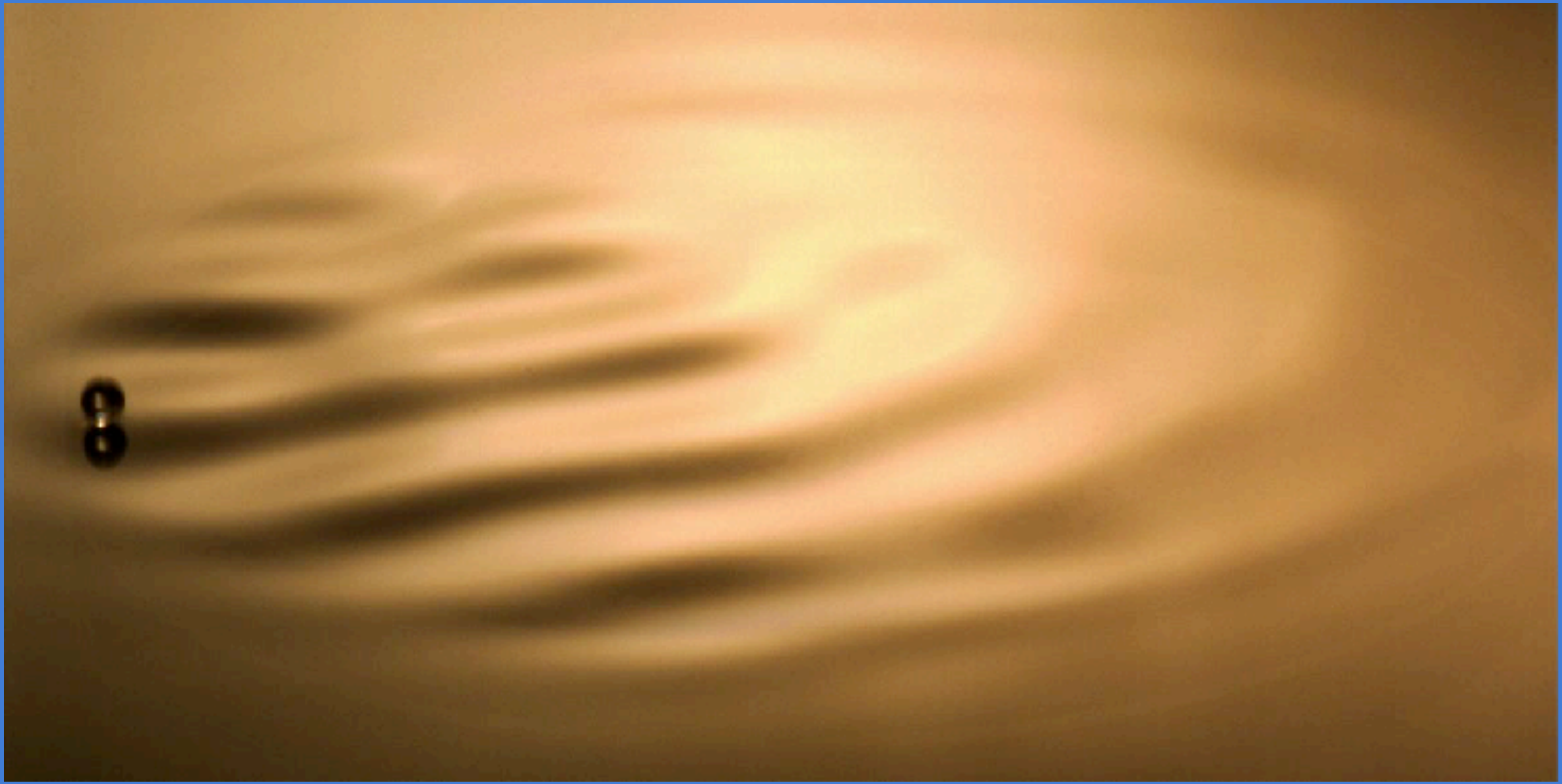
A droplet hovering in a circular corral

- strobe at the wave (and bouncing) frequency, 50 Hz
- fast bouncing dynamics filtered out

- drop appears to glide above the surface
- drop accompanied by a monochromatic wave field

A droplet surfing in a circular corral

- strobe at the wave (and bouncing) frequency, 50 Hz
- fast bouncing dynamics filtered out



- drop appears to surf along the surface

The walker with vertical dynamics resolved

- *resonance condition*: drop bounces at Faraday frequency
- *resonance* allows for a drastic simplification in the modeling

→ **average out the vertical dynamics**

Trajectory equation

$$m\ddot{\mathbf{x}}_p + D\dot{\mathbf{x}}_p = -mg\nabla h(\mathbf{x}_p, t)$$

Drag coefficient: $D = 6\pi\mu_a R + Cmg \cdot \sqrt{\frac{\rho R}{\sigma}}$

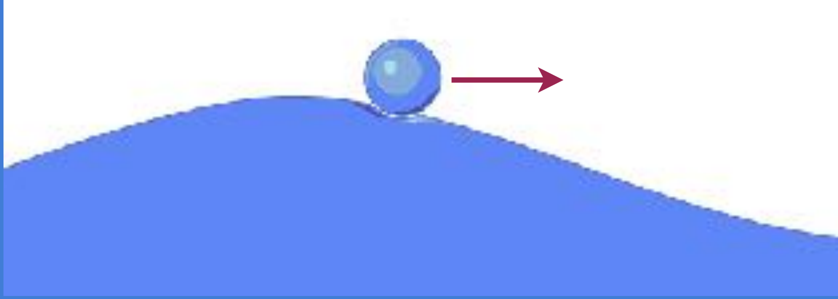
Wave field: $h(\mathbf{x}, t) = A \sum_{k=-\infty}^{\lfloor t/T_F \rfloor} J_0(k_F |\mathbf{x} - \mathbf{x}_p(kT_F)|) e^{-(t-kT_F)/(T_F M_e)}$

Wave amplitude: $A = \frac{4\sqrt{2\pi}}{3} \frac{R^4 k_F^3 \mathbb{O}h_e^{1/2}}{3R^2 k_F^2 + \mathbb{B}o} \cdot \frac{\mathbb{B}o T_F}{\sqrt{\rho R^3/\sigma}} \sin \Phi_I$

Memory parameter: $M_e = \frac{T_d}{T_F(1 - \gamma/\gamma_F)}$ Impact phase: Φ_I

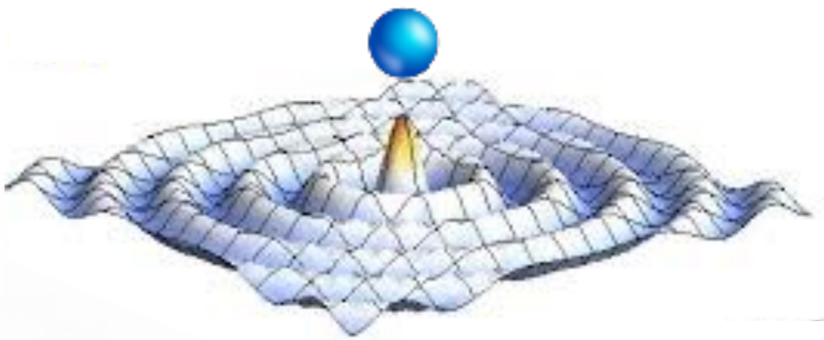
Bond number: $\mathbb{B}o = \frac{\rho g R^2}{\sigma}$ Ohnesorge number: $\mathbb{O}h_e \approx \frac{\mu}{\sqrt{\rho \sigma R}}$

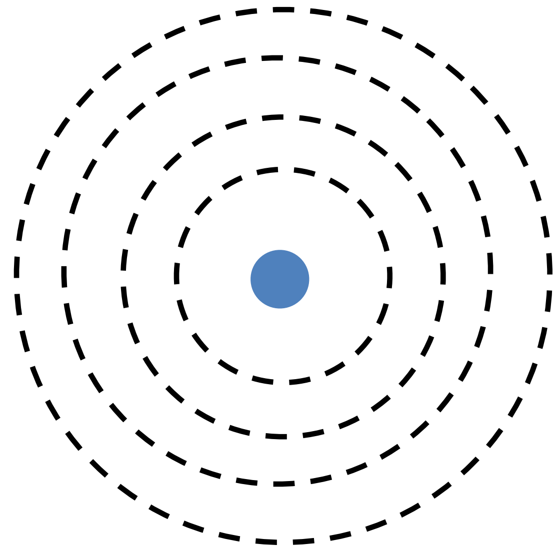
Strobed pilot-wave dynamics



- strobe the system once per bounce cycle
- conceals the vertical dynamics responsible for the guiding wave
- drop appears to surf on the interface, dressed by a quasi-monochromatic pilot-wave field that is stationary in the drop's frame of reference

Standing waves generated by the walker





Wavelength λ_F
Decay time $T_F M_e$

$$h(\mathbf{x},t) = A J_0\left(k_F\left|\mathbf{x}-\mathbf{x}_p\right|\right)\mathrm{e}^{-(t-t_p)/T_F M_e}\cos\left[\frac{\omega_0(t-t_p)}{2}\right]$$

Memory parameter

$$M_e = \frac{T_d}{T_F\left(1-\gamma/\gamma_F\right)}$$

$\boldsymbol{x}_p$: drop position
A: amplitude of single wave
k_F: Faraday wavenumber
J_0: Bessel function of first kind

γ: forcing acceleration
γ_F : Faraday threshold
T_F : bouncing period
T_d: decay time of surface waves

The stroboscopic model

Oza, Rosales & Bush (2013)

$$m\ddot{\mathbf{x}}_p + D\dot{\mathbf{x}}_p = -mg\nabla h(\mathbf{x}_p, t)$$

MEMORY TERM

Approximate discrete sum as integral:

$$\nabla h(\mathbf{x},t) = -Ak_F \int_{-\infty}^{t} \frac{J_1(k_F\,|\mathbf{x}-\mathbf{x}_p(s)|)}{|\mathbf{x}-\mathbf{x}_p(s)|}(\mathbf{x}-\mathbf{x}_p(s))e^{-(t-s)/(T_F M_e)}\,ds$$

Valid for high-frequency bouncing:

$$T_F \ll \lambda_F/\,|\dot{\mathbf{x}}_p|$$

$$F = mgAk_F$$

$$m\ddot{\mathbf{x}}_p + D\dot{\mathbf{x}}_p = \frac{F}{T_F}\int_{-\infty}^{t} \frac{J_1(k_F\,|\mathbf{x}_p(t)-\mathbf{x}_p(s)|)}{|\mathbf{x}_p(t)-\mathbf{x}_p(s)|}\,(\mathbf{x}_p(t)-\mathbf{x}_p(s))\,e^{-(t-s)/(T_F M_e)}\,ds$$

MEMORY TERM

- integral-differential equation describes horizontal motion in the strobed frame

Walking states

MEMORY FORCE

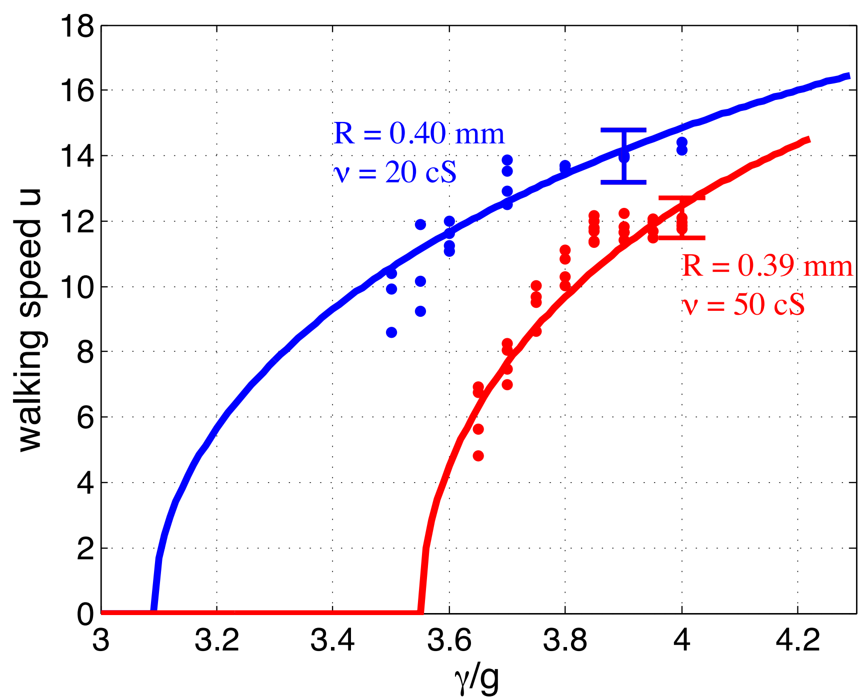
$$m\ddot{x}_p + D\dot{x}_p = \frac{F}{T_F}\int_{-\infty}^{t} J_1(k_F(x_p(t) - x_p(s)))e^{-(t-s)/(T_F M_e)}\, ds$$

Seek steady walking solution : $x_p = u\ t$

$$u = \frac{1}{k_F T_d}\left(1 - \frac{\gamma}{\gamma_F}\right)\left\{\frac{1}{4}\left[-1 + \sqrt{1 + 8\left(\frac{\gamma_F - \gamma_W}{\gamma_F - \gamma}\right)^2}\right]^2 - 1\right\}^{1/2}$$

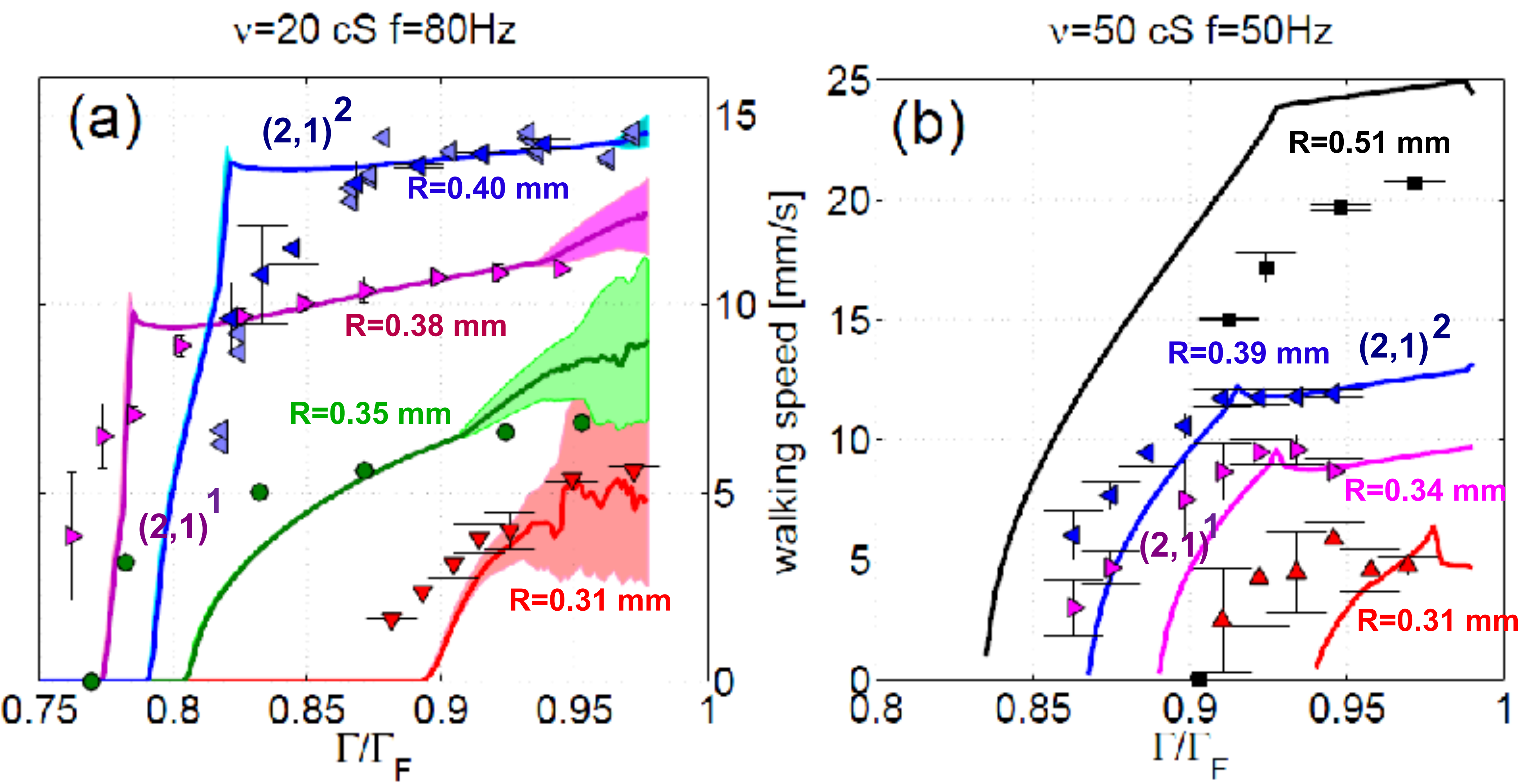
Walking threshold :

$$\gamma_W = \gamma_F\left(1 - \sqrt{\frac{F k_F T_d^2}{2 D T_F}}\right)$$



- use best fit for bouncing phase parameter

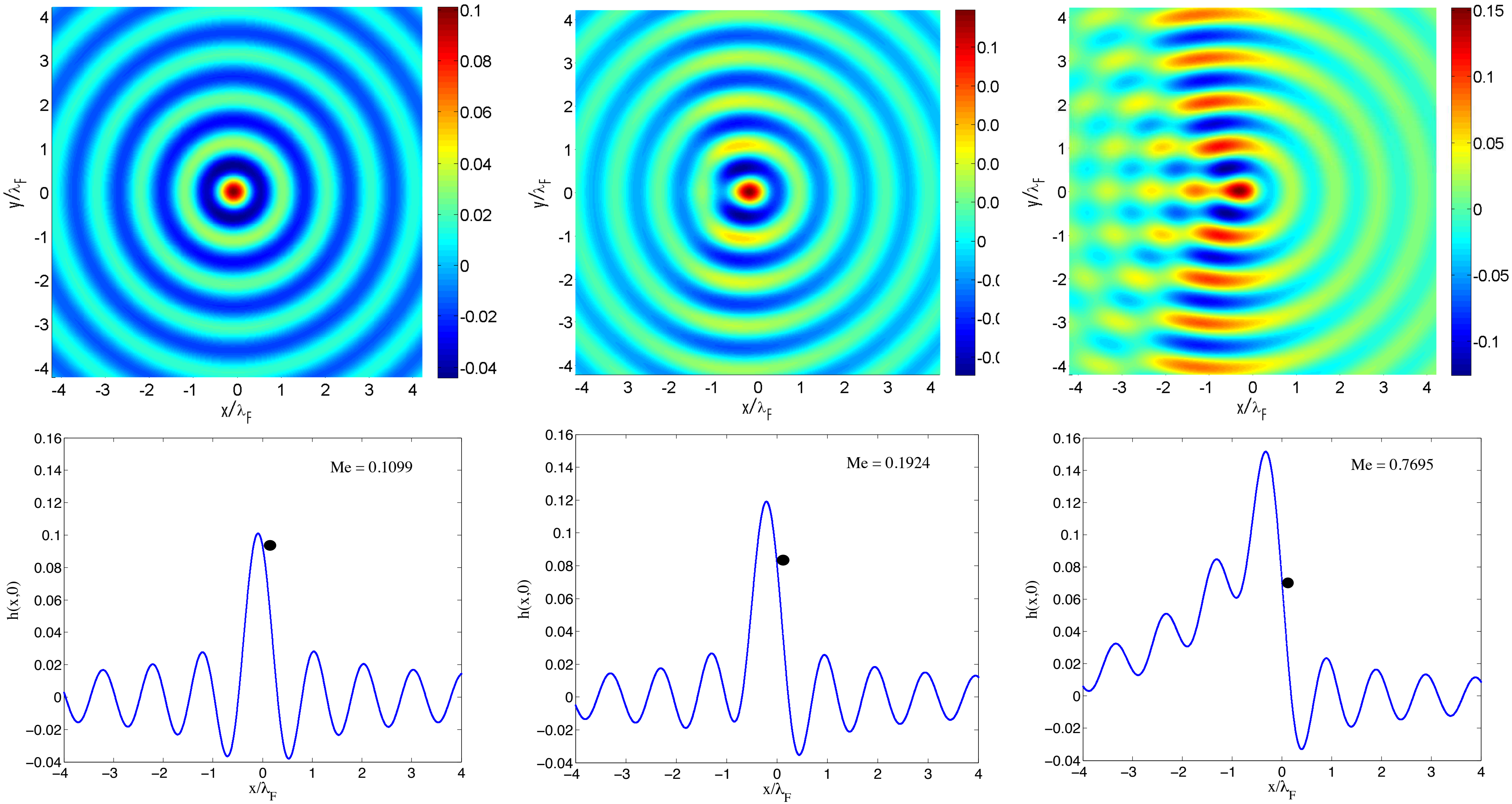
Predicted walking speeds



- discontinuities associated with transition to more energetic walking state:

$(2,1)^1 \longrightarrow (2,1)^2$

Pilot-wave field of walking droplets



- the walker surfs on its pilot wave, moving down the wave, faster as Me increases

Energetics of the stroboscopic model

Durey & Bush (202X)

What are the relative magnitudes of drop KE, drop GPE and wave energy?

Trajectory: $$m\ddot{\boldsymbol{x}}_p + D\dot{\boldsymbol{x}}_p = -mg\nabla h(\boldsymbol{x}_p, t) + \boldsymbol{F}$$

Wave field: $$h(\boldsymbol{x}, t) = \frac{A}{T_F}\int_{-\infty}^{t} \mathscr{H}(|\boldsymbol{x} - \boldsymbol{x}_p(s)|)\mathrm{e}^{-(t-s)/T_M}\,\mathrm{d}s.$$

Wave kernel: $\mathscr{H}(r)$ bounded, quasi-monochromatic, $\mathscr{H}(0) = 1$

Wave amplitude beneath walker, $H(t) = h(\boldsymbol{x}_p, t)$, and bouncer $H_B = AT_M/T_F$

Chain rule: $$\dot{H} = \partial_t h(\boldsymbol{x}_p, t) + \dot{\boldsymbol{x}}_p \cdot \nabla h(\boldsymbol{x}_p, t)$$

Sub into trajectory eqn: $$\dot{H} = \frac{A}{T_F} - \frac{H}{T_M} + \frac{1}{mg}\dot{\boldsymbol{x}}_p \cdot \left(\boldsymbol{F} - m\ddot{\boldsymbol{x}}_p - D\dot{\boldsymbol{x}}_p\right)$$

→ $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m|\dot{\boldsymbol{x}}_p|^2 + mgH\right) = \dot{\boldsymbol{x}}_p \cdot \boldsymbol{F} + \frac{mgA}{T_F}\left(\gamma_D(|\dot{\boldsymbol{x}}_p|) - \frac{H}{H_B}\right)$$

where $\gamma_D(v) = 1 - \dfrac{v^2}{c^2}$ and the drop speed limit $c = \sqrt{mgA/DT_F}$

Energetics of the stroboscopic model

Work equation: $$\dot{E}_p = \frac{mgA}{T_F}\left(\gamma_D(|\dot{\boldsymbol{x}}_p|) - \frac{H}{H_B}\right)$$

Drop energy: $$E_p = \frac{1}{2}m|\dot{\boldsymbol{x}}_p|^2 + V(\boldsymbol{x}_p) + mgH \quad \text{where} \quad \boldsymbol{F} = -\nabla V(\boldsymbol{x}_p)$$

Steady state: $$\boxed{\frac{H}{H_B} = \gamma_D(v) = 1 - \frac{v^2}{c^2}}$$ relates drop GPE to KE

Wave energy: $$E(t) = \iint_{\mathbb{R}^2} \frac{\rho g}{2} h^2(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x} + \iint_{\mathbb{R}^2} \frac{\sigma}{2}|\nabla h|^2\, \mathrm{d}\boldsymbol{x}$$

For a nearly monochromatic wave field, Matt Durey has shown

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{2E_B}{T_M}\left(\frac{H(t)}{H_B} - \frac{E(t)}{E_B}\right)$$

Steady state:

$$\boxed{\frac{H}{H_B} = \frac{E}{E_B} = \gamma_D(v) = 1 - \frac{v^2}{c^2}}$$ relates wave energy, GPE, KE

The stroboscopic model: advantages

- a significant simplification relative to full model of Molacek

  → continuous rather than discrete

- allows for analysis of stability of bouncing and walking states

- allows for analysis of stability of orbital motion in rotating frame, in SHO and spin states (at low Me)

- allows for characterization of transitions to chaos in rotating frame, SHO and Coulomb potentials

- allows for analysis of interacting droplets, provided they are in resonance

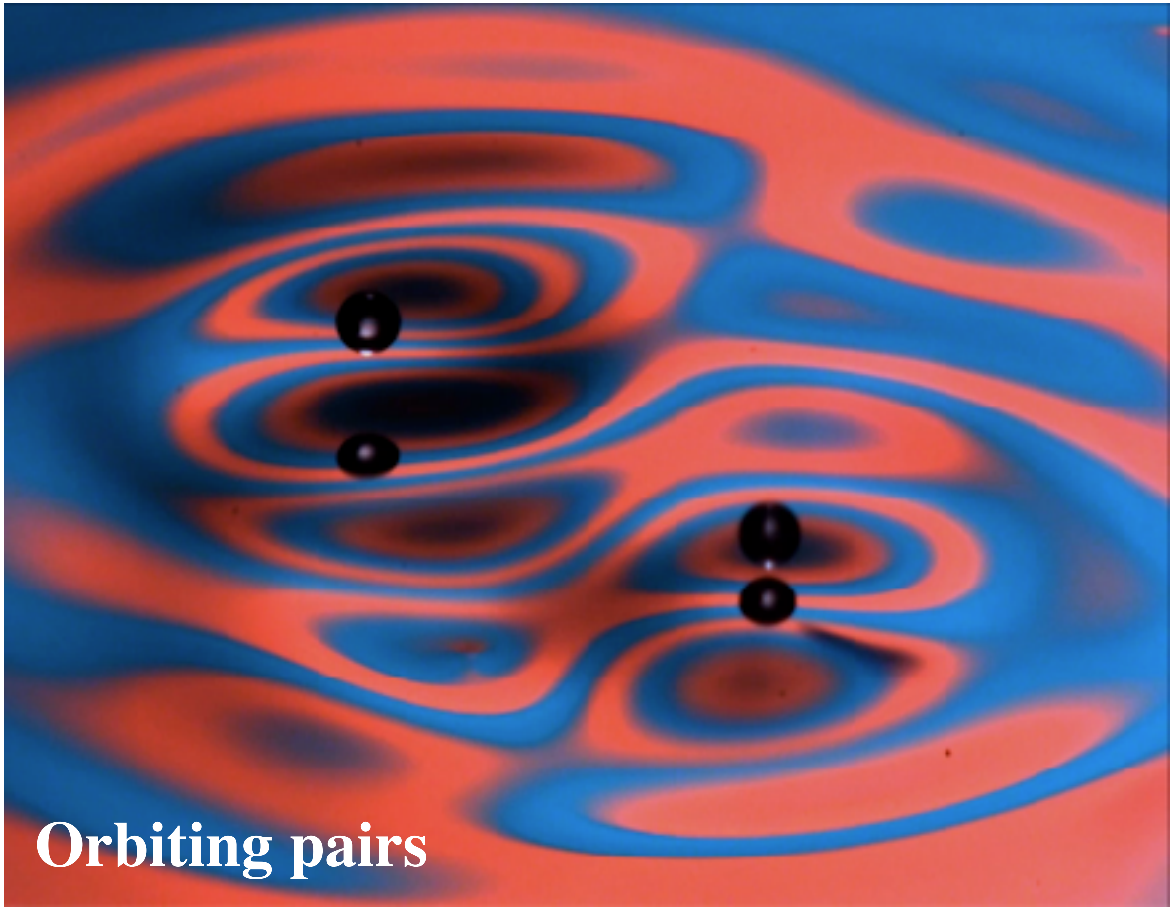
The stroboscopic model: shortcomings

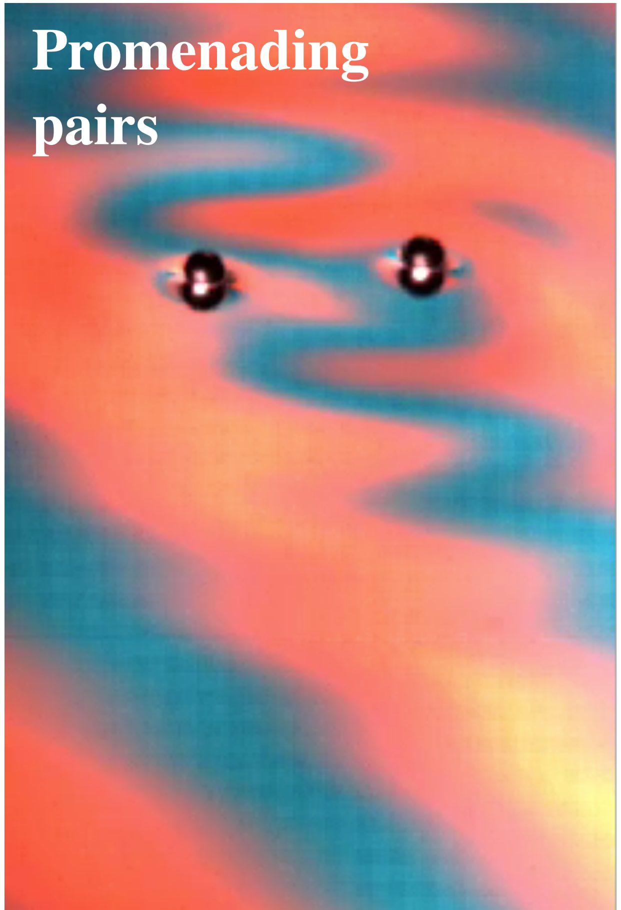
- only applies in situations where bouncing phase is constant
- bouncing phase variations may be induced by wave fields associated with:

    1) neighboring droplets (both *transient* and standing waves)

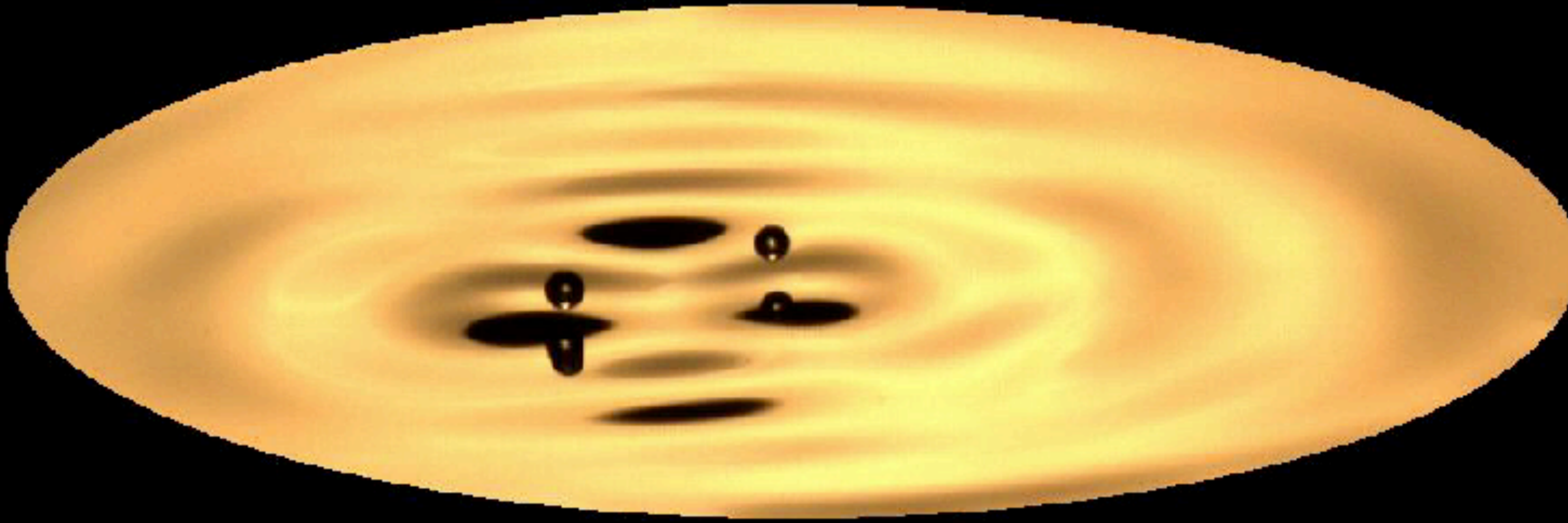
    2) standing waves associated with deep regions (above Faraday threshold)

    3) the droplet's own history in closed domains



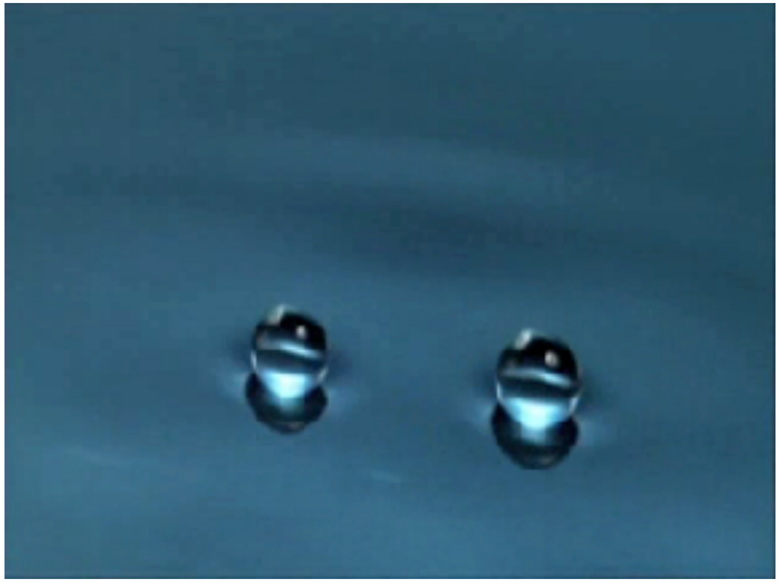


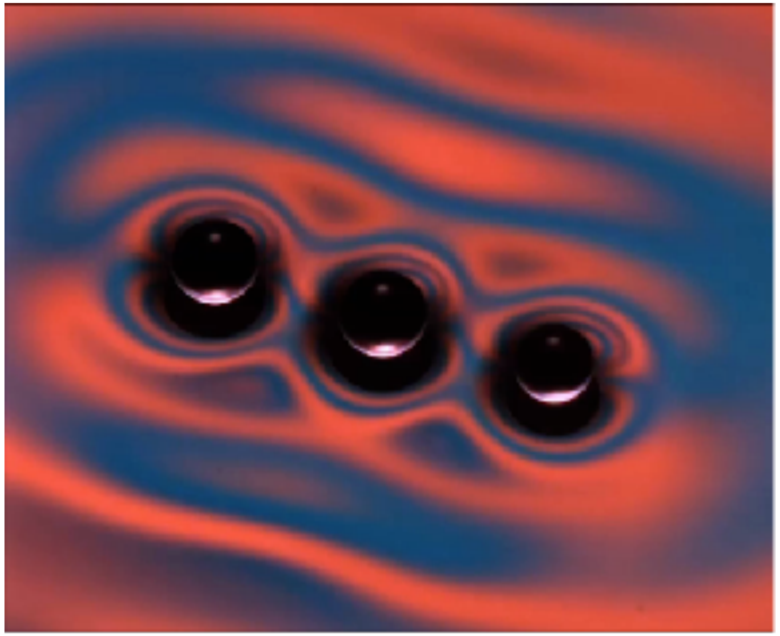
A flagrantly non-resonant orbiting pair

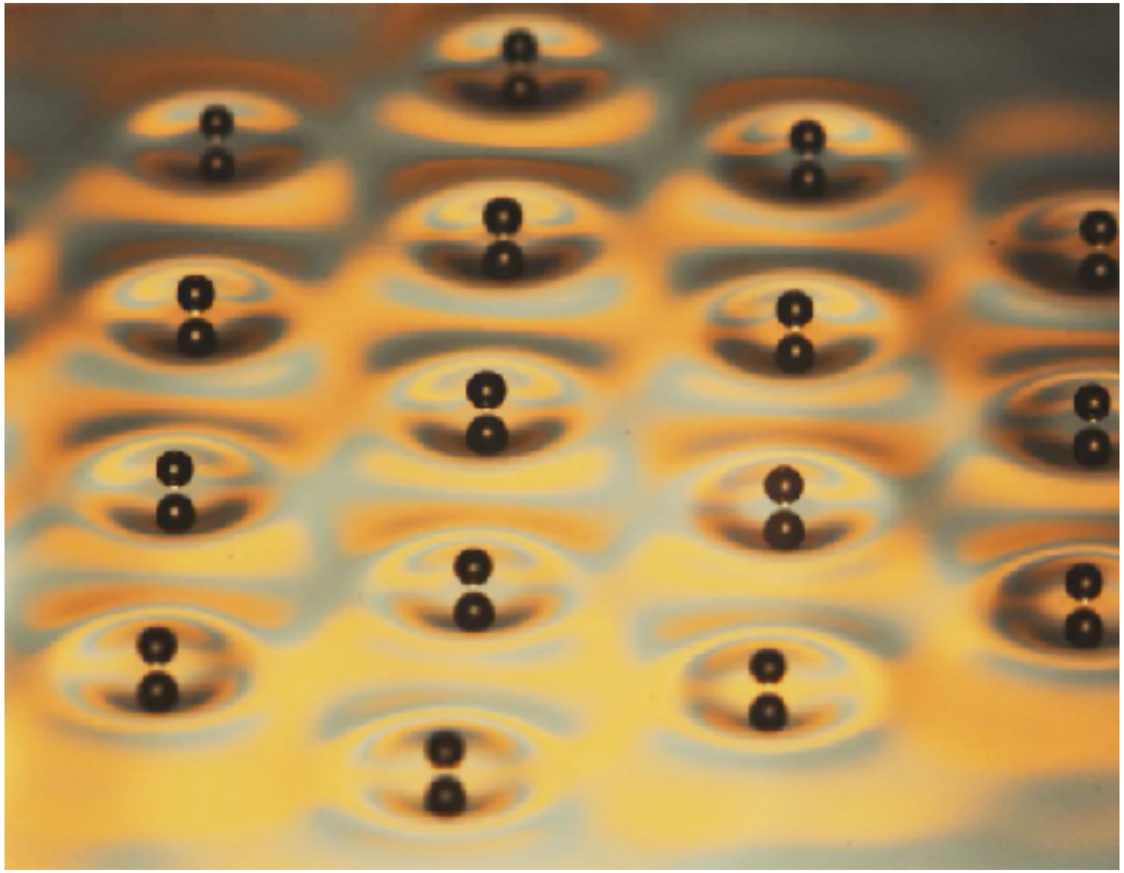


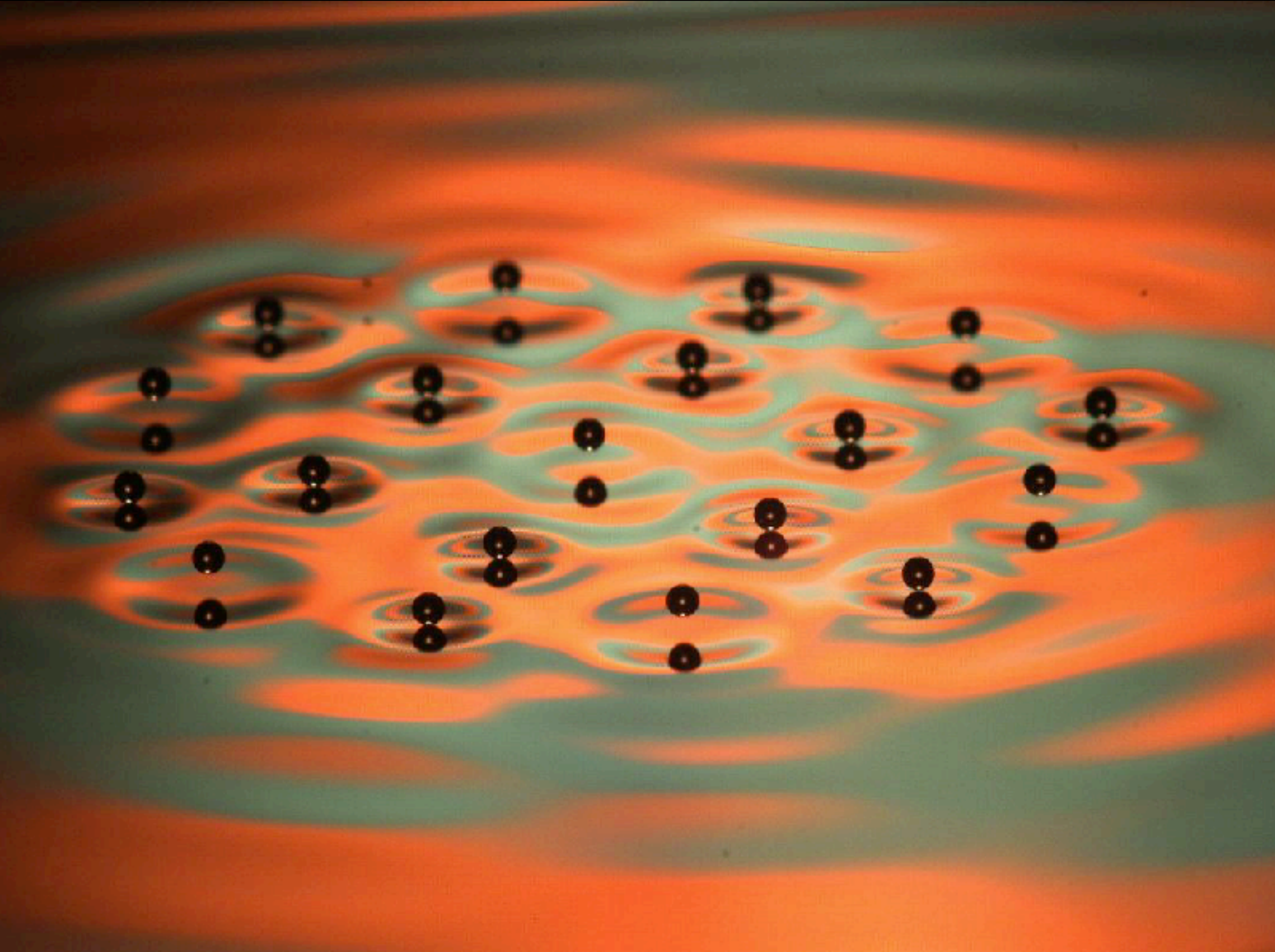
Static bound states

- multiple droplets bounce in resonance, but instability may break resonance



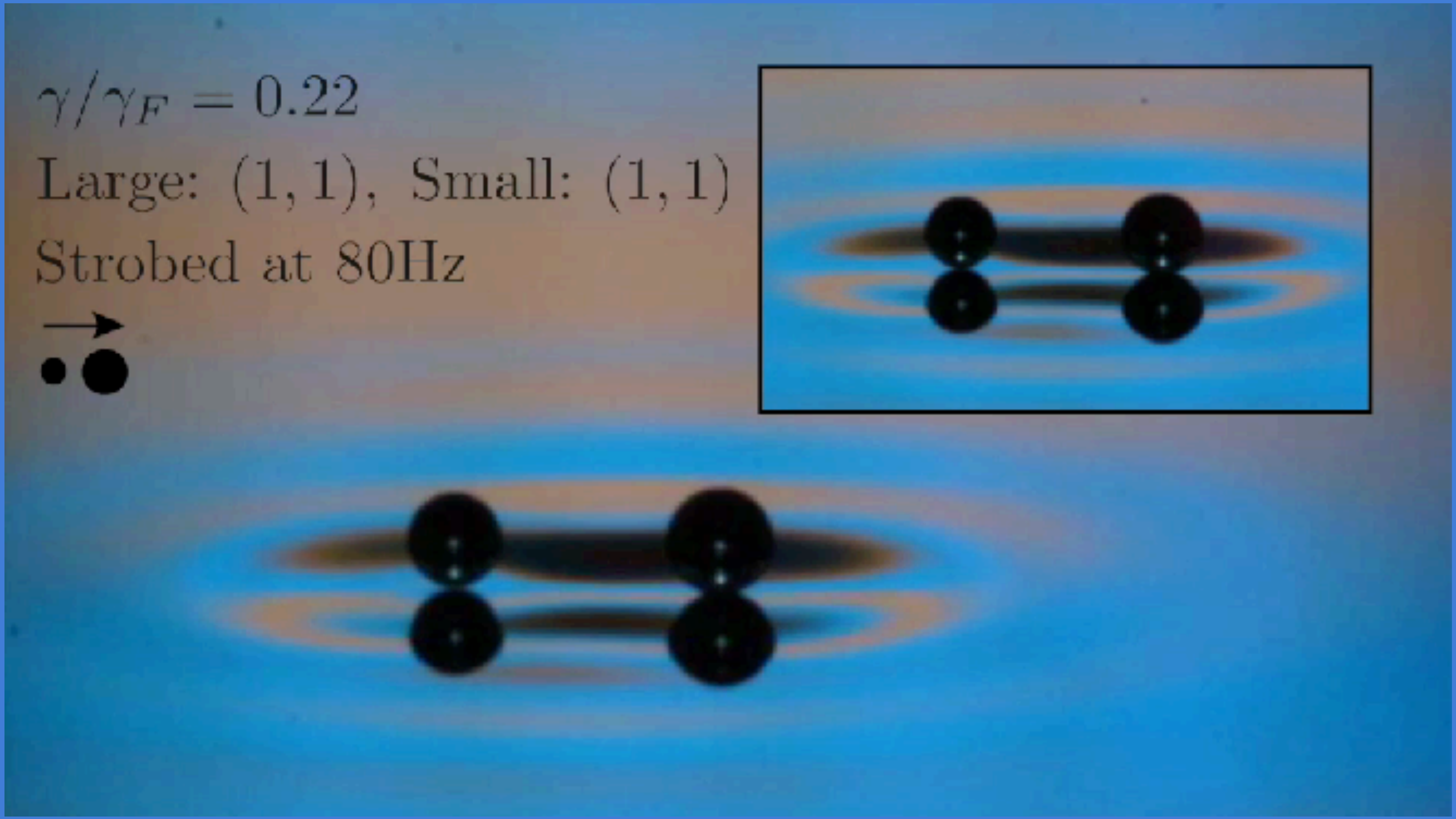






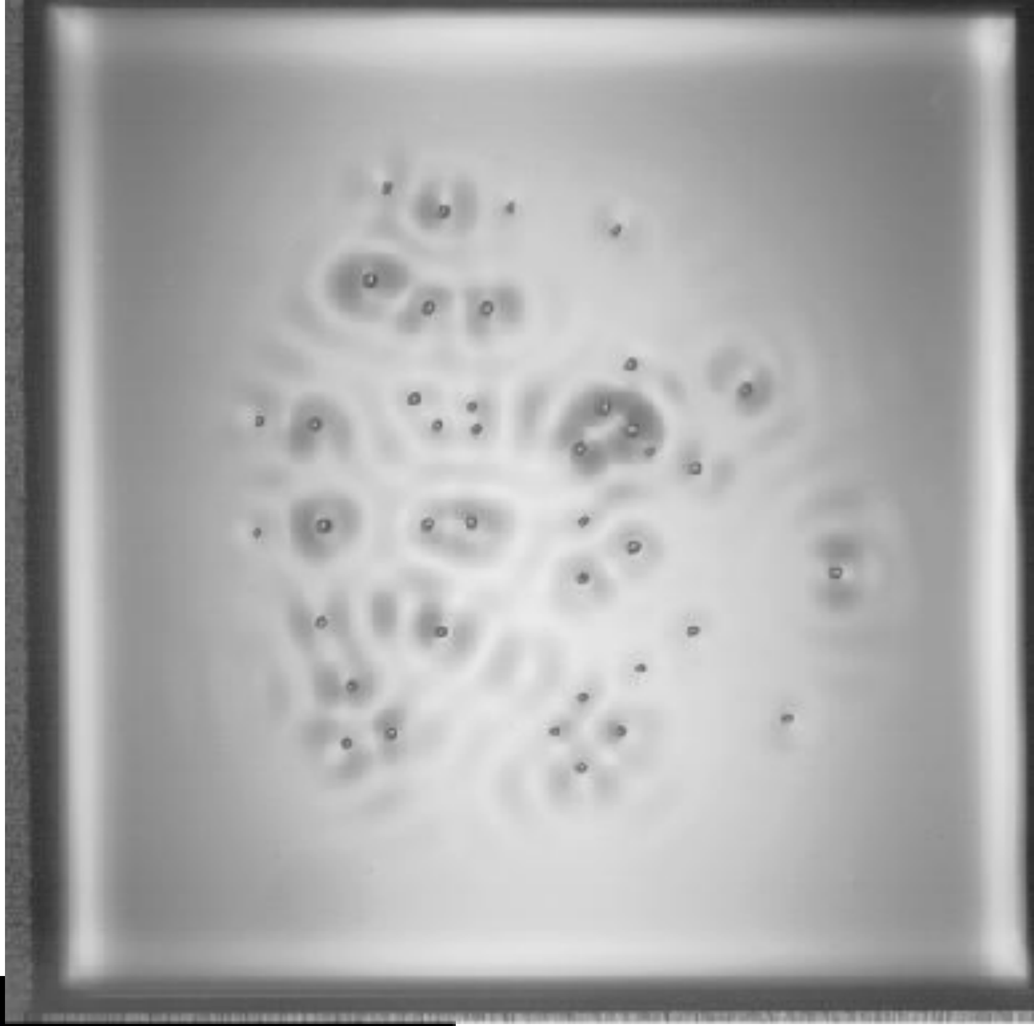
Ratcheting pairs (Eddi et al. 2010, Galeano Rios et al. 2018)

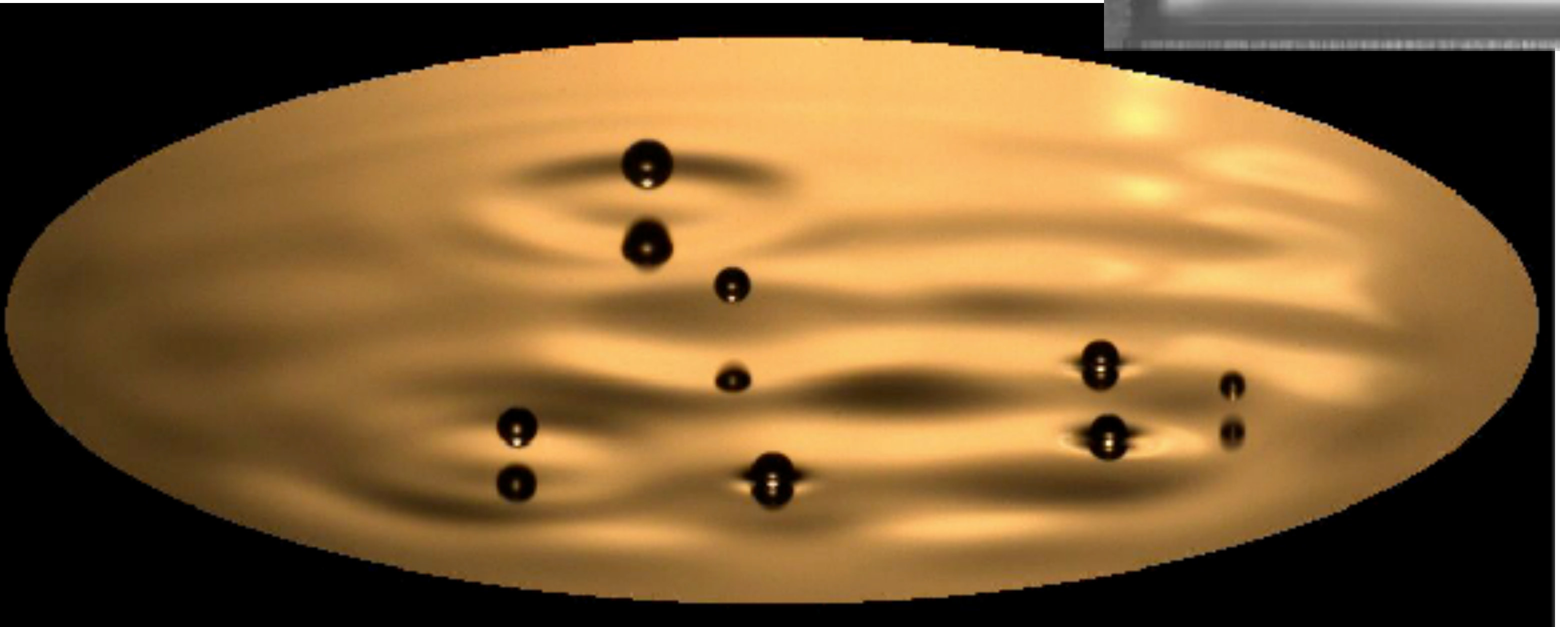
- unequal pairs self propel by virtue of the asymmetry in their wave fields



- observed behavior *not* well captured by strobe models (Oza et al. 2017)
- indicates importance of phase variation and transient waves on stability

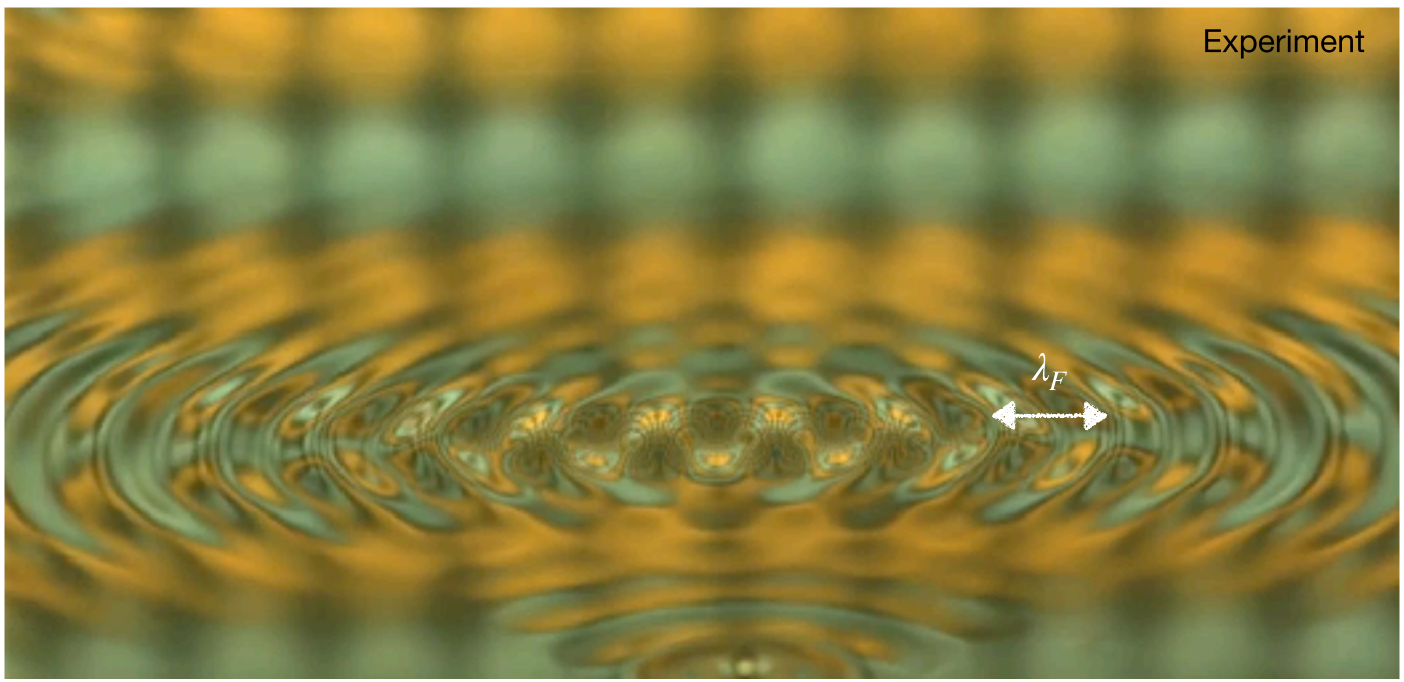
The interaction of many walkers

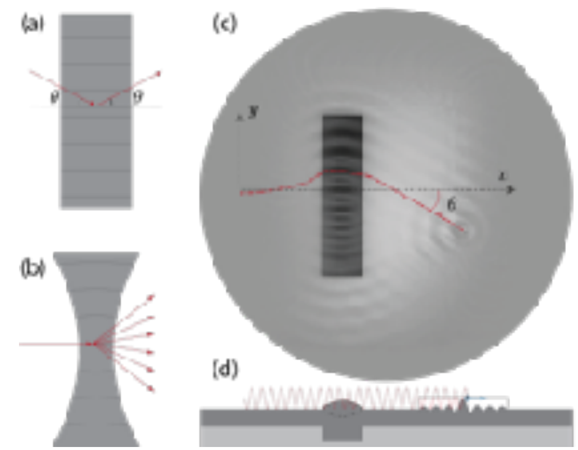


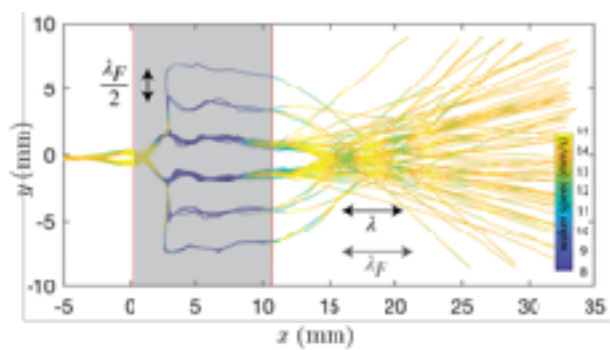


A walker traverses a standing Faraday wave field

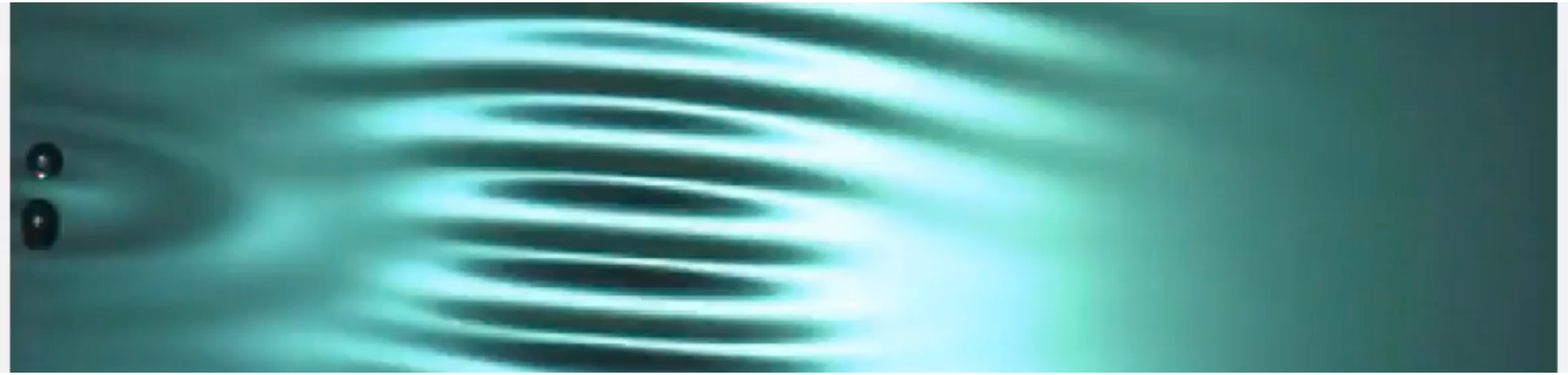
Strobed motion

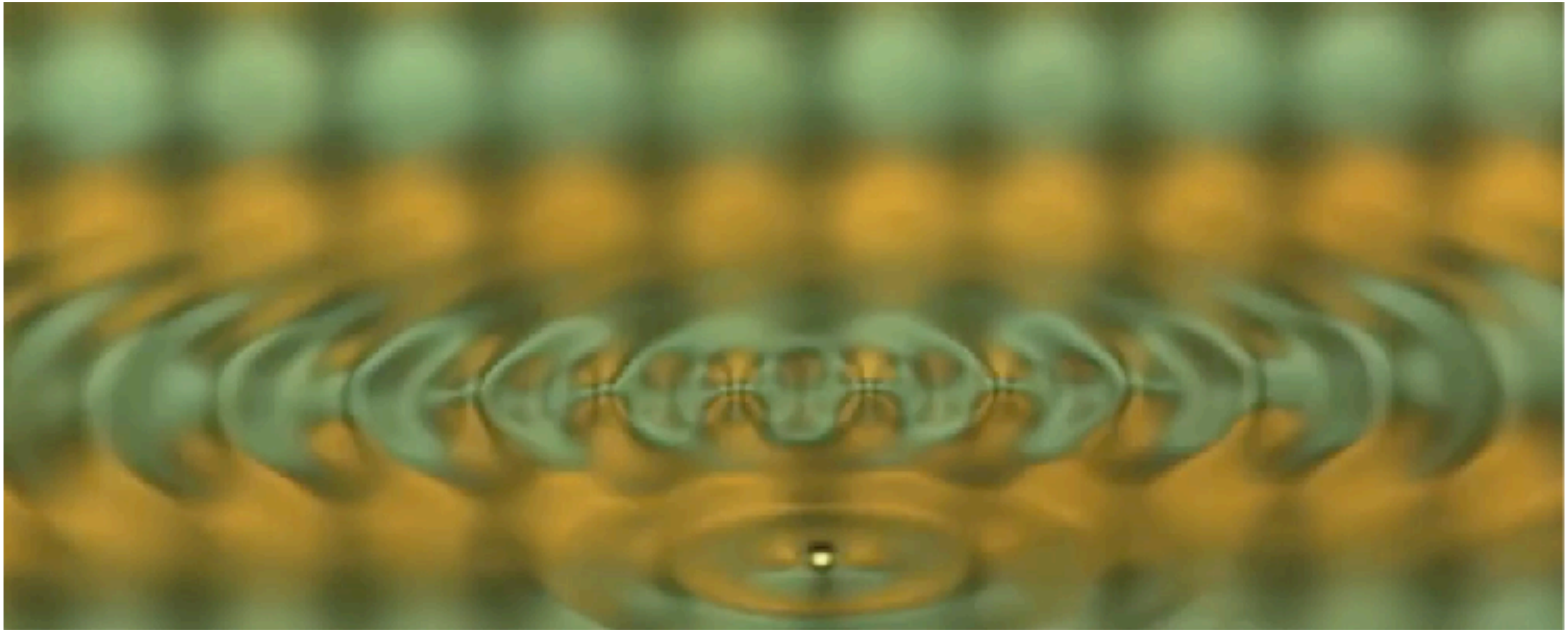






Walker loses resonance owing to interaction with standing waves

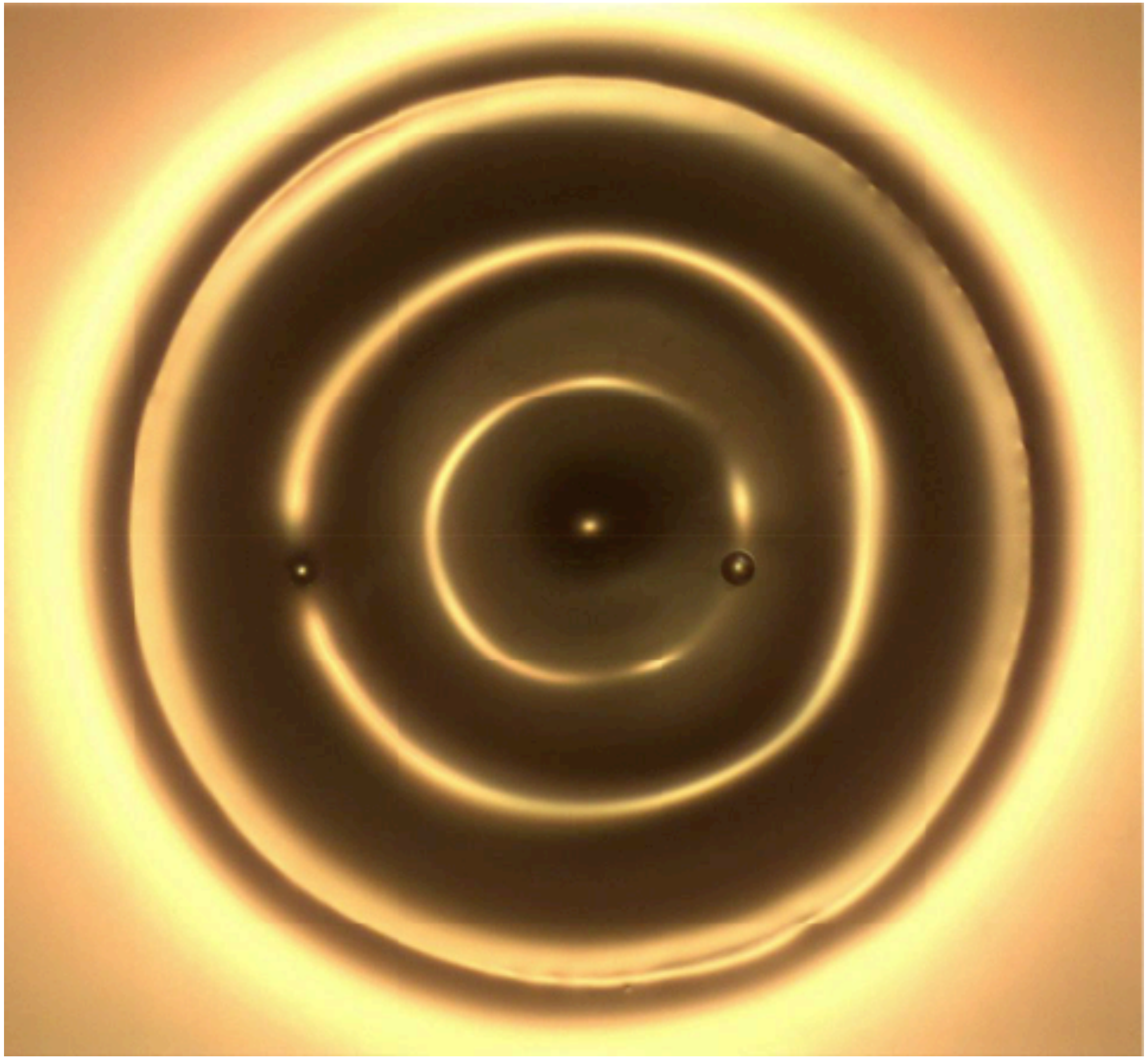




- dynamics cannot be captured by the stroboscopic model

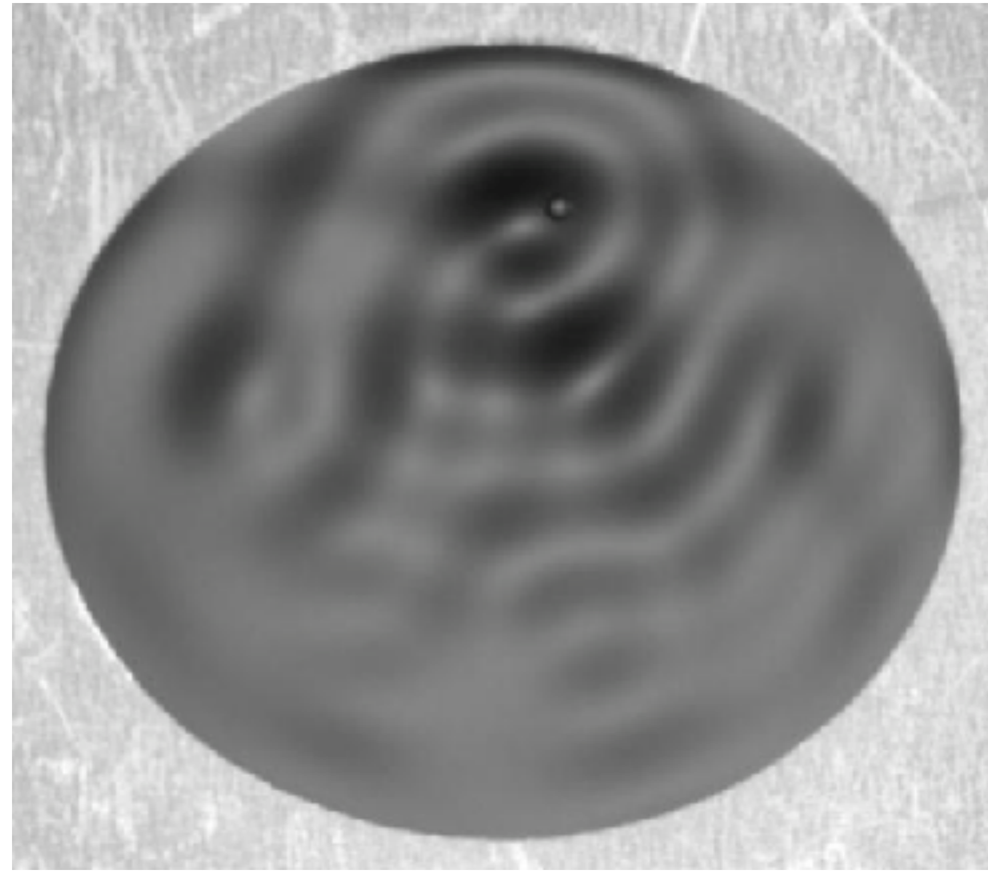
Two walkers in a corral above threshold

- bouncing phase variations induced by background wave field and partner

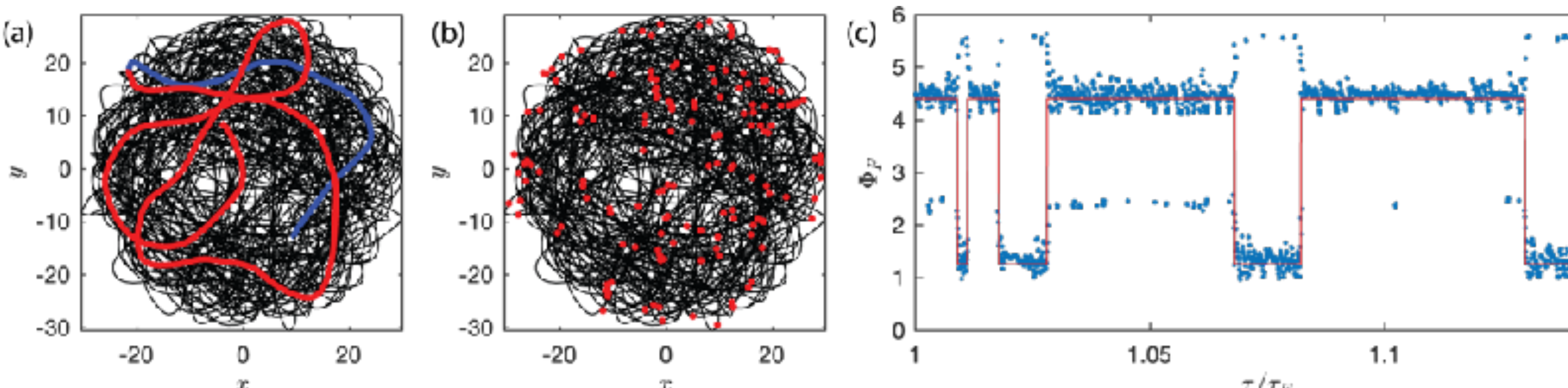


Walker in a corral

- explores its own pilot-wave field
- variations of bouncing phase induced by its pilot-wave field
- dynamics, statistics *not* captured by the stroboscopic model

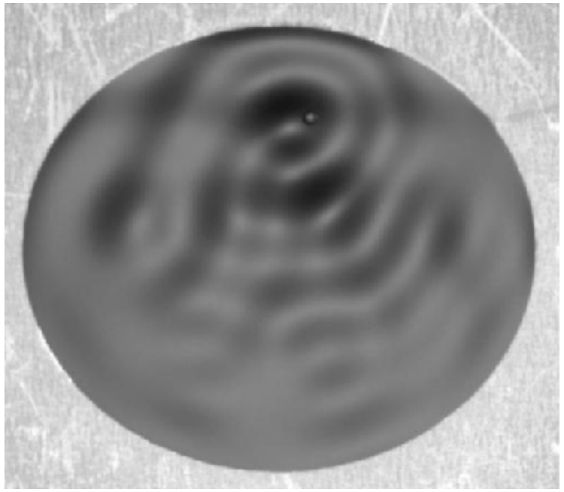


Phase variations



Current state of the art (see Bauyrzhan)

- vertical dynamics, non-resonance effects important at high Me, in confinement



Vertical dynamics:

$$\ddot{z}_p = F_N(\tau) - Bo$$

Horizontal dynamics:

$$\ddot{\mathbf{x}}_p + (\mathscr{D}_h F_N(\tau) + \mathscr{D}_a)\dot{\mathbf{x}}_p = -F_N(\tau)\nabla h$$

Normal force: $F_N(\tau) = -\mathscr{H}(-z_p + z_b + h)[\mathscr{D}_v(\dot{z}_p - \dot{z}_b - \dot{h}) + \mathscr{C}_v(z_p - z_b - h)]$

Linear drag

Surface tension: linear spring