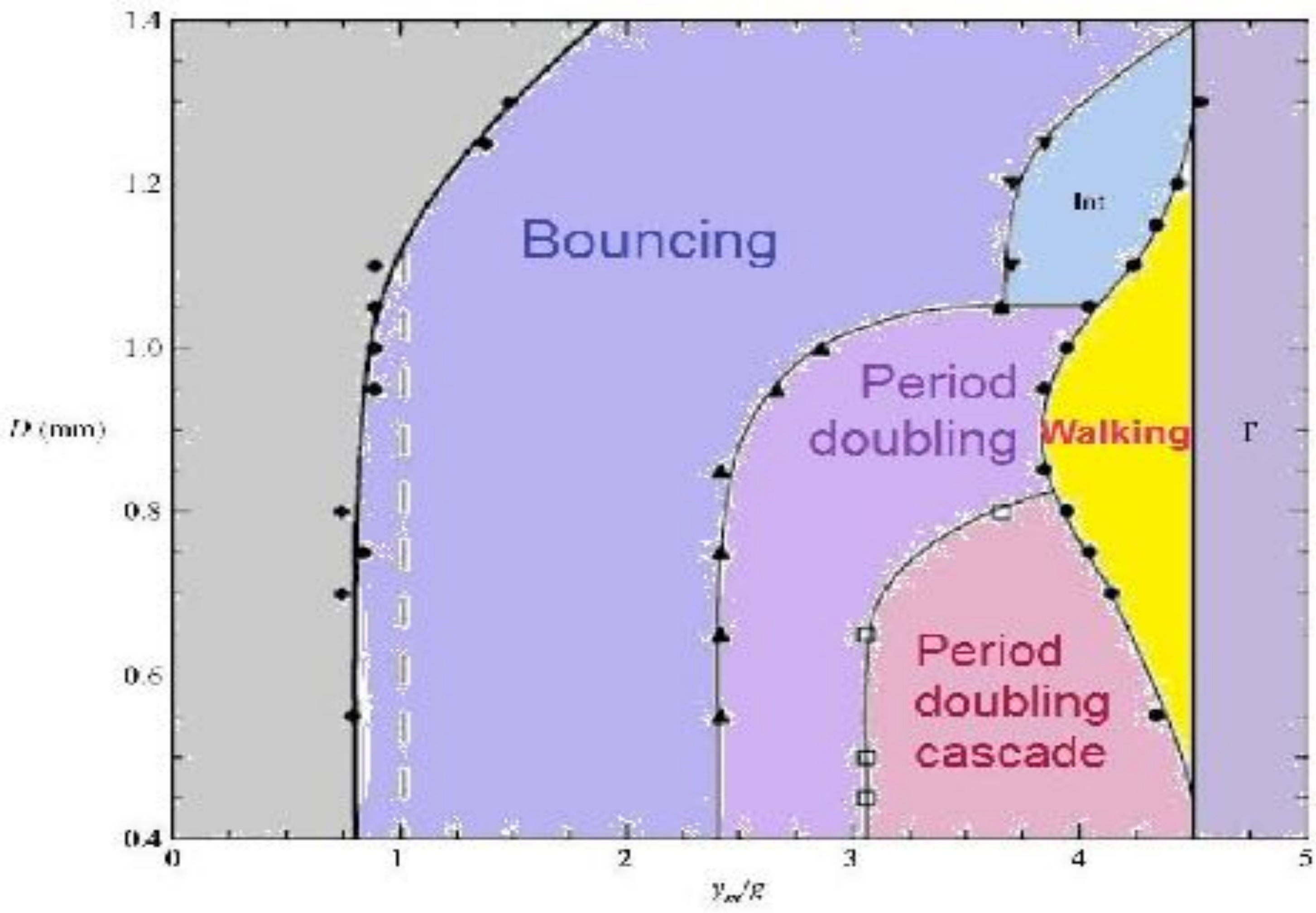
HQA Lecture 10 :

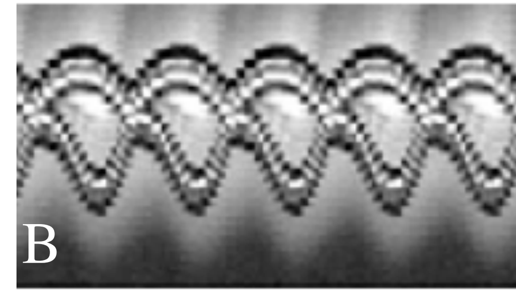
Bouncing and walking droplets

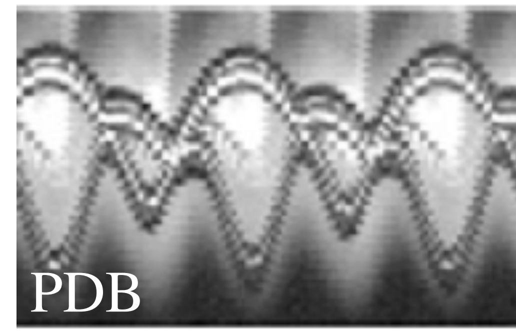
Jan Molacek Dan Harris Anand Oza

Couder's phase diagram

Protiere at al (2006)







$$\Gamma = A(2\pi f)^2/g$$

$$f = 50Hz$$
$$\nu = 50cSt$$

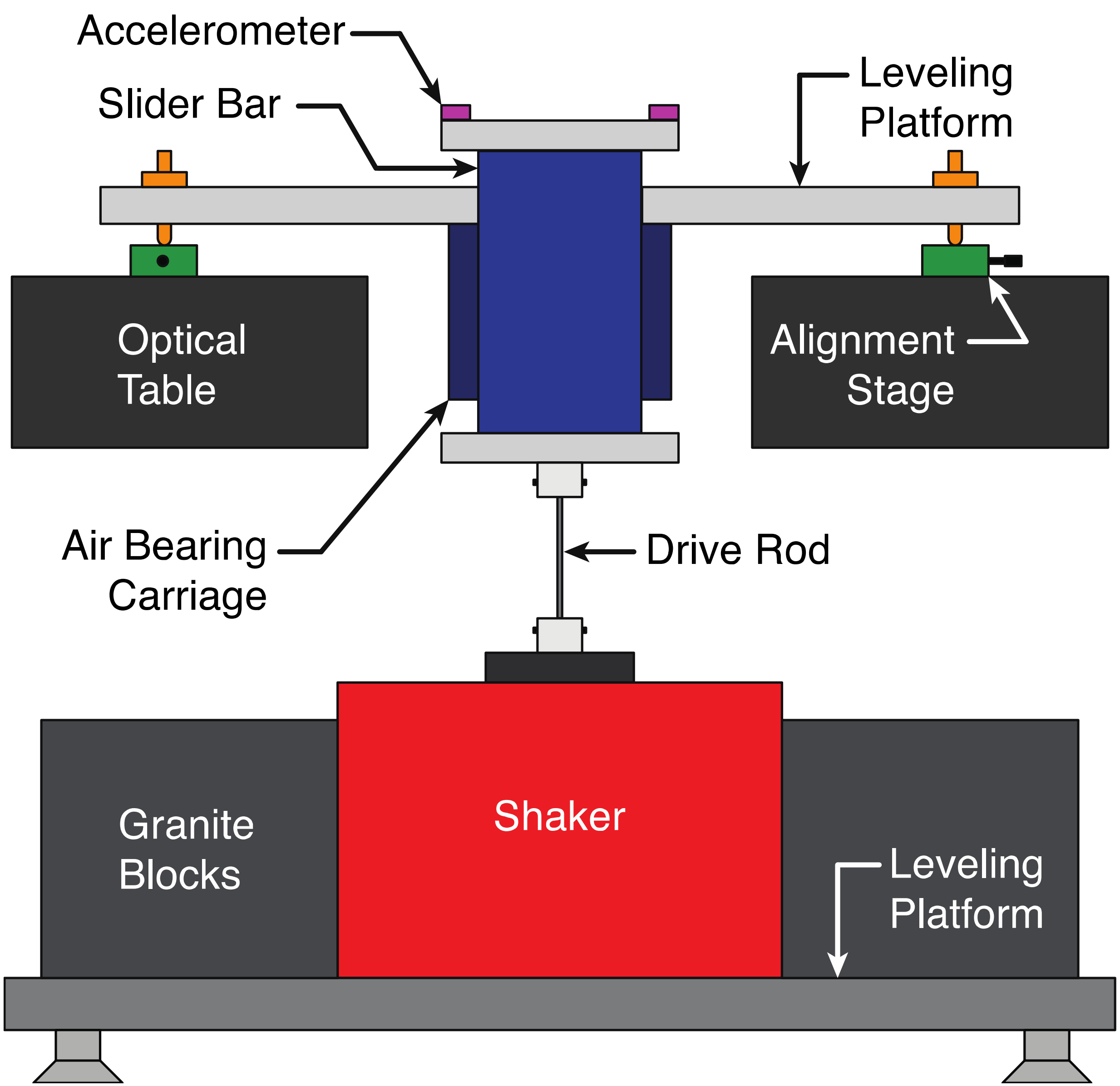
Can we rationalize these curves?

The PhD thesis of Jan Molacek

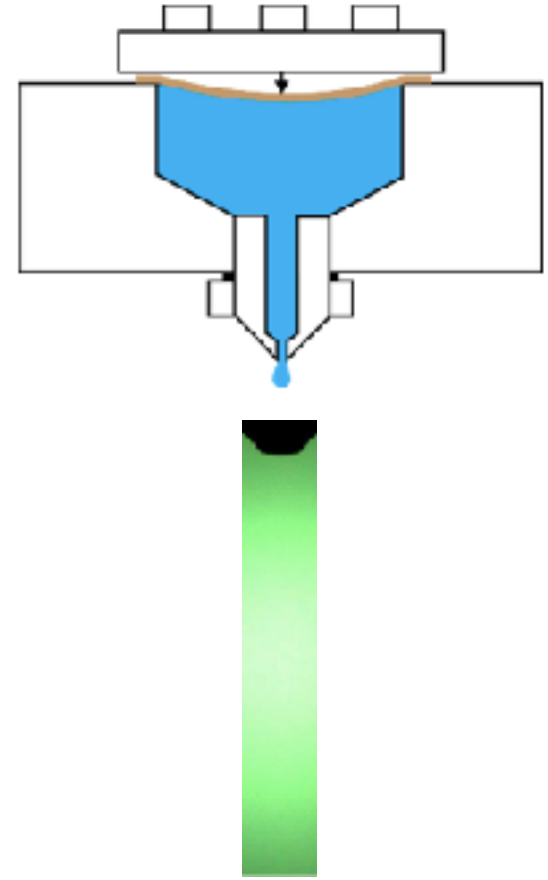
- an integration of experiment and mathematical modeling
- developed theoretical description of **bouncing**, **walking** drops
- rationalized the regime diagrams of bouncing modes
- developed a model for the standing waves created by impact
- rationalized extent of walking regimes, predict walking speeds
- developed a **trajectory equation** for the walking droplets
- provided the basis for the stroboscopic model of Anand Oza

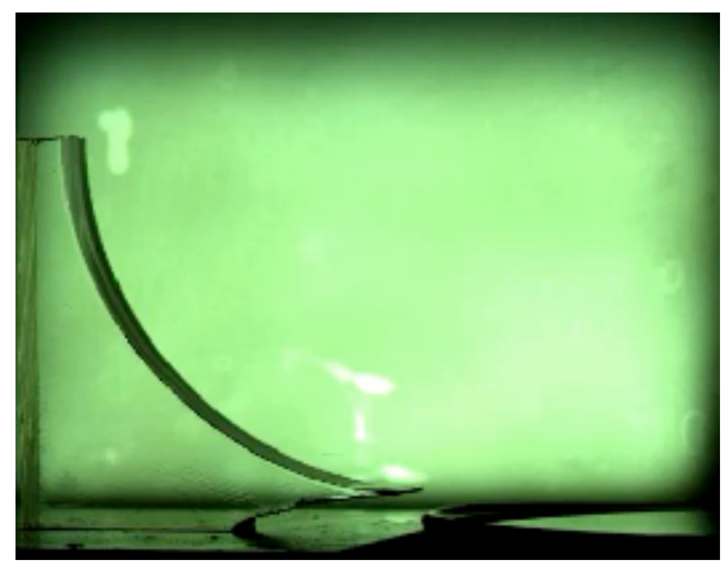
Refined experimental apparatus

- precise control of bath vibration and drop size
- influence of ambient air currents eliminated



Piezo-electric drop generator



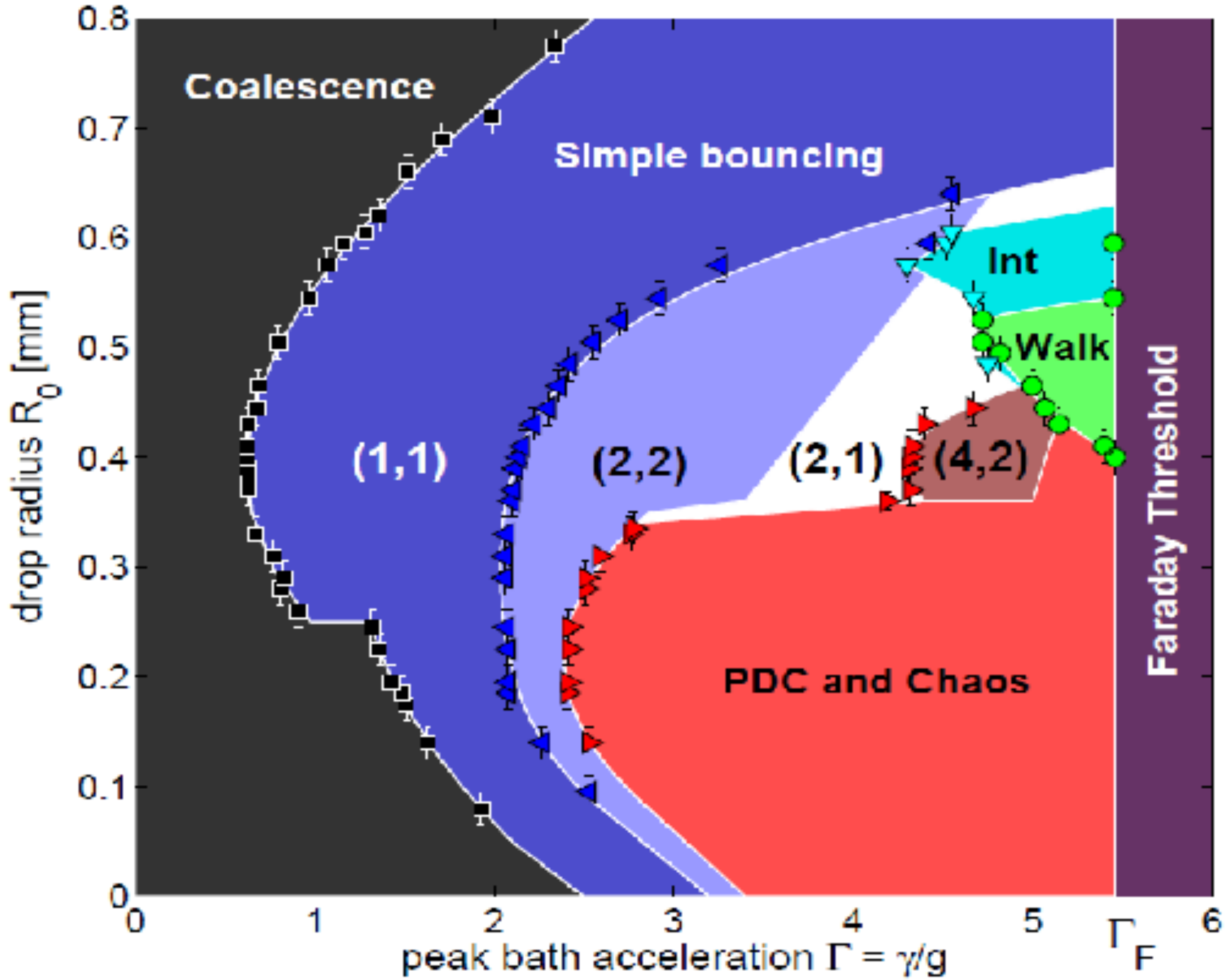


Harris & Bush (2015), Harris, Liu & Bush (2015)

Regime diagrams

50 cS 60 Hz

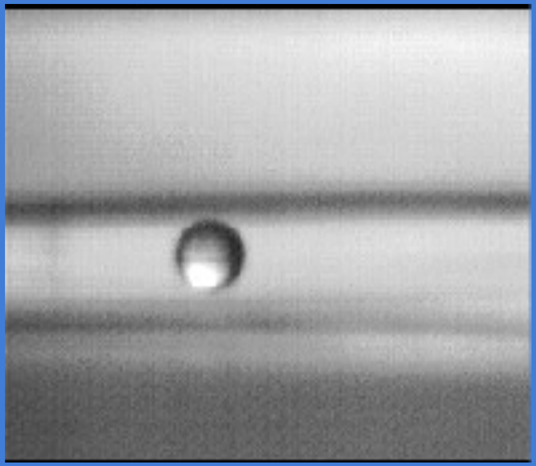
- extended range of drop sizes, fluid viscosities, driving frequencies



Dimensional analysis and pilot-wave triggers

If there are two physical constants in the Universe, ρ and σ , what is the natural frequency of oscillation of a drop of radius a ?

$$\boxed{\omega_d = \sqrt{\frac{\sigma}{\rho a^3}}}$$



The dynamics of bouncing and walking drops

Molacek & Bush (2013ab)

- rationalized dependence of droplet dynamics on system parameters
- developed quasi-static impact model: drop and interface take quasi-static forms
- highlighted the importance of the **Vibration Number**:

$$\Omega = \frac{2\pi f}{(\sigma/\rho R_0^3)^{1/2}} = \frac{\text{forcing frequency}}{\text{drop's natural frequency}}$$

Bouncing states: nomenclature

A bouncing state (m, n) bounces n times in m forcing periods.

i.e. one period of the trajectory corresponds to m forcing periods and n bounces

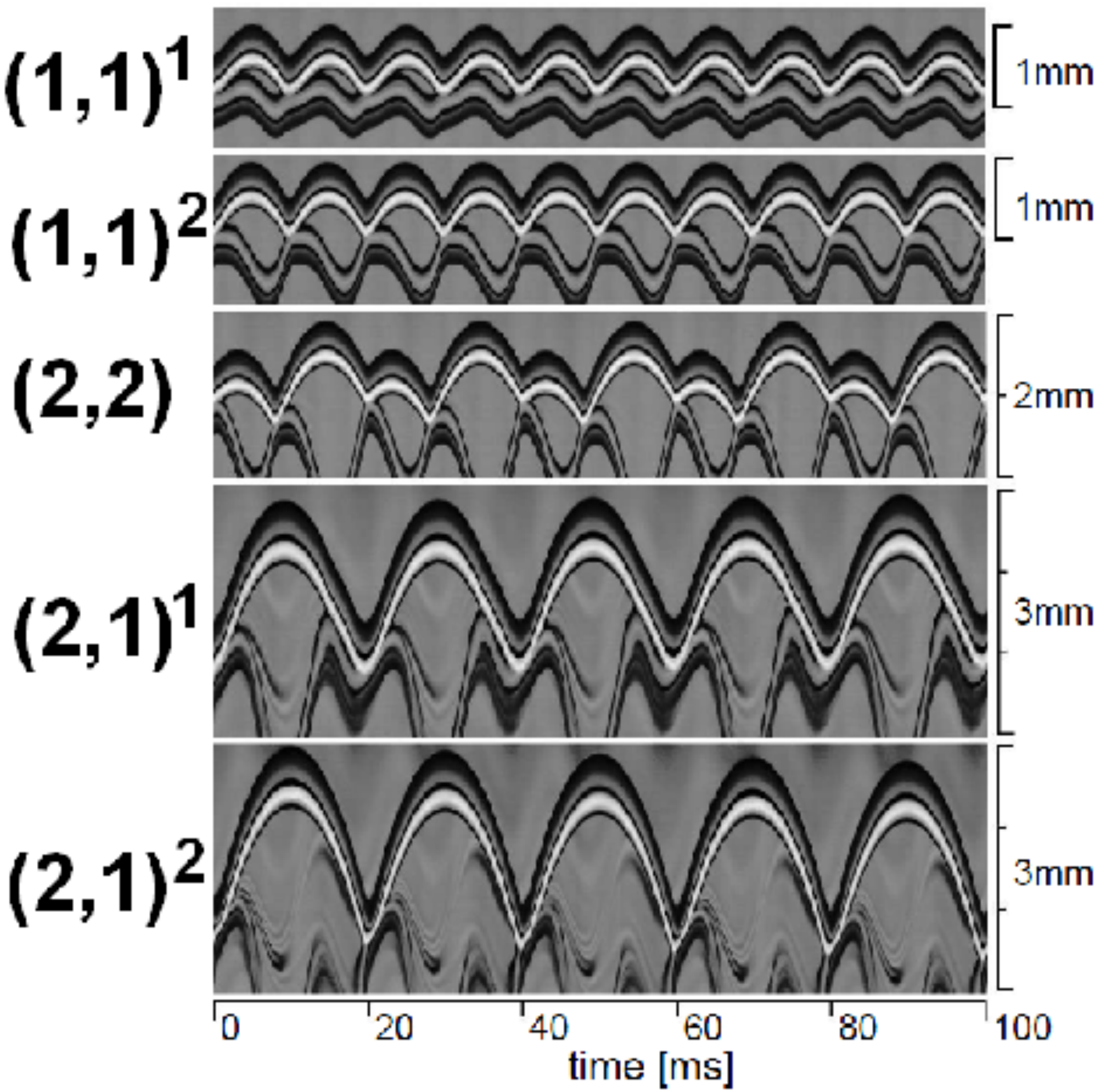
- to distinguish between degeneracy of bouncing states, use superscripts: $\boxed{(m,n)^{i}}$

Energy of $(m,n)^{\mathbf{1}}$ < Energy of $(m,n)^{\mathbf{2}}$ < ..... etc.

Bouncing modes

$\nu = 50cS, \quad f = 50Hz$

In (m, n) mode, a drop bounces n times in m forcing periods.

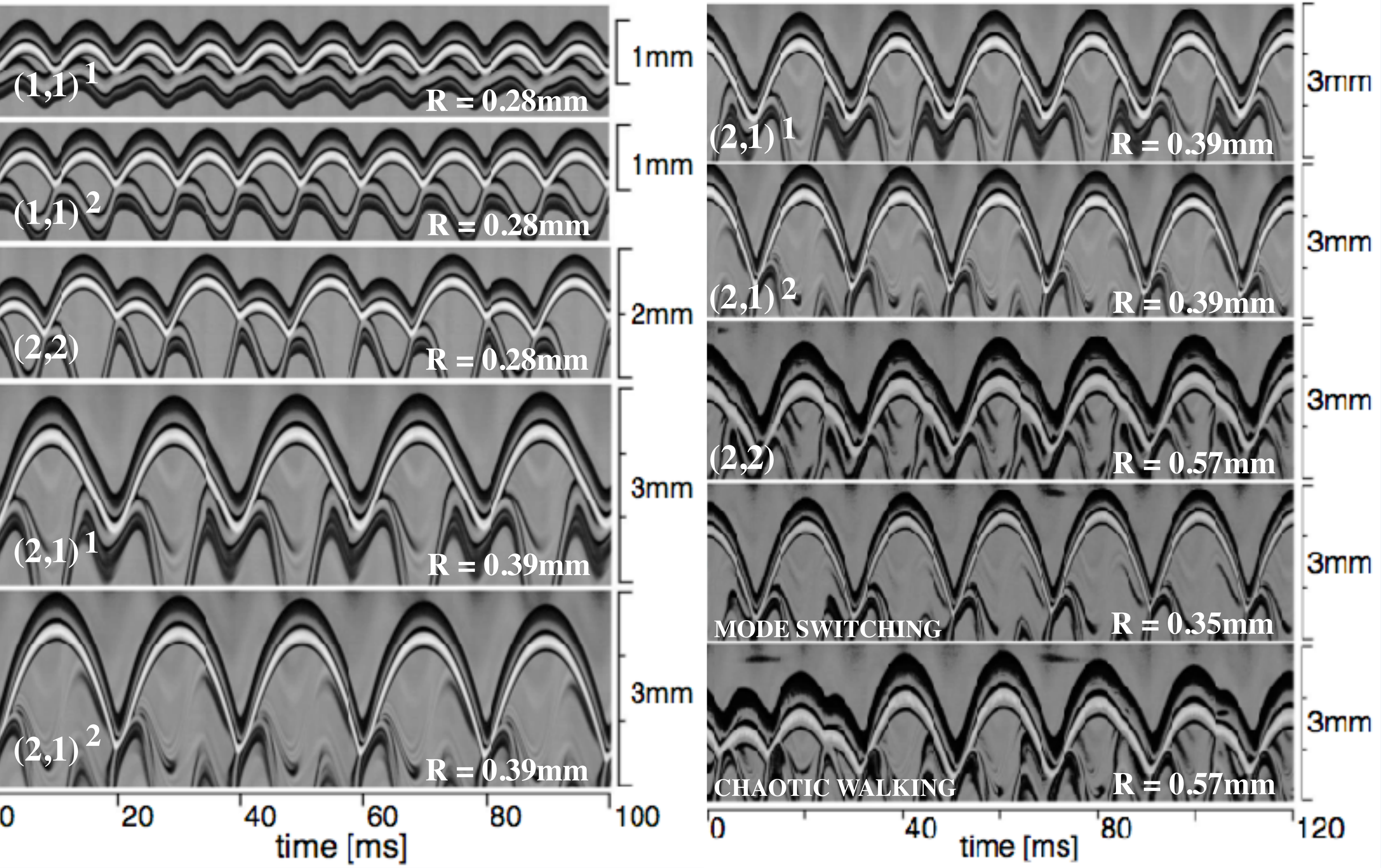


driving Γ increases

- mechanical energy of $(m, n)^2 > (m, n)^1$

local forcing at frequency of most unstable Faraday waves — may destabilize into walkers

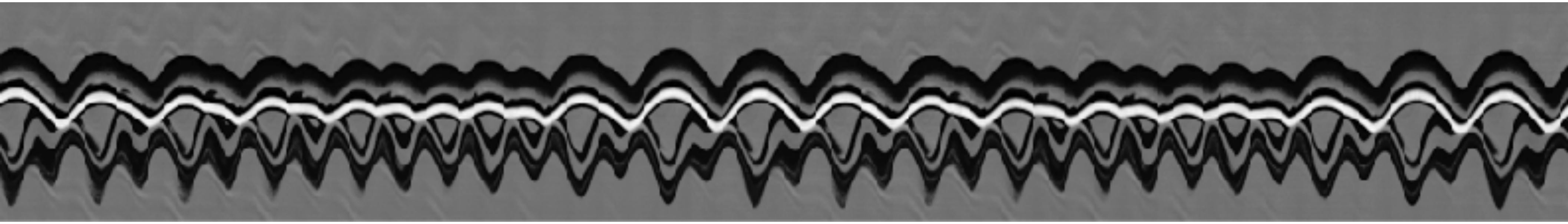
Bouncing modes $\nu = 50cS, \quad f = 50Hz$ Walking modes



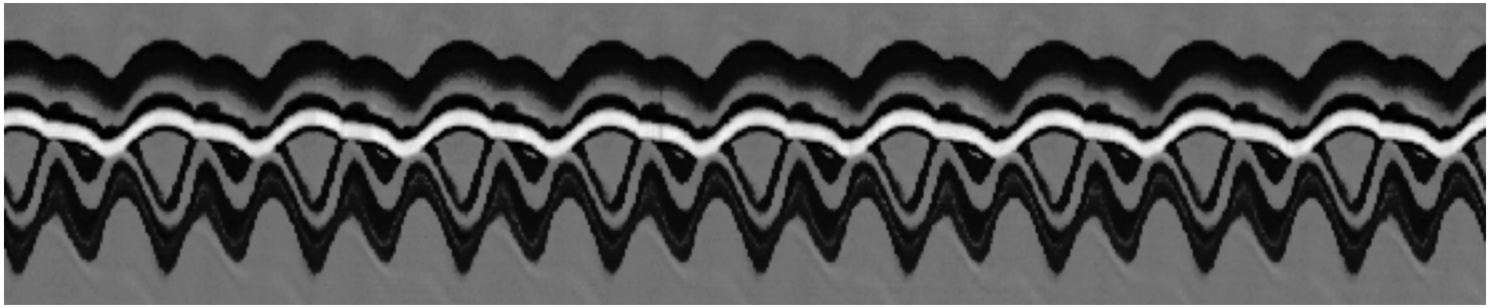
In (m, n) mode, a drop bounces n times in m forcing periods.

Bouncing Modes in 20cS, 80 Hz

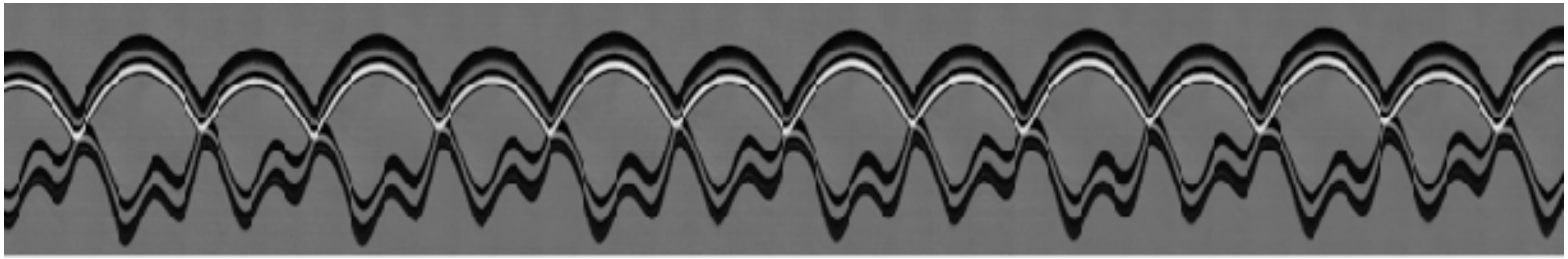
Chaotic bouncing (a040)



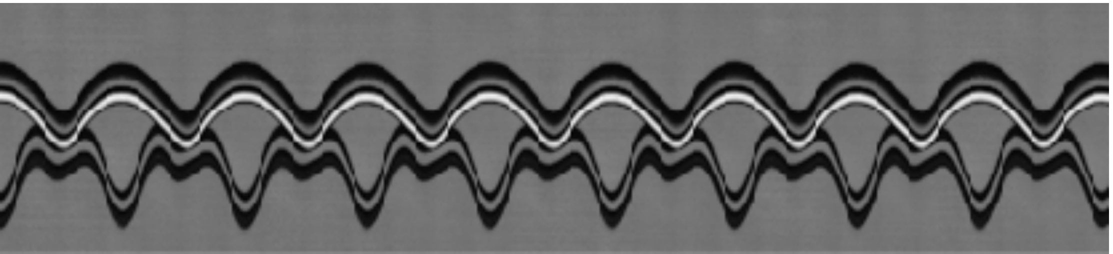
Bouncing (2,2) (a038)

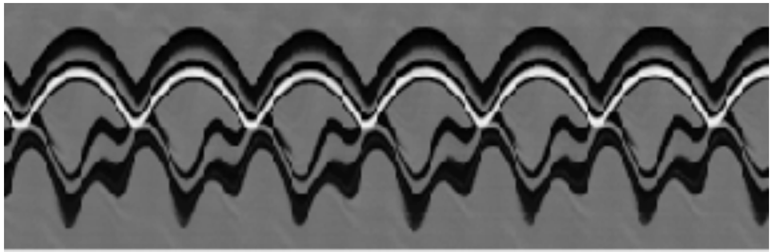


Bouncing (4,2) (a059)



Bouncing (2,1)^1 and (2,1)^2 (a058 and a051)

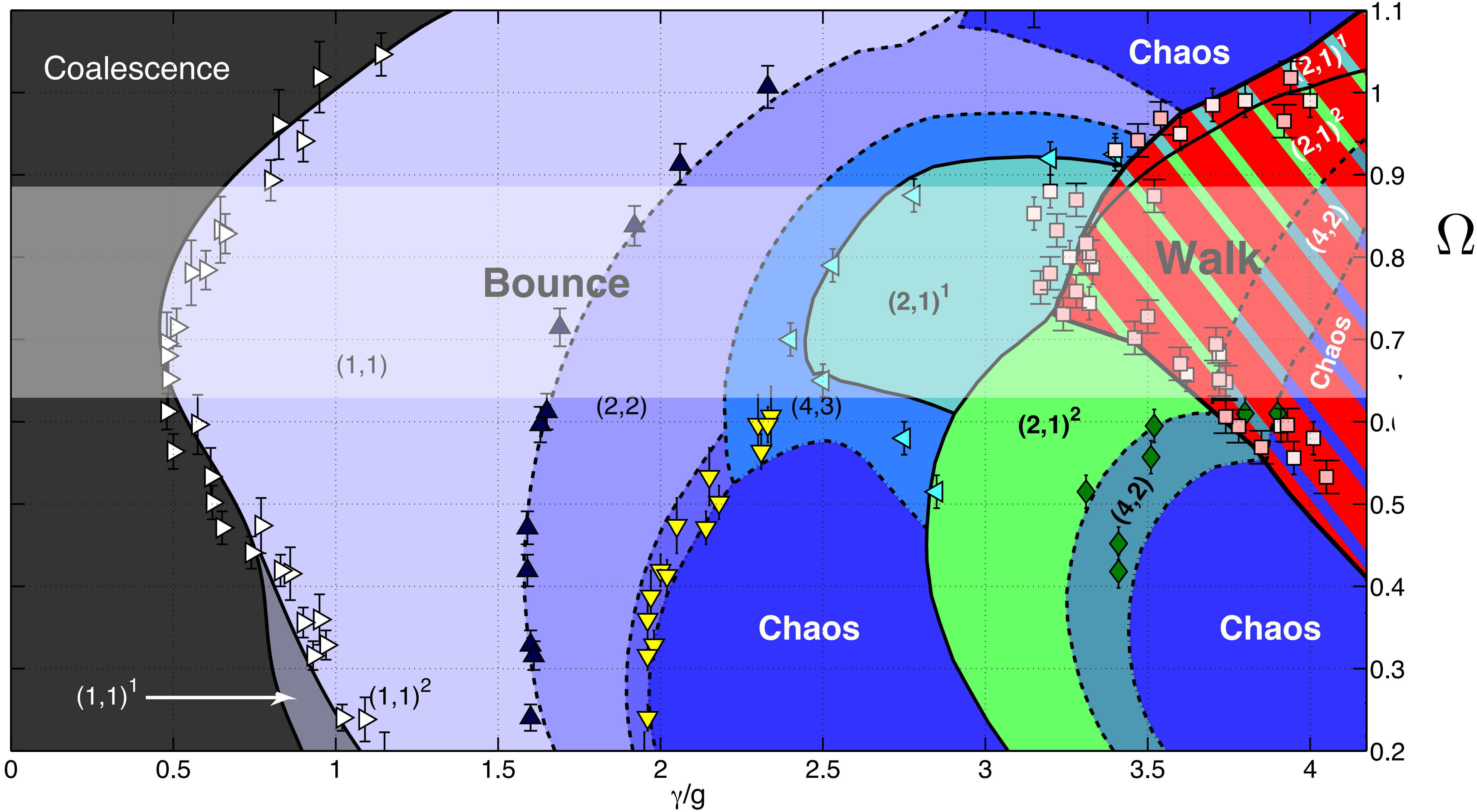




Regime diagram (20cS, 80Hz)

Molacek & Bush (2013ab)

$$\Omega = 2\pi f(\rho R_0^3/\sigma)^{1/2}$$



In (m, n) mode, a drop bounces n times in m forcing periods.

- drops most readily bounce, walk when forced at their natural frequency

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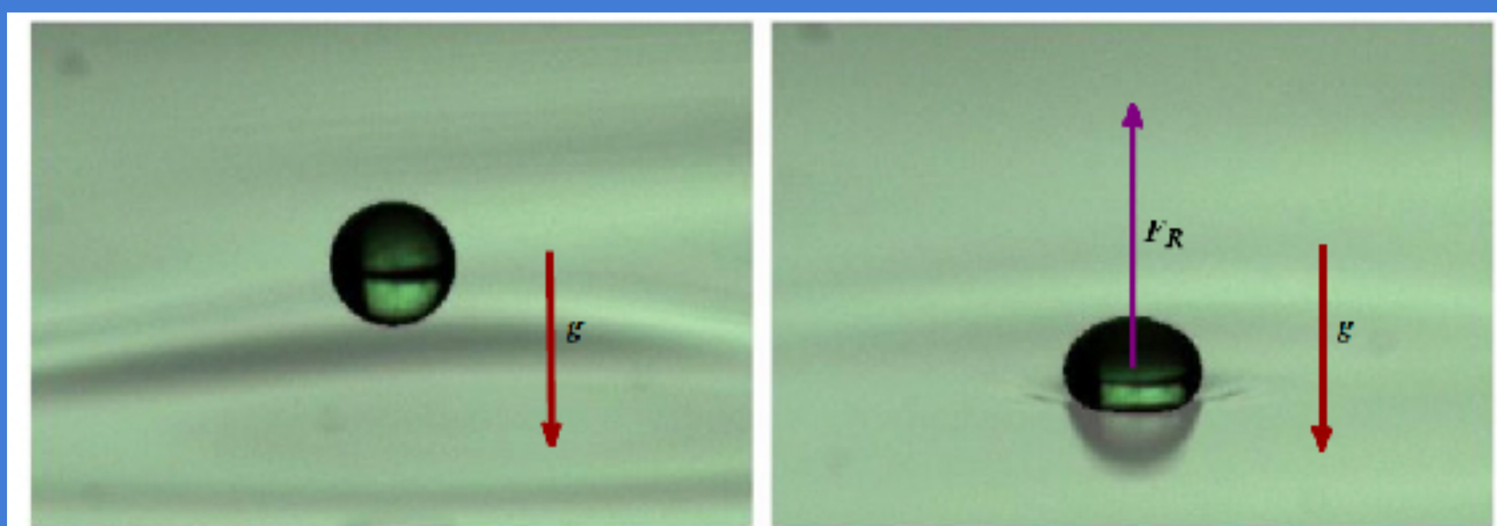
Drops bouncing on a vibrating bath

Jan Moláček and John W. M. Bush†

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA

Bouncing droplets

- except at the highest memory, can neglect influence of wave field
- assume surface recovers during a bounce, reverts to a flat surface

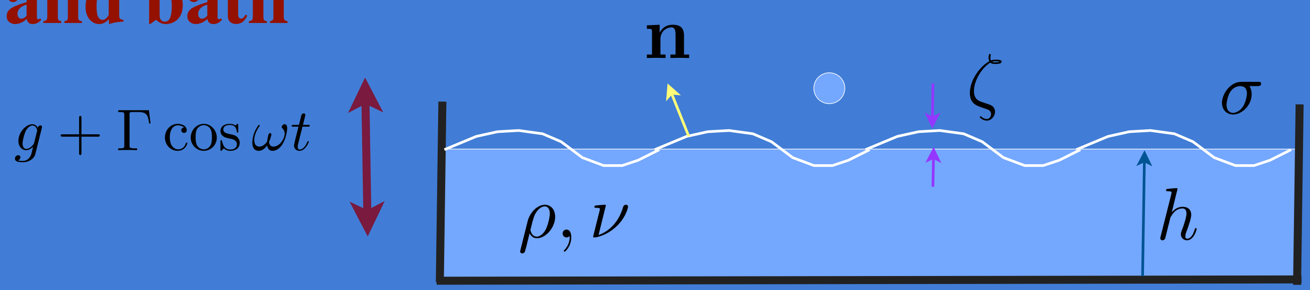


Notation

Symbol	Meaning	Typical value	
R_0	drop radius	$0.07 - 0.8$	mm
ρ	silicone oil density	$949 - 960$	kg/m^3
σ	drop surface tension	$20 - 21$	mN/m
g	gravitational acceleration	9.81	m/s^2
V_{in}	drop incoming speed	$0.1 \quad 1$	m/s
V_{out}	drop outgoing speed	$0.01 - 1$	m/s
μ	drop dynamic viscosity	$10^{-3} - 10^{-1}$	kg/(m·s)
ν	drop kinematic viscosity	$10 - 100$	cS
T_C	contact time	$1 - 20$	ms
C_R	$= V_{in}/V_{out}$ coefficient of restitution	$0 - 0.4$	
f	bath shaking frequency	$40 - 200$	Hz
ω	$= 2\pi f$ bath angular frequency	$250 - 1250$	$rad{\cdot}s^{-1}$
γ	peak bath acceleration	$0 \quad 70$	m/s^2
ω_D	$= (\sigma/\rho R_0^3)^{1/2}$ char. drop oscillation frequency	$300 - 5000$	s^{-1}

$\mathcal{W}e$	$= \rho R_0 V_{in}^2/\sigma$ Weber number	$0.01 - 1$
$\mathcal{B}o$	$= \rho g R_0^2/\sigma$ Bond number	$10^{-3} \quad 0.4$
$\mathcal{O}h$	$= \mu(\sigma\rho R_0)^{-1/2}$ drop Ohnesorge number	$0.004 - 2$
Ω	$= 2\pi f\sqrt{\rho R_0^3/\sigma}$ vibration number	$0 - 1.4$
Γ	$= \gamma/g$ peak nondim. bath acceleration	$0 - 7$

The coupling of drop and bath



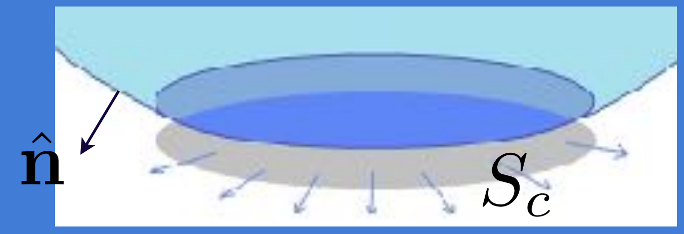
Bernoulli at free surface:

$$\rho \frac{\partial \phi}{\partial t} \;+\; \frac{1}{2}\,\rho\,\mathbf{u}^2 \;+\; \sigma \nabla \cdot \mathbf{n} + (g + \Gamma \cos \omega t)\zeta \;=\; -\,p_s$$

where drop applies $p_s(\mathbf{x}, t) \sim \cos \omega t/2$ at Faraday frequency

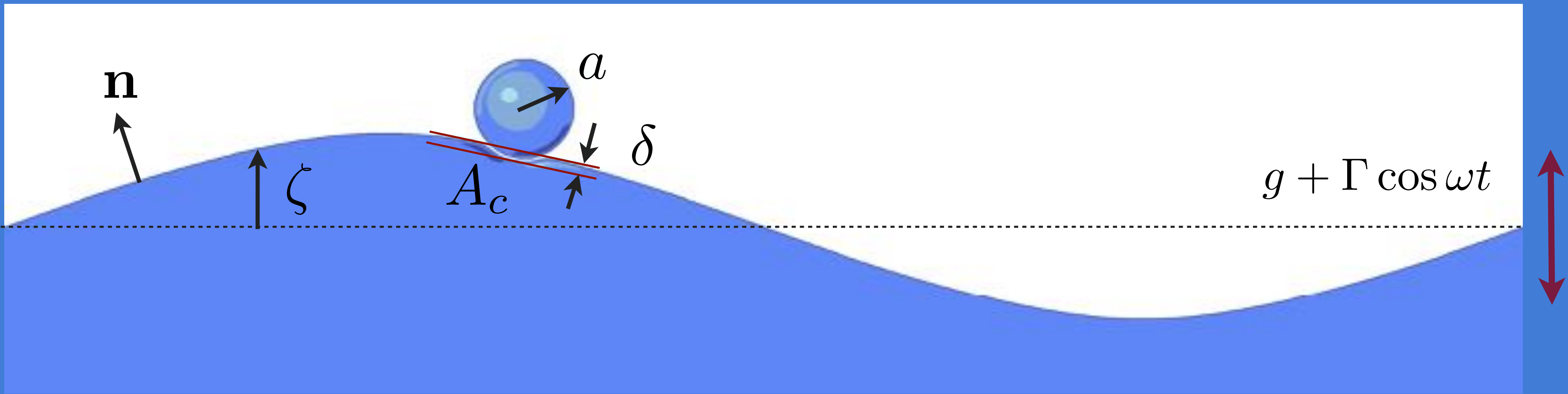
Force on drop: $\mathbf{F} \;=\; \int_S \mathbf{T} \cdot \hat{\mathbf{n}}\; dS$ where $T = -p\;\mathbf{I} + 2\mu\;\mathbf{E}$

Drop trajectory: free flight plus impact



- lubrication layer communicates p_s between droplet and bath, and applies tangential viscous force

Normal force during impact: $\mathbf{F}_n \;=\; \int_{S_c} -p_s\;\hat{\mathbf{n}}\;dS$



$$\mathbf{F}_n = -\int_{S_c} \left[\rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho\, \mathbf{u}^2 + \sigma \nabla \cdot \mathbf{n} + \cancel{\rho(g + \Gamma \cos \omega t)\zeta} \right] \hat{\mathbf{n}}\, dS$$

INERTIA CURVATURE HYDROSTATIC

Scaling

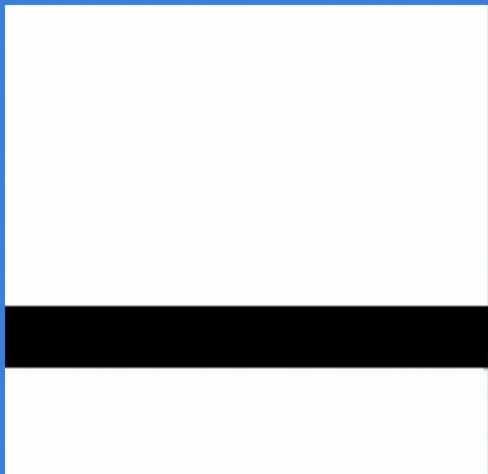
Bond number: $$Bo = \frac{\text{HYDROSTATIC}}{\text{CURVATURE}} \sim \frac{\rho \Gamma \zeta}{\sigma / a} \sim 0.1$$

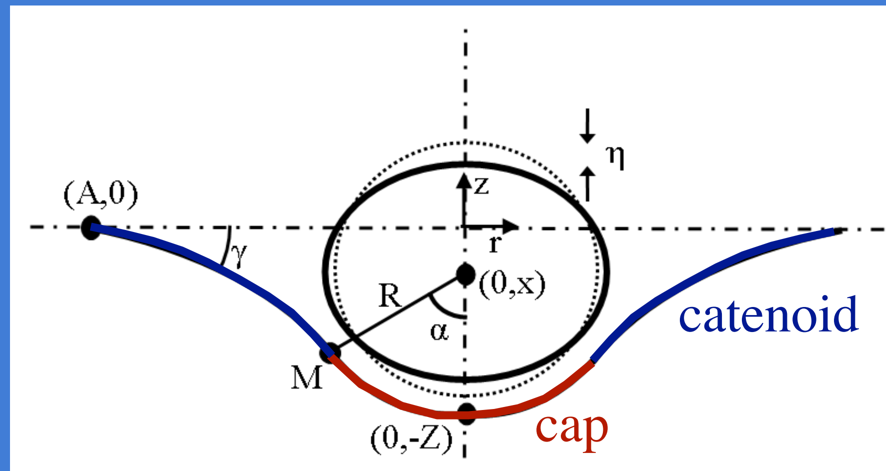
$$We = \frac{\text{INERTIA}}{\text{CURVATURE}} \sim \frac{\rho u^2 / 2}{\sigma / a} \sim 0.3$$

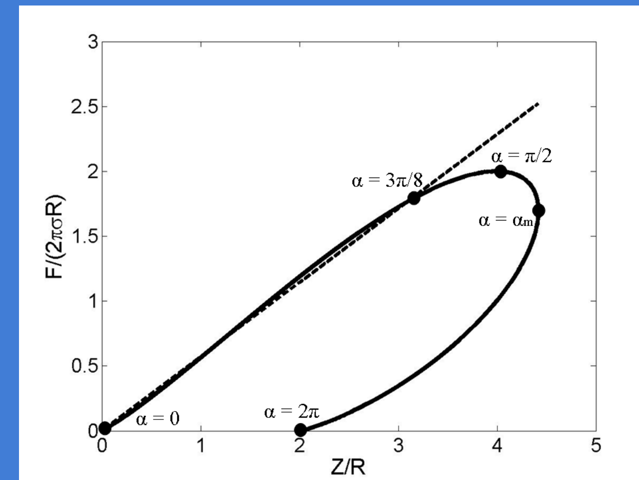
Note: in limit of $(Bo, We) \ll 1$, curvature pressure dominates

Droplets bouncing on a vibrating soap film

Gilet & Bush (2009ab)







- quasi-static interface: consists of spherical cap plus catenoid
- interface behaves like linear spring with spring constant $k \sim \sigma$
- form of dissipation indicated by experiment

$$\boxed{m\ddot{Z} = mg - kZH(-Z) - DH(-Z)\dot{Z}|\dot{Z}| - mg\Gamma\cos(\Omega t + \phi)}$$

SPRING **DISSIPATION** **DRIVING**

- adequately described all bouncing states observed on a driven soap film

Linear Spring Model

- follow Gilet & Bush's modeling of the interface as a linear spring
- assume interface, drop both recover between impacts, air drag negligible

$$\boxed{m_D \ddot{Z} + \mathrm{H}(-Z)\left[\sqrt{m_D \sigma} D \dot{Z} + \sigma C Z\right] = -m_D g \left(1 + \Gamma \sin \omega t\right)}$$

drop acceleration

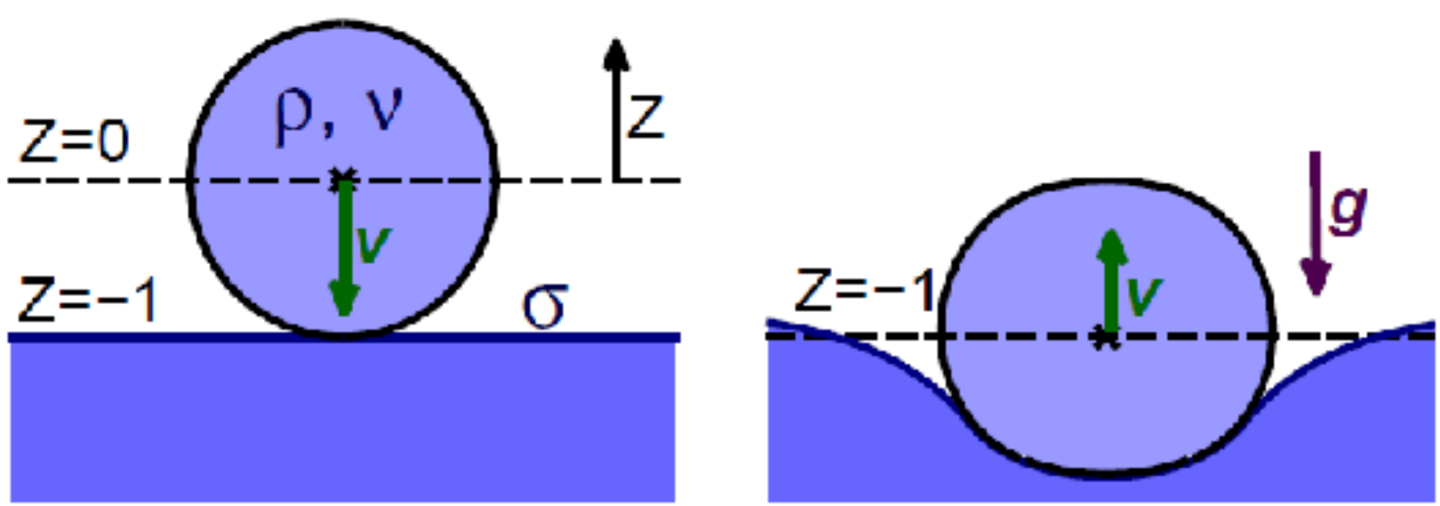
Heaviside function: surface only acts during contact

Drag proportional to speed

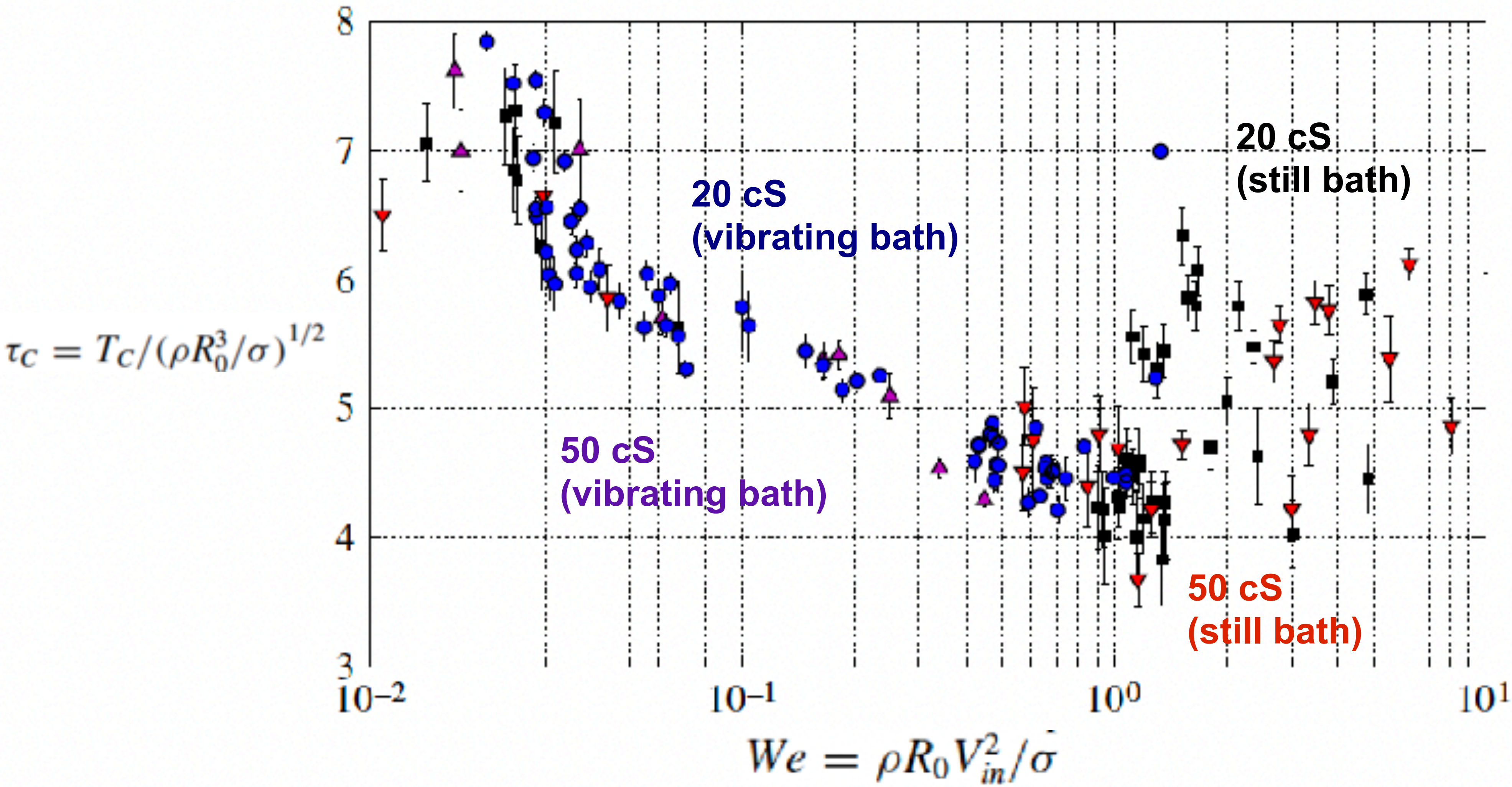
Surface tension: linear spring

gravity in bath frame of reference

where C, D are constants deduced by matching data for $T_C, C_R(We)$



Contact time

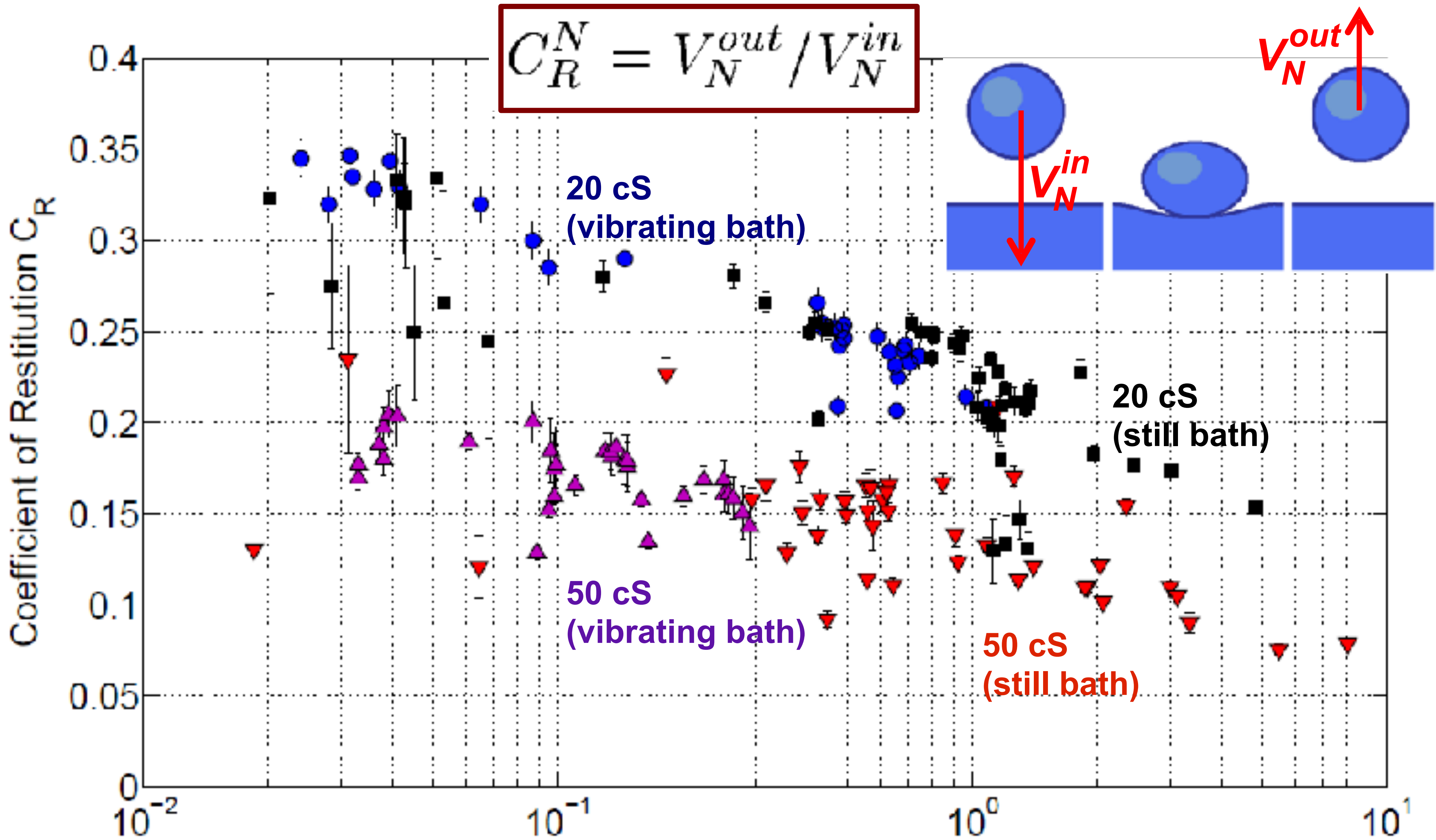


Stationary bath: 20 cSt (■) and 50 cSt (▼)

Vibrating bath: 20 cSt (●) and 50 cSt (▲)

Normal Coefficient of Restitution

$$C_R^N = V_N^{out} / V_N^{in}$$



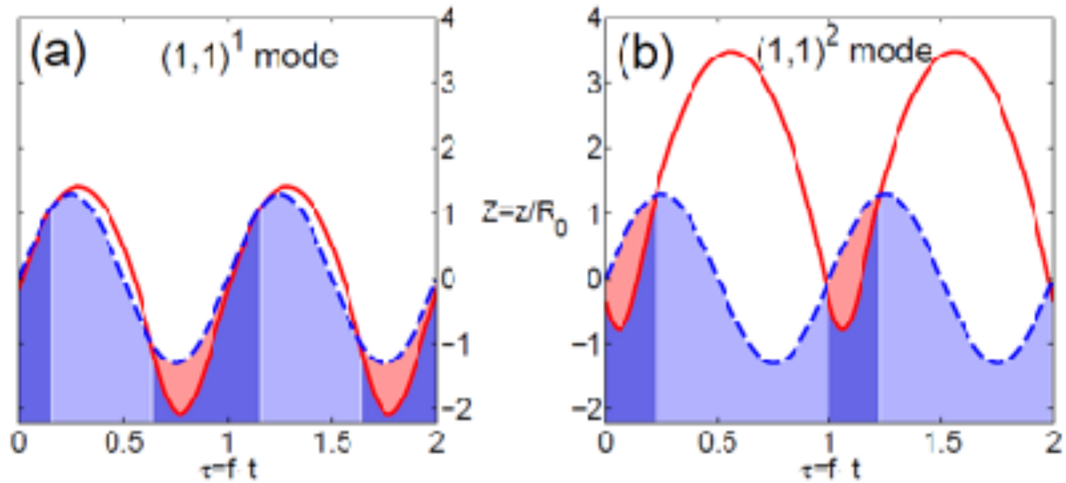
$$We = \rho R_0 V_{in}^2 / \sigma$$

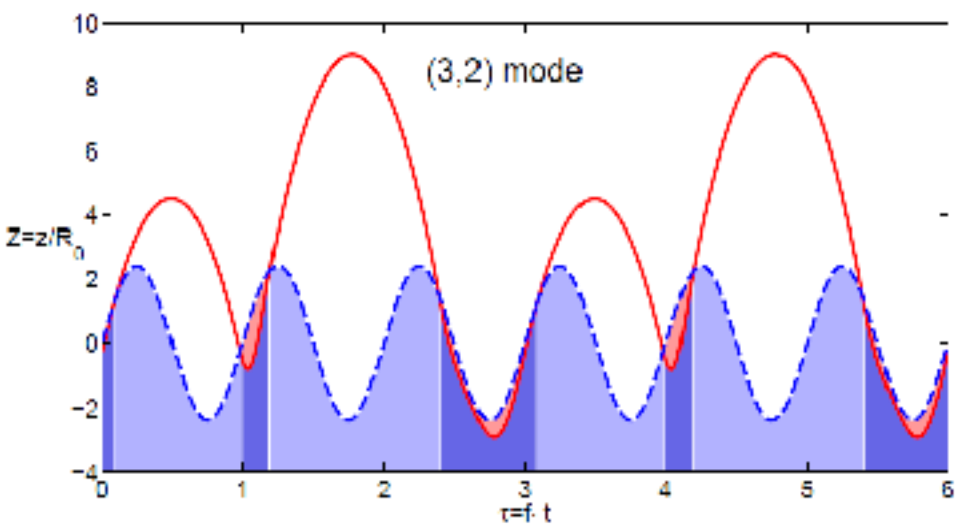
Linear spring model captures bouncing modes

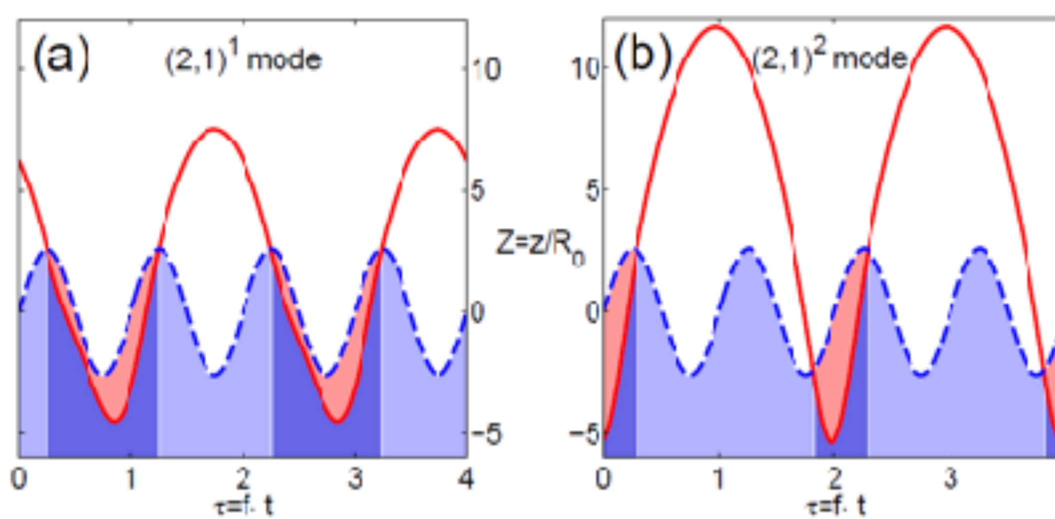
- in the (m, n) modes, the drop bounces n times in m driving periods
- multiple (m, n) modes may coexist: differentiated with superscript
- mechanical energy of

$$(m, n)^2 > (m, n)^1$$

Lower energy **Higher energy**

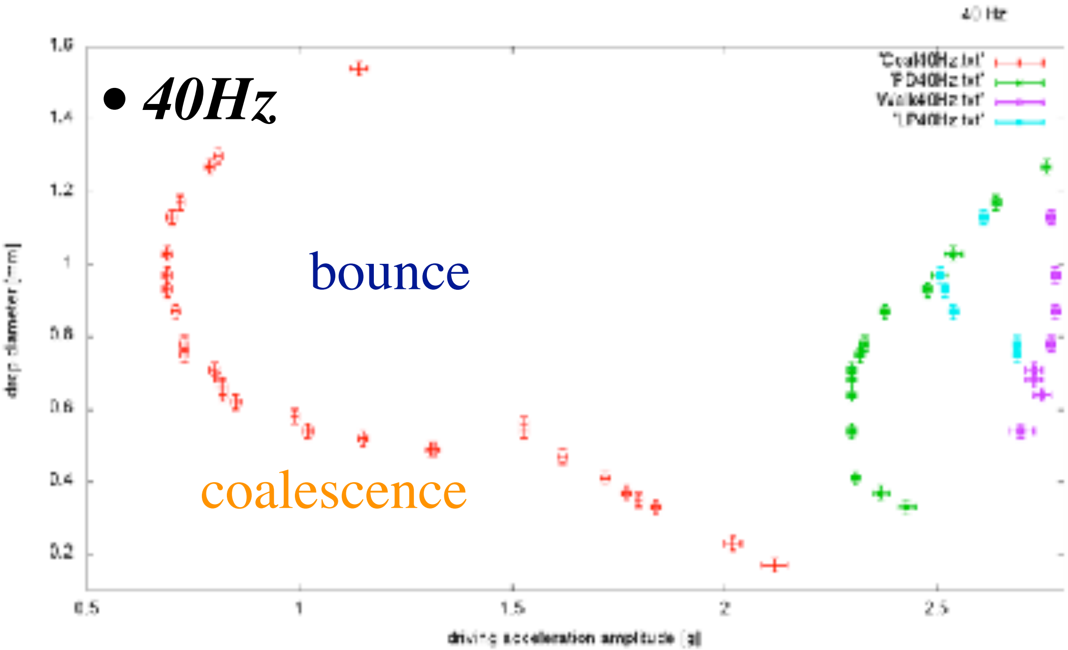


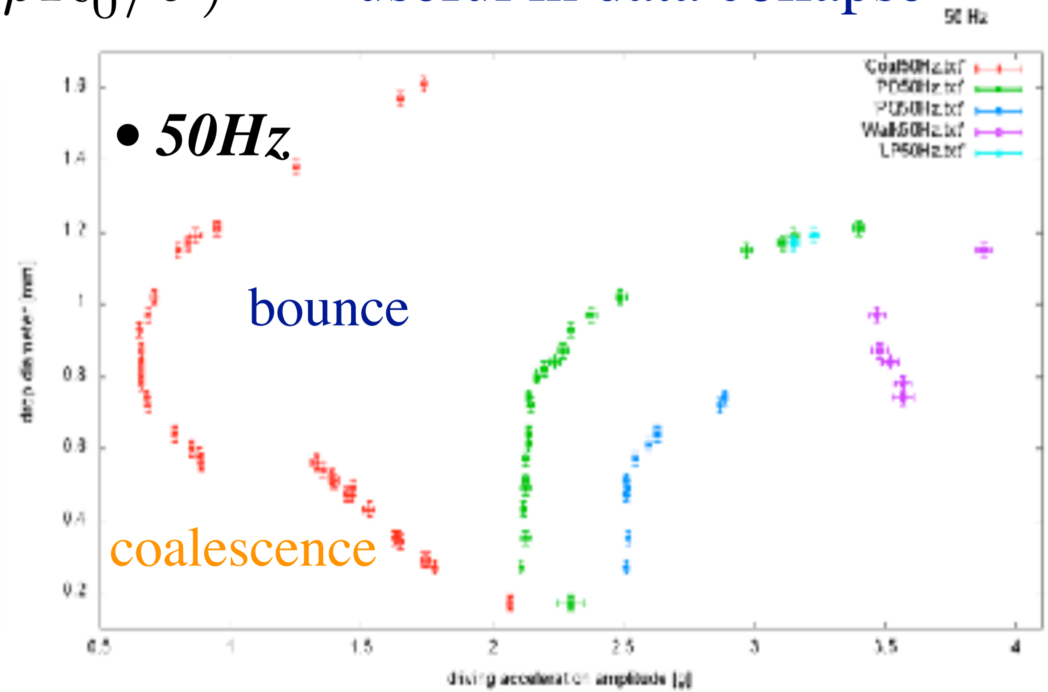


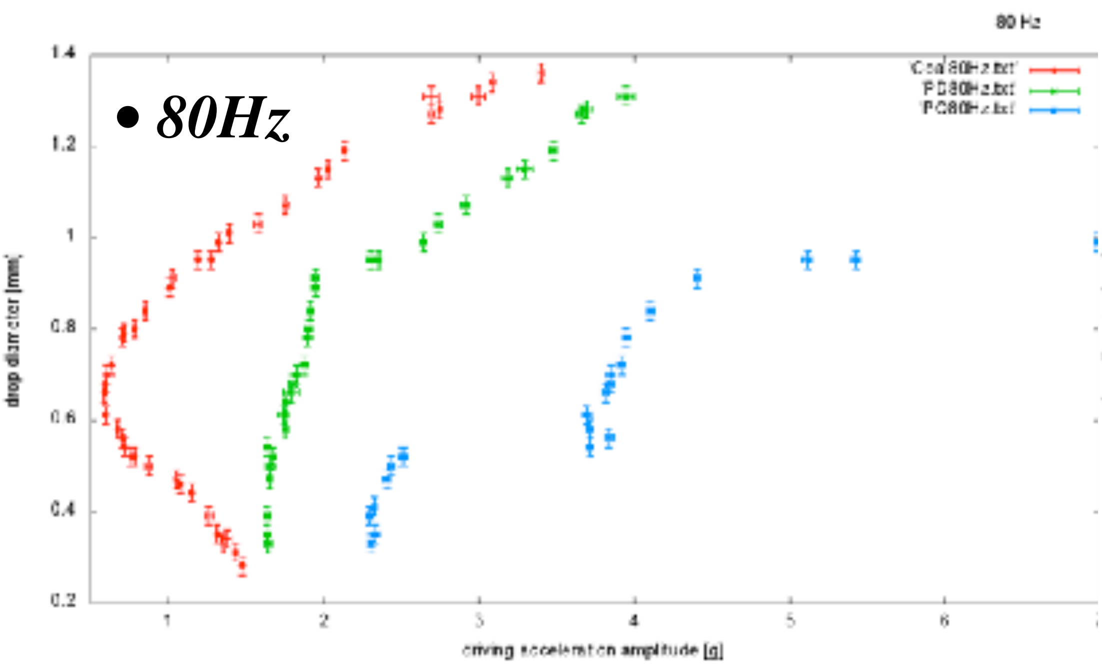


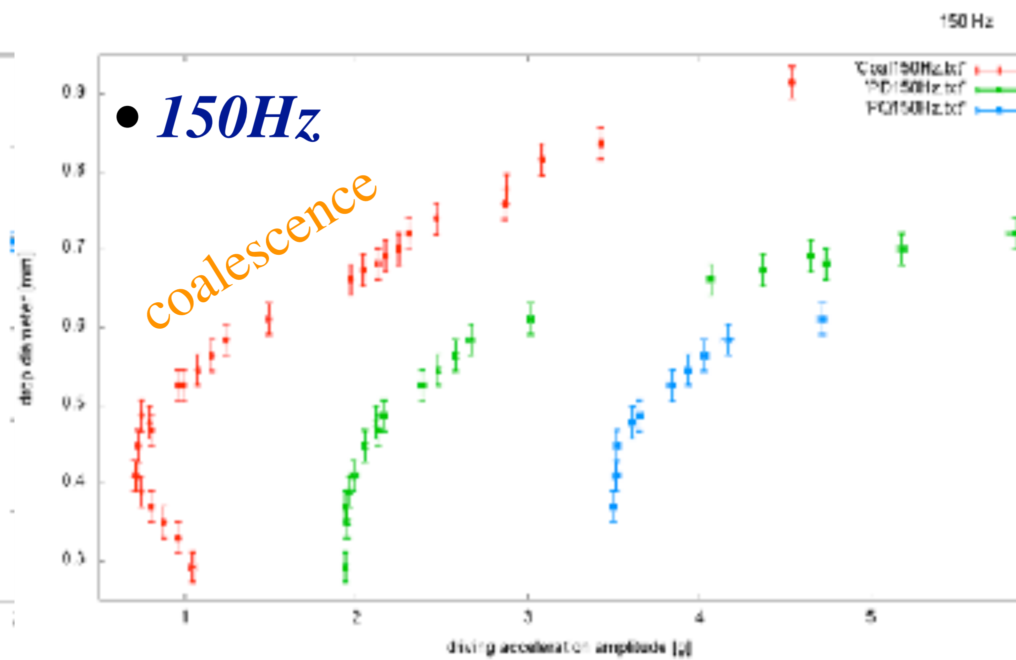
Coalescence threshold

Vibration number: $\Omega = 2\pi f(\rho R_0^3/\sigma)^{1/2}$ useful in data collapse



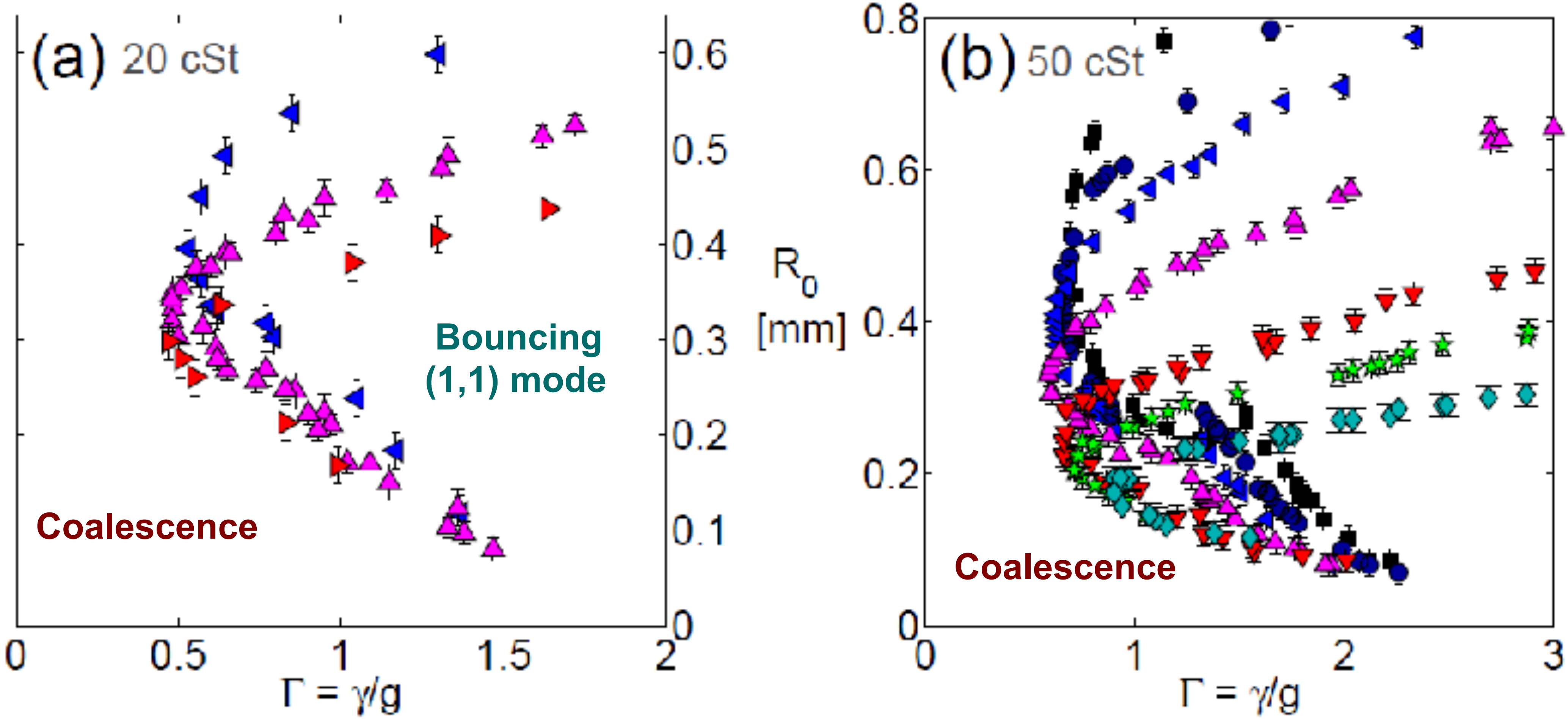






Bouncing thresholds

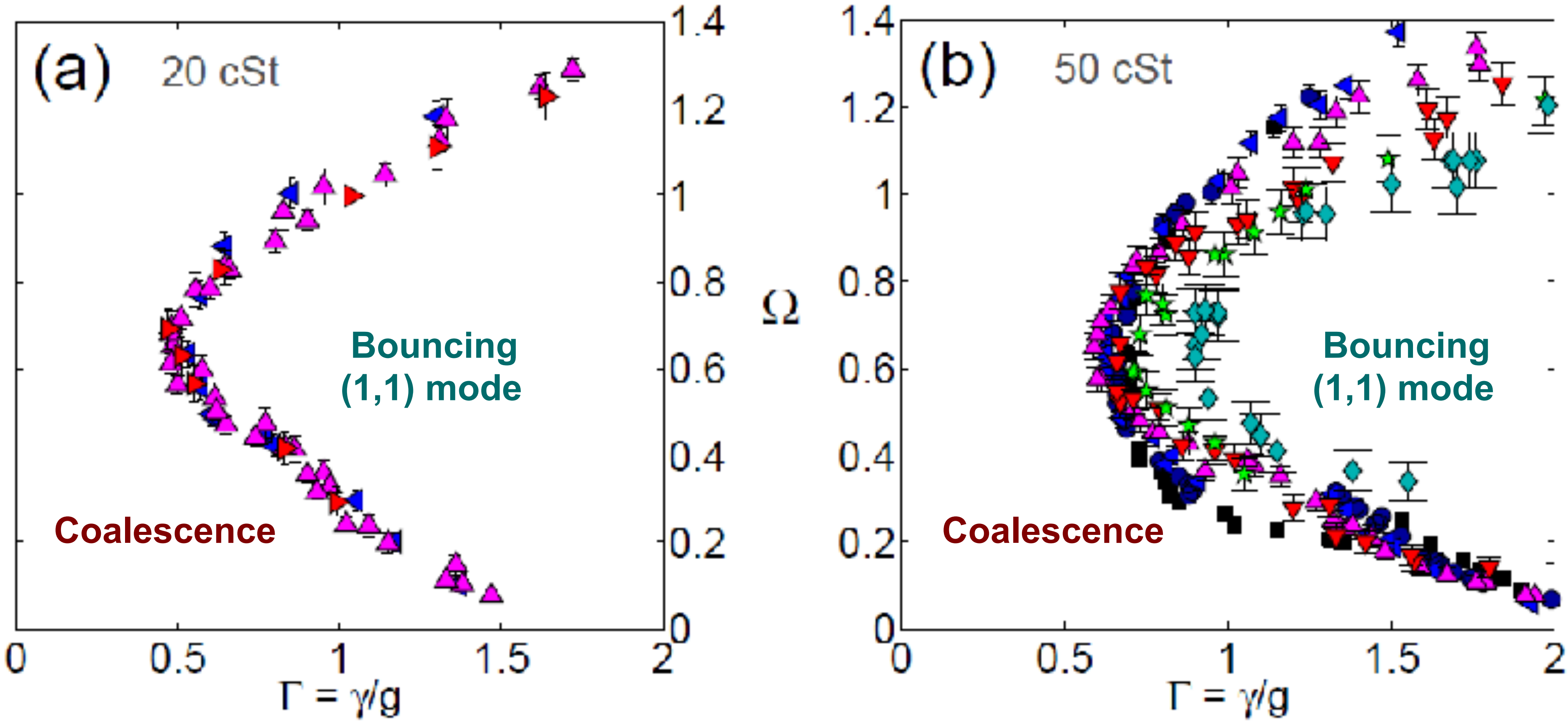
- critical $\Gamma = \gamma/g$ below which drops coalesce, above which they bounce



f=40 Hz (■), 50 Hz (●), 60 Hz (◄), 80 Hz (▲), 100 Hz (►), 120 Hz (▼), 150 Hz (★) and 200 Hz (♦).

Bouncing thresholds

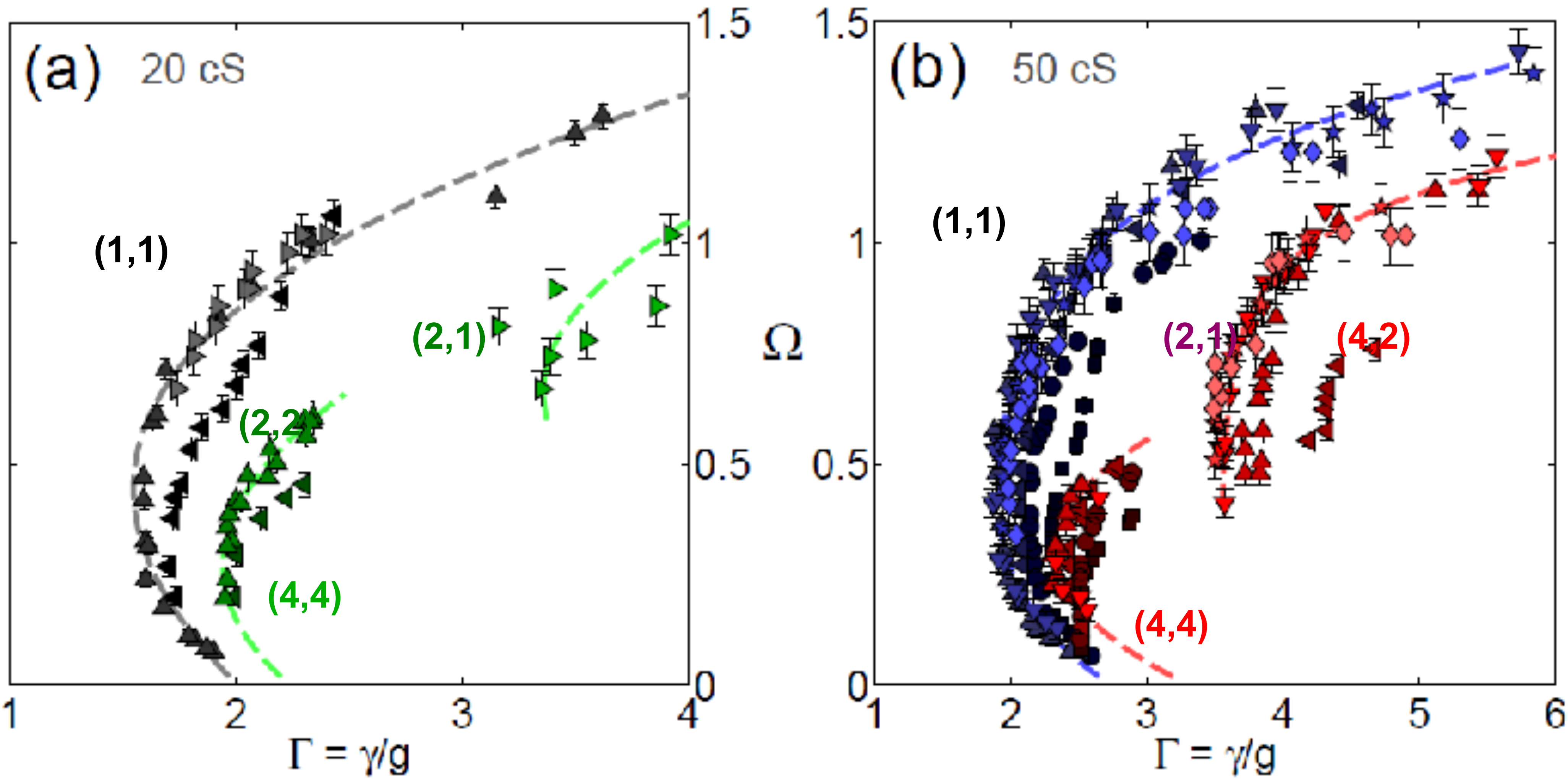
- critical $\Gamma = \gamma/g$ below which drops coalesce, above which they bounce



f=40 Hz (■), 50 Hz (●), 60 Hz (◄), 80 Hz (▲), 100 Hz (►), 120 Hz (▼), 150 Hz (★) and 200 Hz (♦).

Vibration number: $\Omega = 2\pi f(\rho R_0^3/\sigma)^{1/2}$ useful in data collapse

Period-doubling transitions



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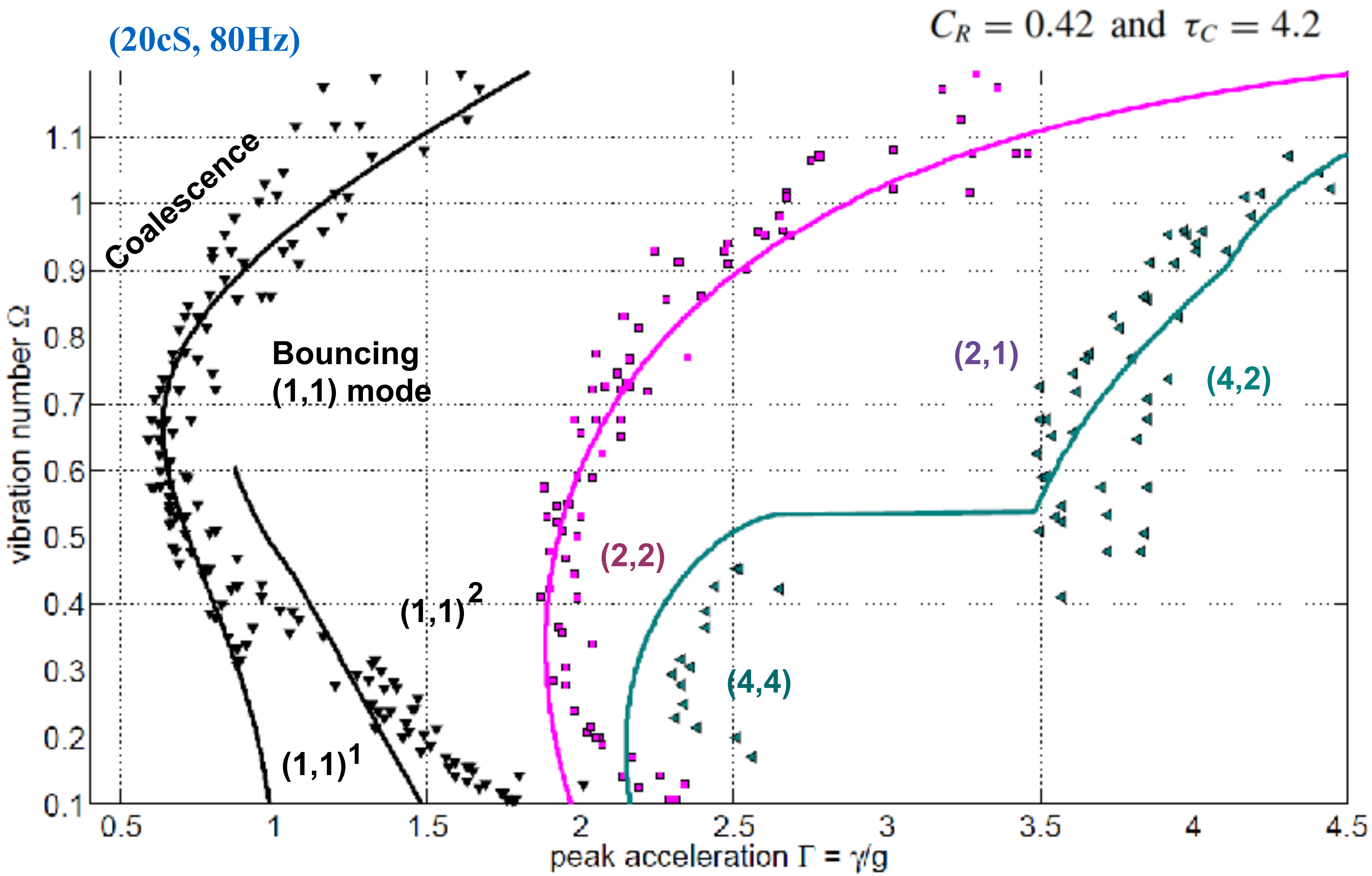
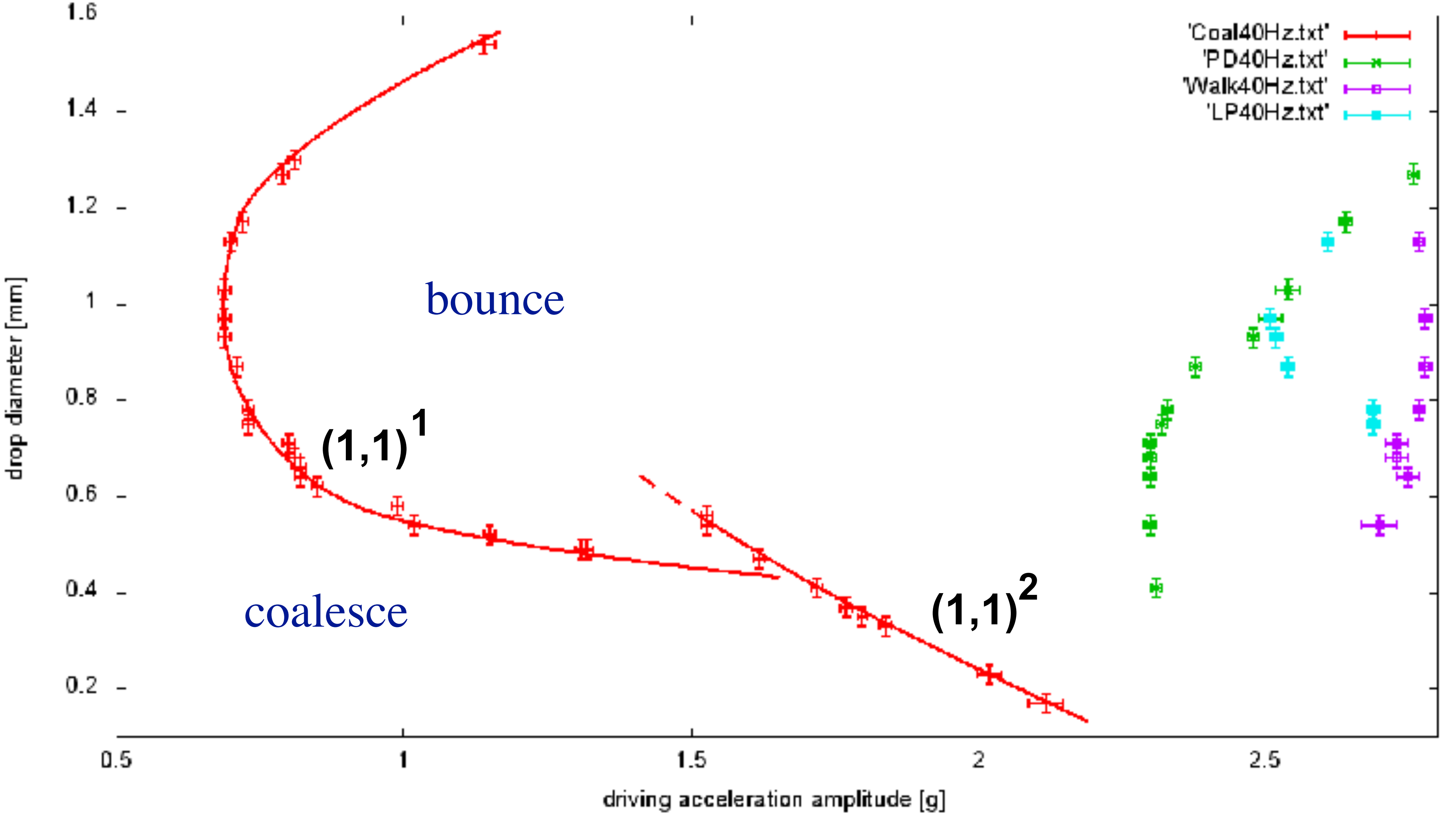
Vibration number: $\Omega = 2\pi f(\rho R_0^3/\sigma)^{1/2}$

Linear Spring Model



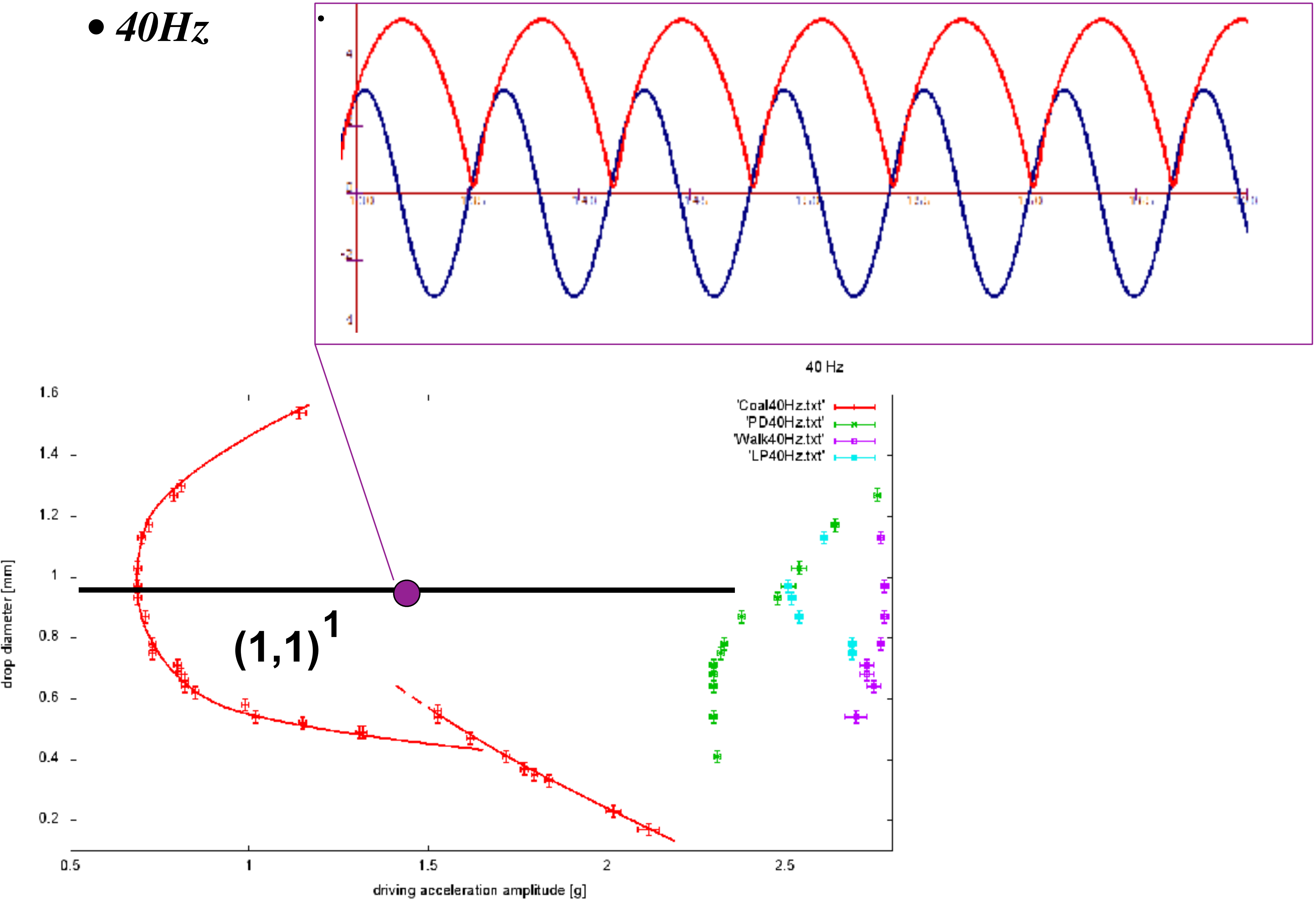
Bouncing threshold discontinuous

- *40Hz*

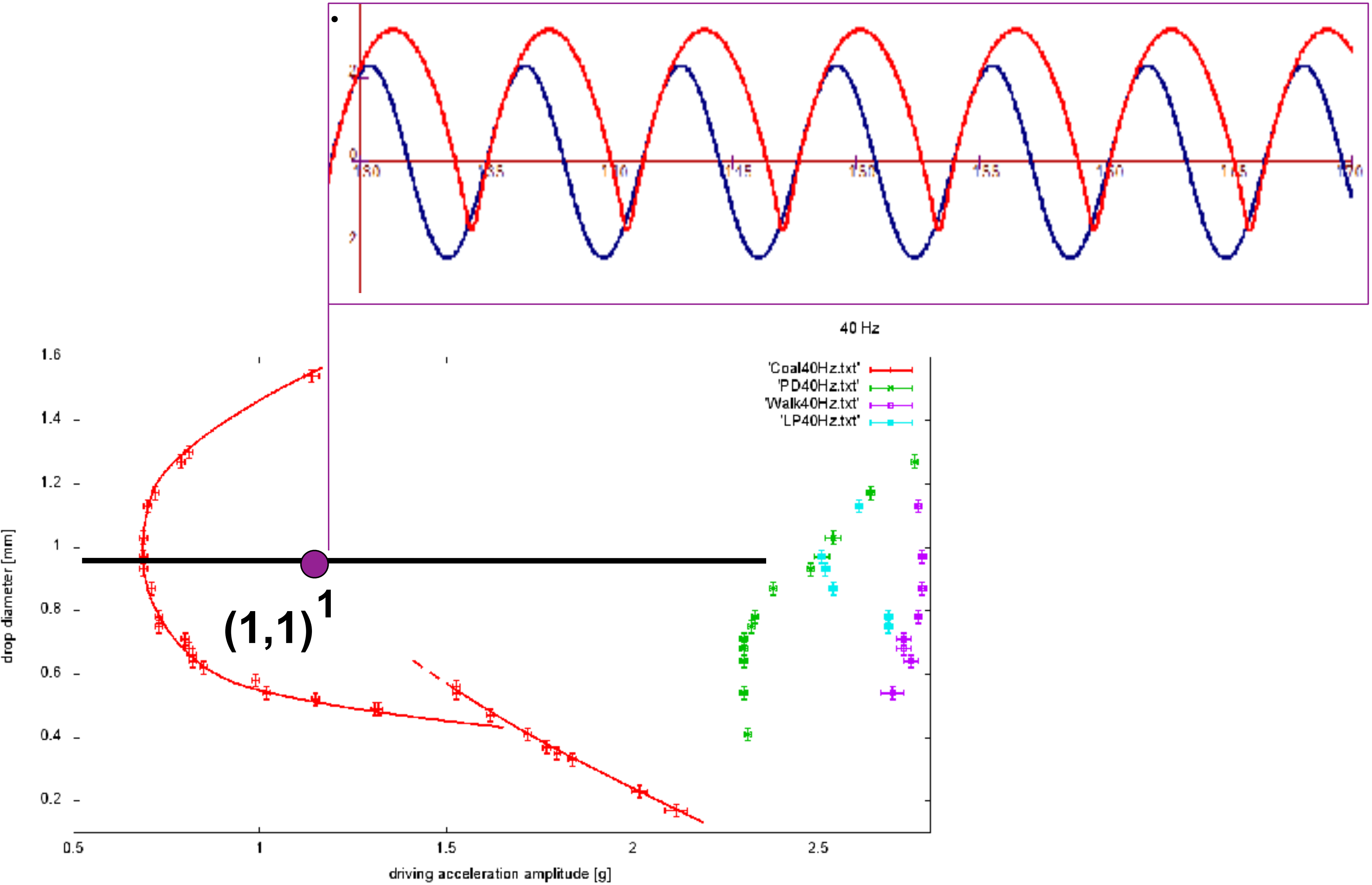


Approach to coalescence from (1,1)[1] state

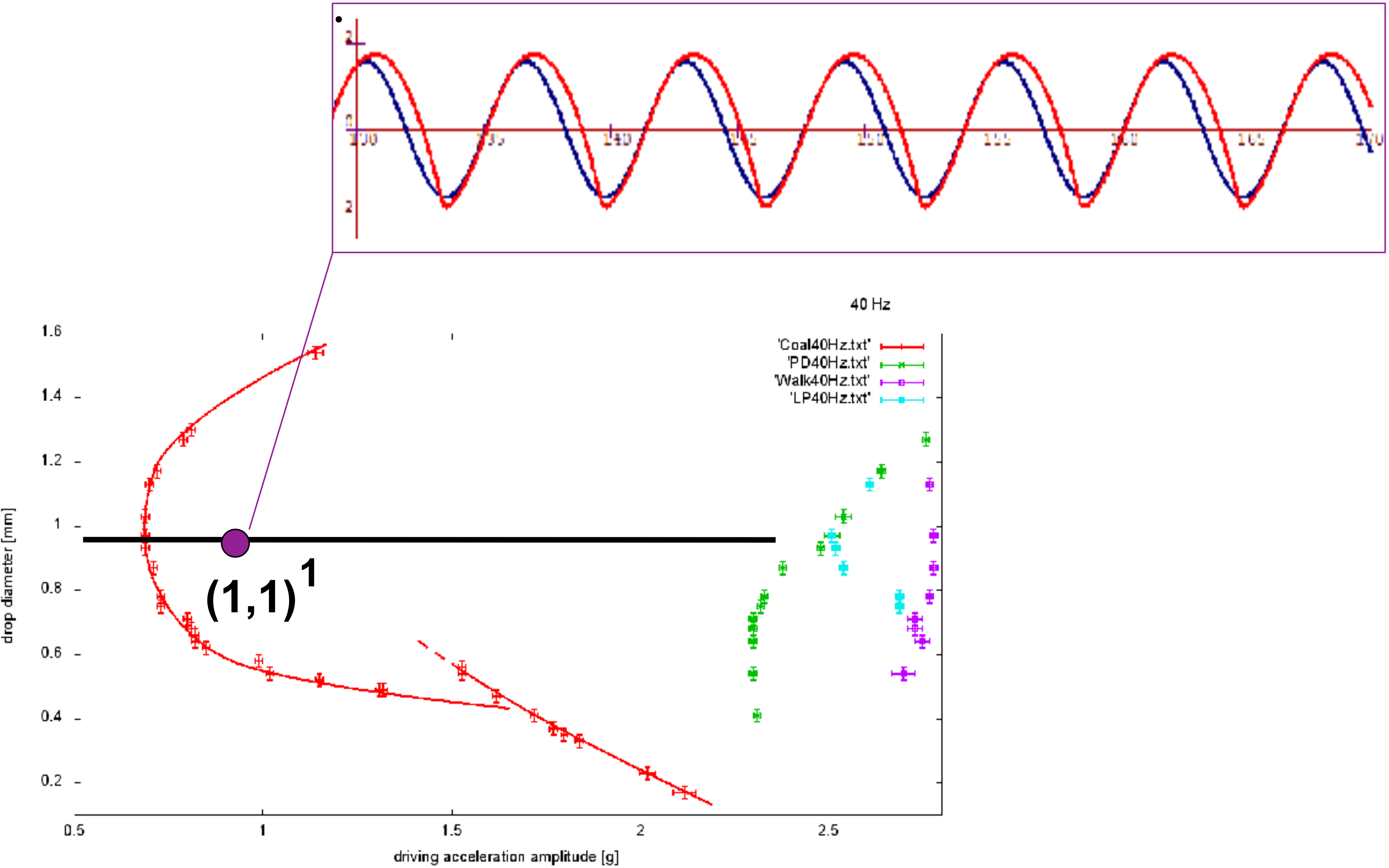
- *40Hz*



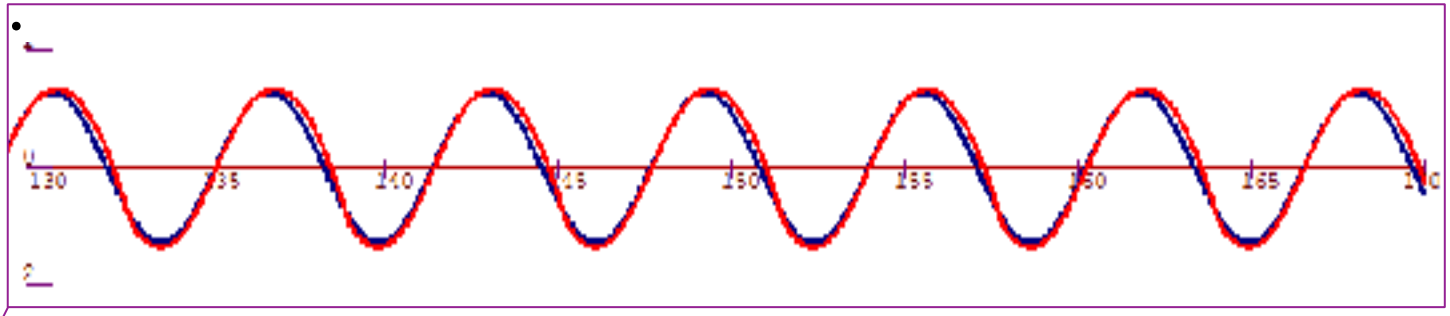
Approach to coalescence from (1,1)[1] state

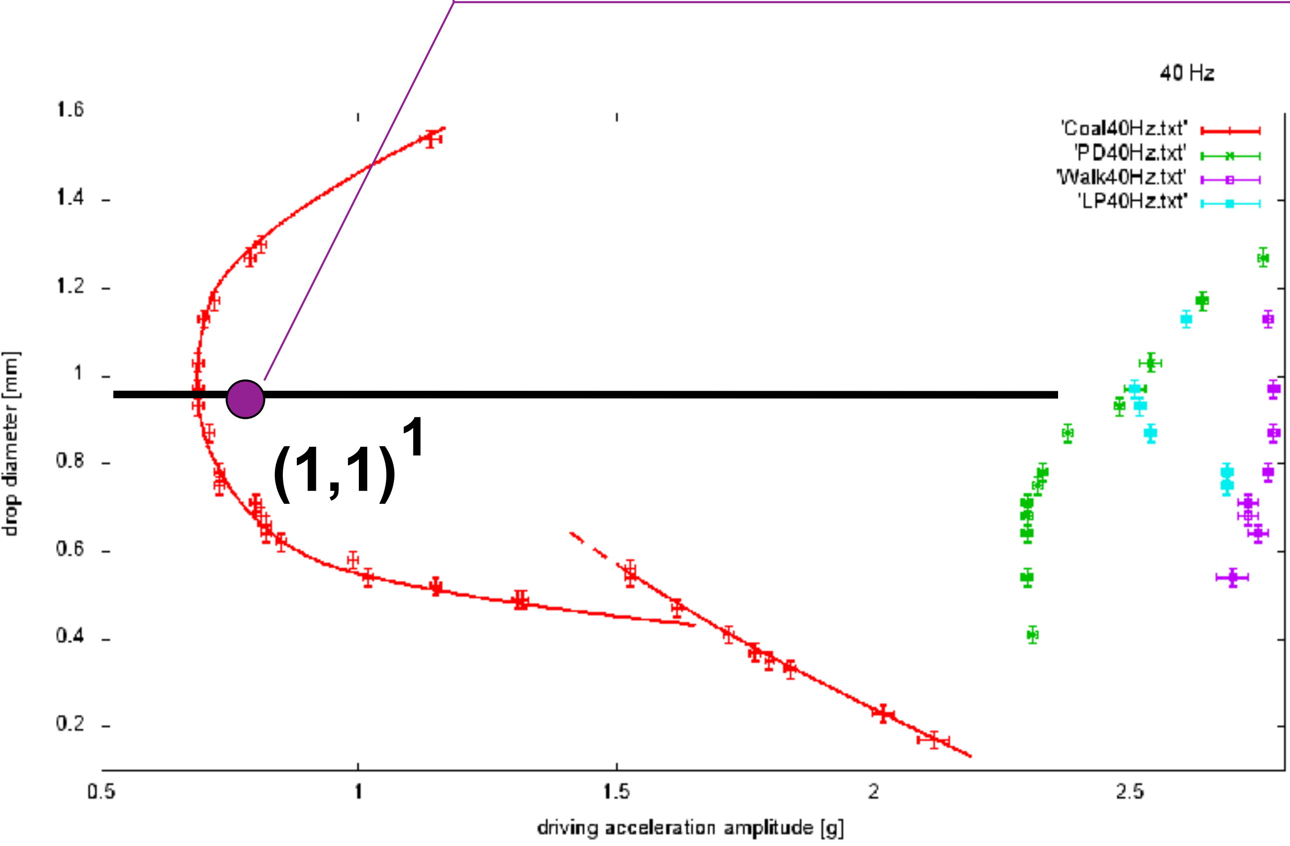


Approach to coalescence from $(1,1)^1$ state

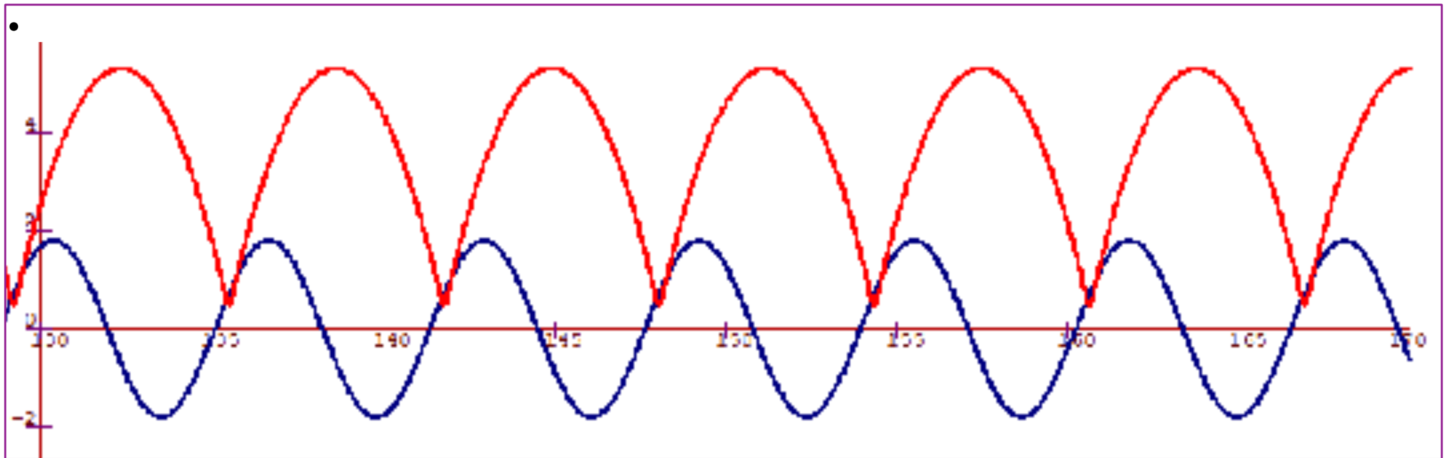


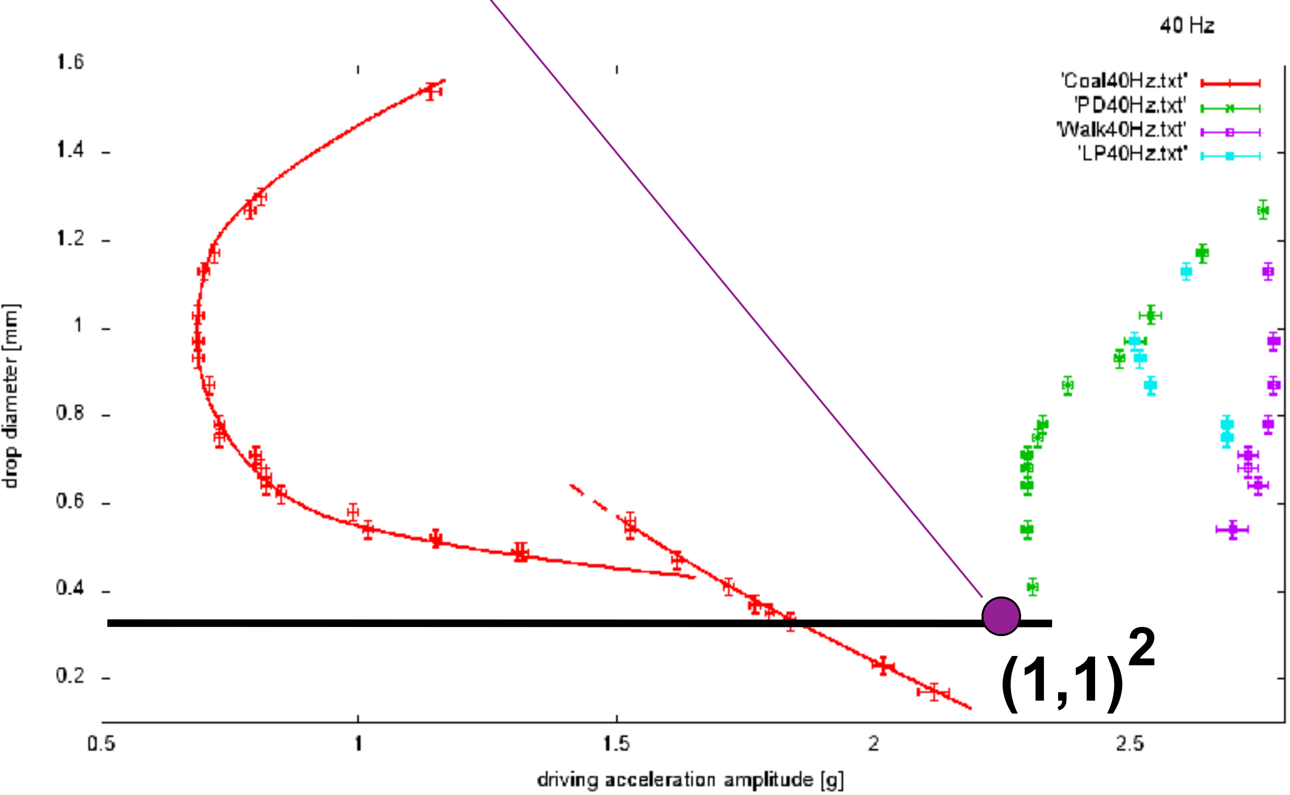
Approach to coalescence from $(1,1)^1$ state



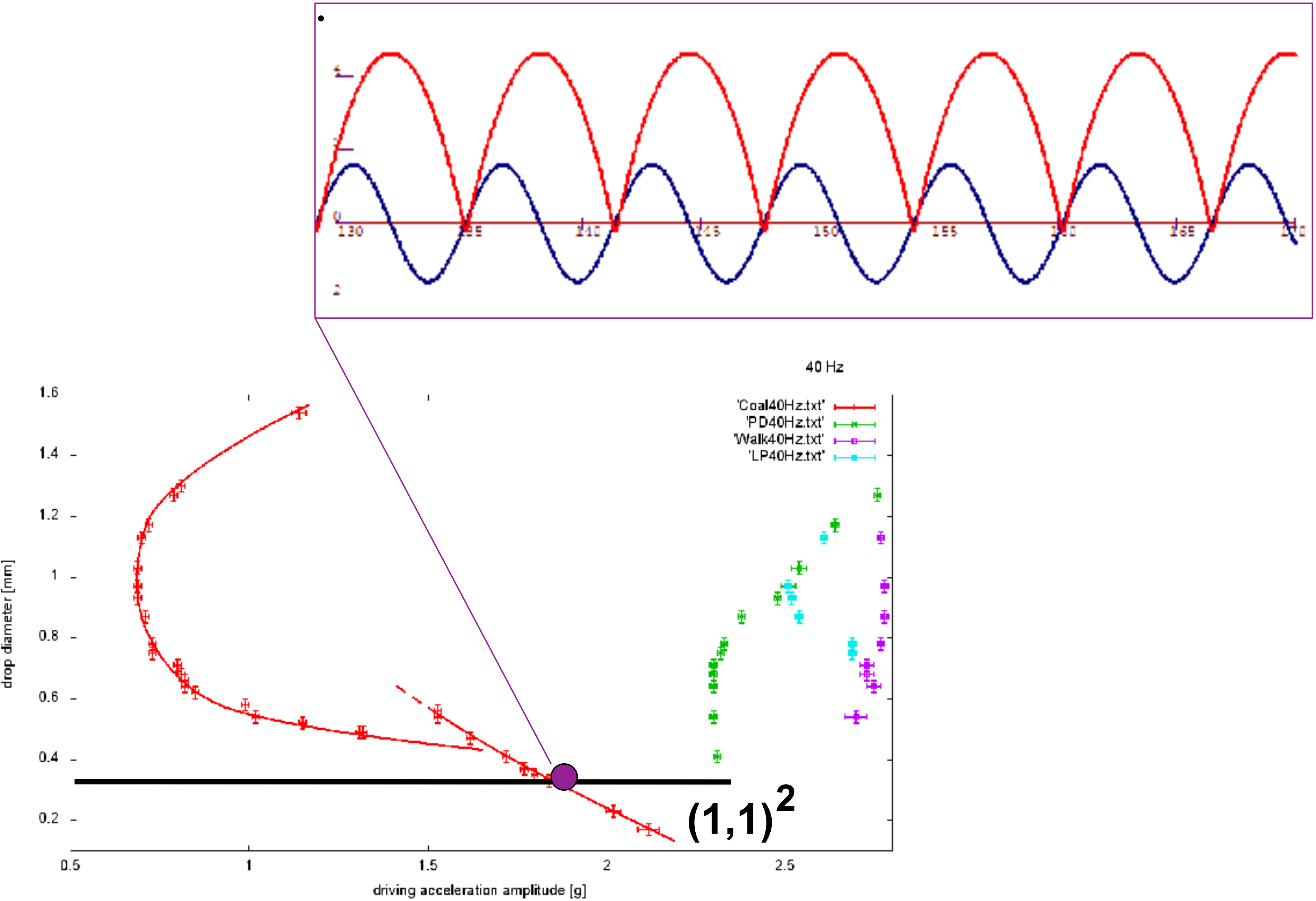


Approach to coalescence from $(1,1)^2$ state

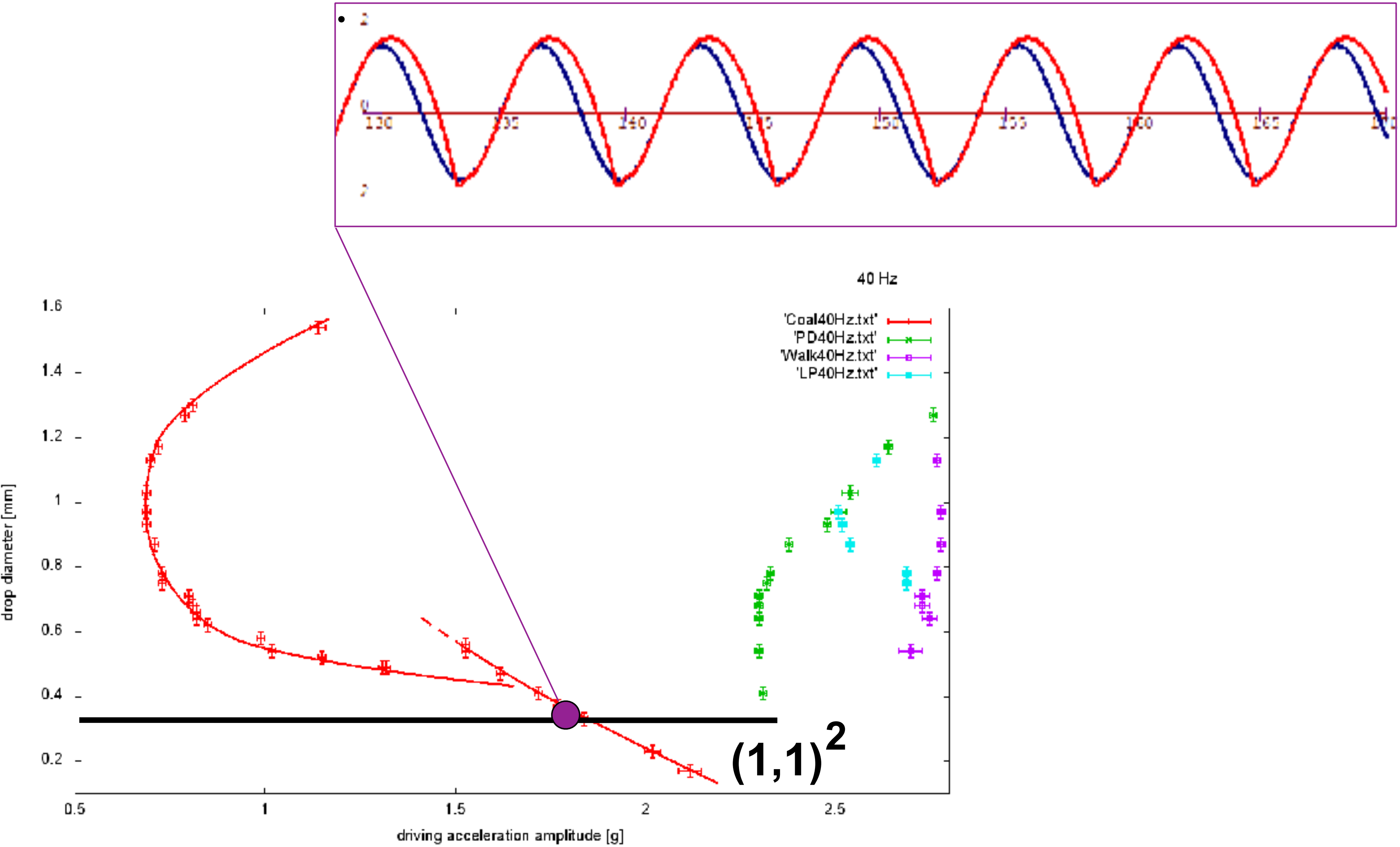




Approach to coalescence from $(1,1)^2$ state



Approach to coalescence from $(1,1)^2$ state

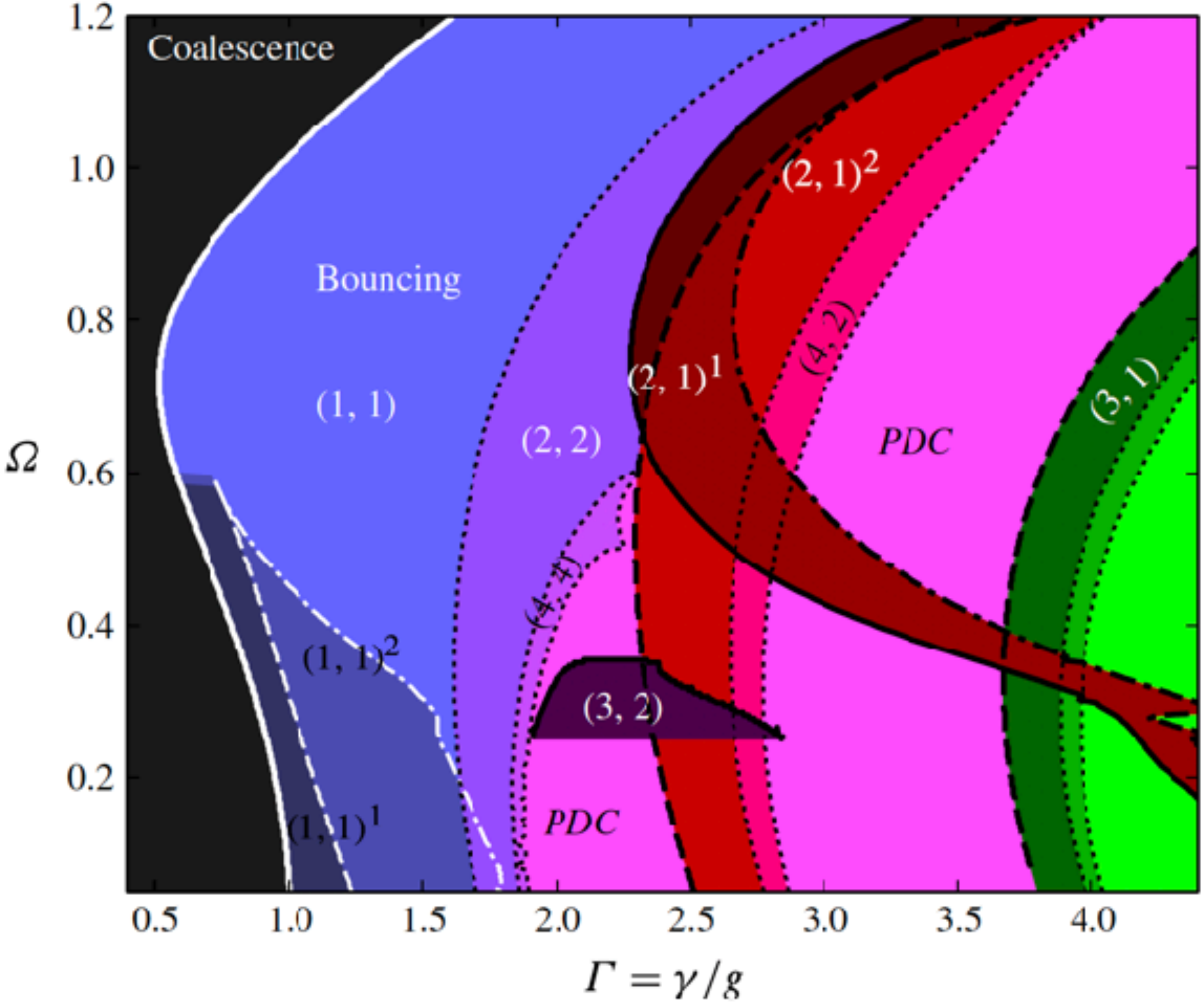


Molacek & Bush (2013a)

Regime diagram: predictions of linear spring

$$\Omega = 2\pi f(\rho R_0^3/\sigma)^{1/2}$$

(20cS, 80Hz)

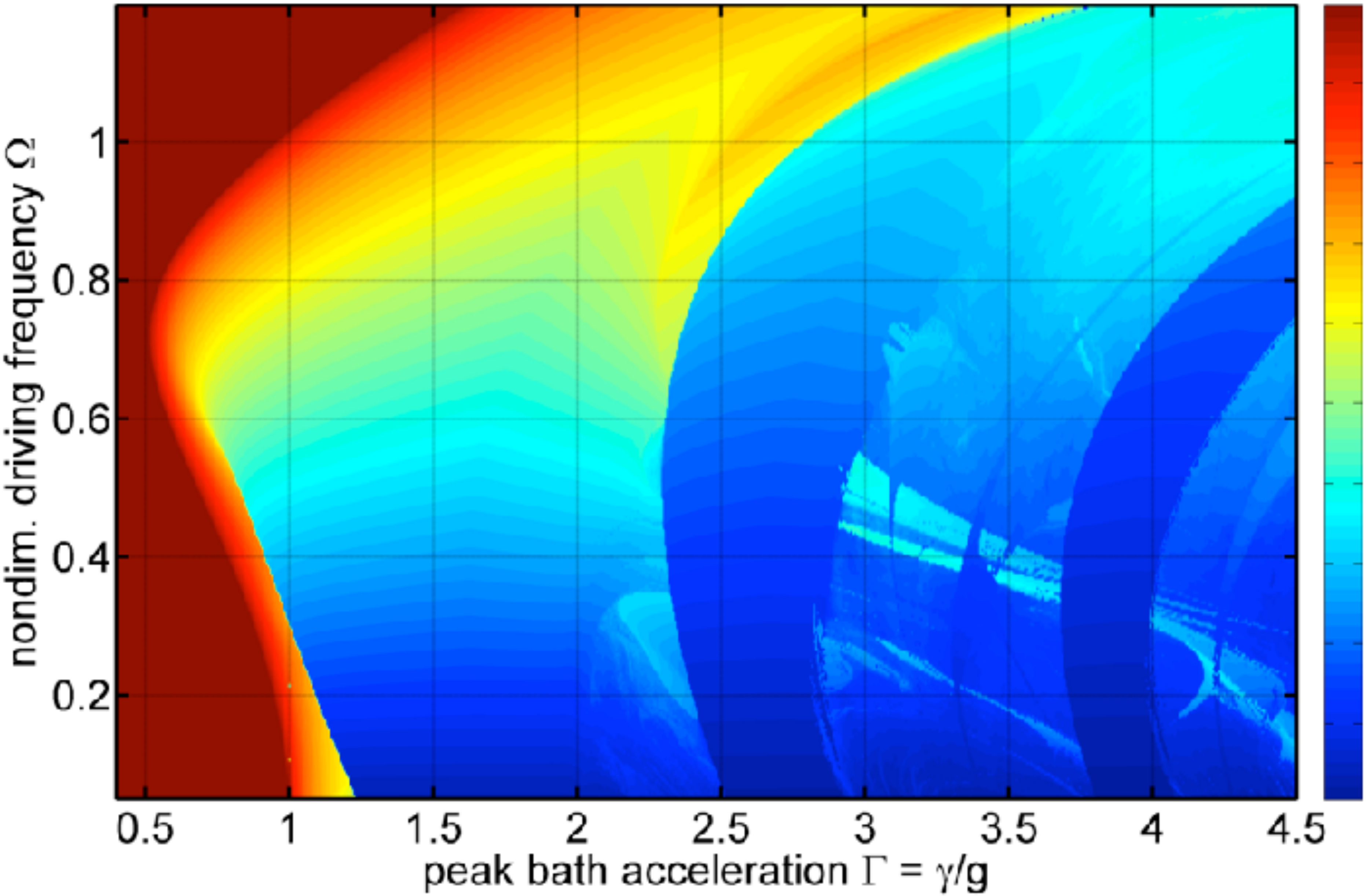


(20cS, 80Hz)

Contact time

$$\Omega = 2\pi f(\rho R_0^3/\sigma)^{1/2}$$

- **predicted on the basis of the linear spring model for the interface**



Molacek & Bush (2012a)

Logarithmic Spring Model

- developed to address perceived shortcomings in linear spring model
- spring softer for smaller deformations

$$\boxed{m_D \ddot{Z}\left(1+\frac{c_3}{Q^2(Z)}\right)+\frac{\mu R_0 c_2}{Q(Z)}\dot{Z}+\frac{3\sigma Z}{2Q(Z)}=-m_D g\left(1+\Gamma\sin\omega t\right)}$$

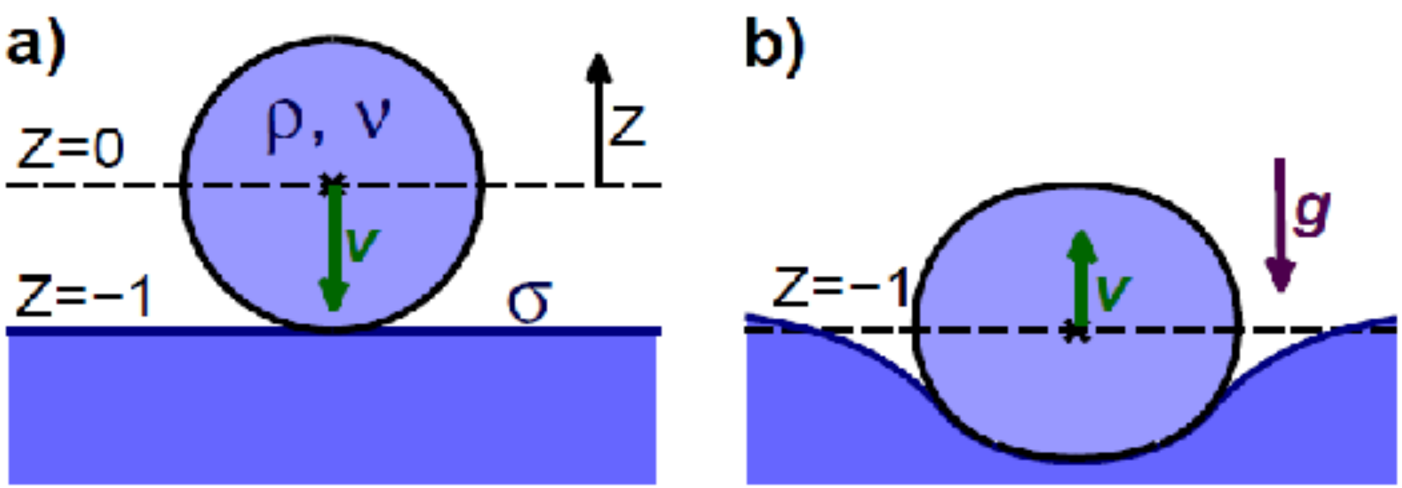
internal fluid motion

nonlinear drag

logarithmic spring

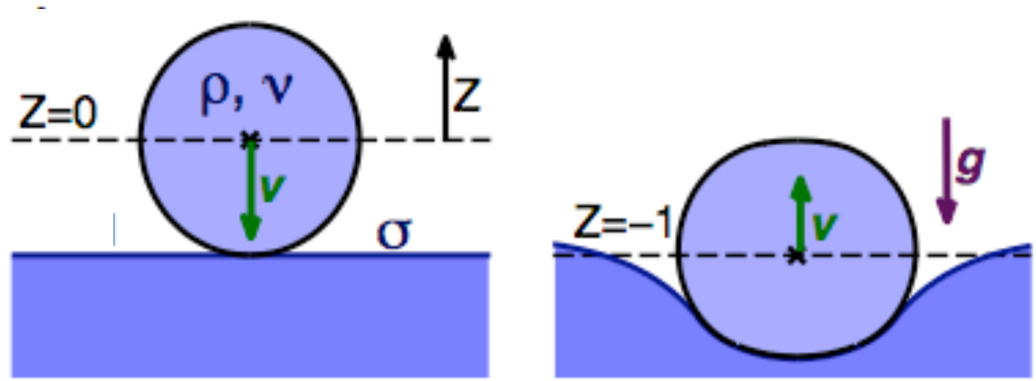
gravity in bath frame of reference

- Z denotes penetration depth, and $Q(Z)=\ln\left(c_1 R_0/|Z|\right)$



Logarithmic spring

- assume both bath and drop take quasi-static forms
- incorporate effects of fluid inertia



Molacek & Bush (2013)

Free flight: $$m\ddot{z} = -mg^*(t)$$

Impact: $$\left(1+\frac{c_3}{\ln^2\left|\frac{c_1 R_0}{Z}\right|}\right)m\ddot{z} + \frac{4}{3}\frac{\pi\mu R_0 c_2(\nu)}{\ln\left|\frac{c_1 R_0}{Z}\right|}\dot{Z} + \frac{2\pi\sigma Z}{\ln\left|\frac{c_1 R_0}{Z}\right|} = -mg^*(t)$$

ADDED MASS — DRAG — LOGARITHMIC SPRING — GRAVITY

where $g^*(t) = g + \gamma\sin(2\pi f t)$ is the effective gravity

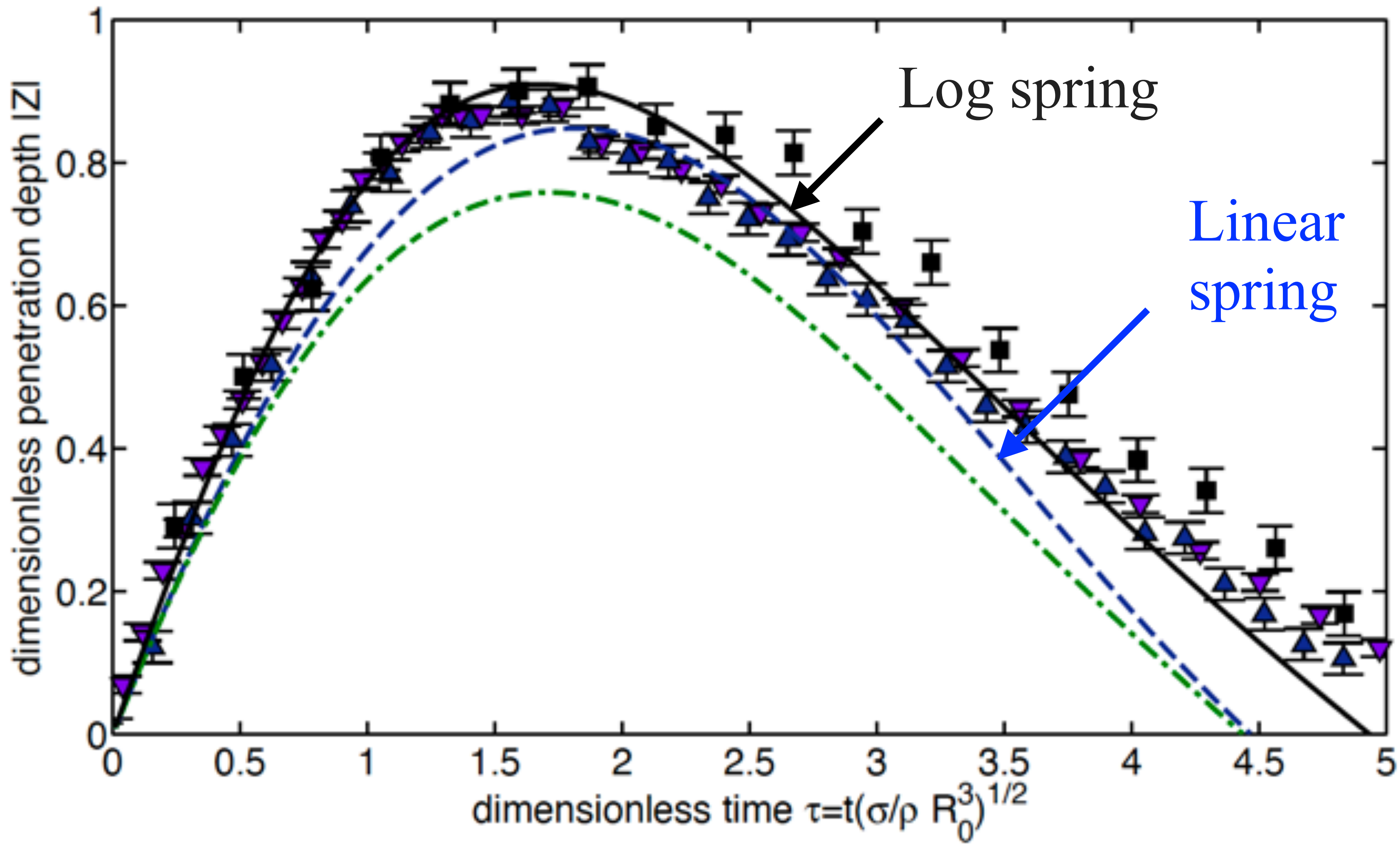
$c_1, c_2(\nu), c_3$ are constants deduced by matching data for $T_C, C_R(We)$

- assume interface recovers between impacts: allows for characterization of low energy bouncing states

Penetration depth

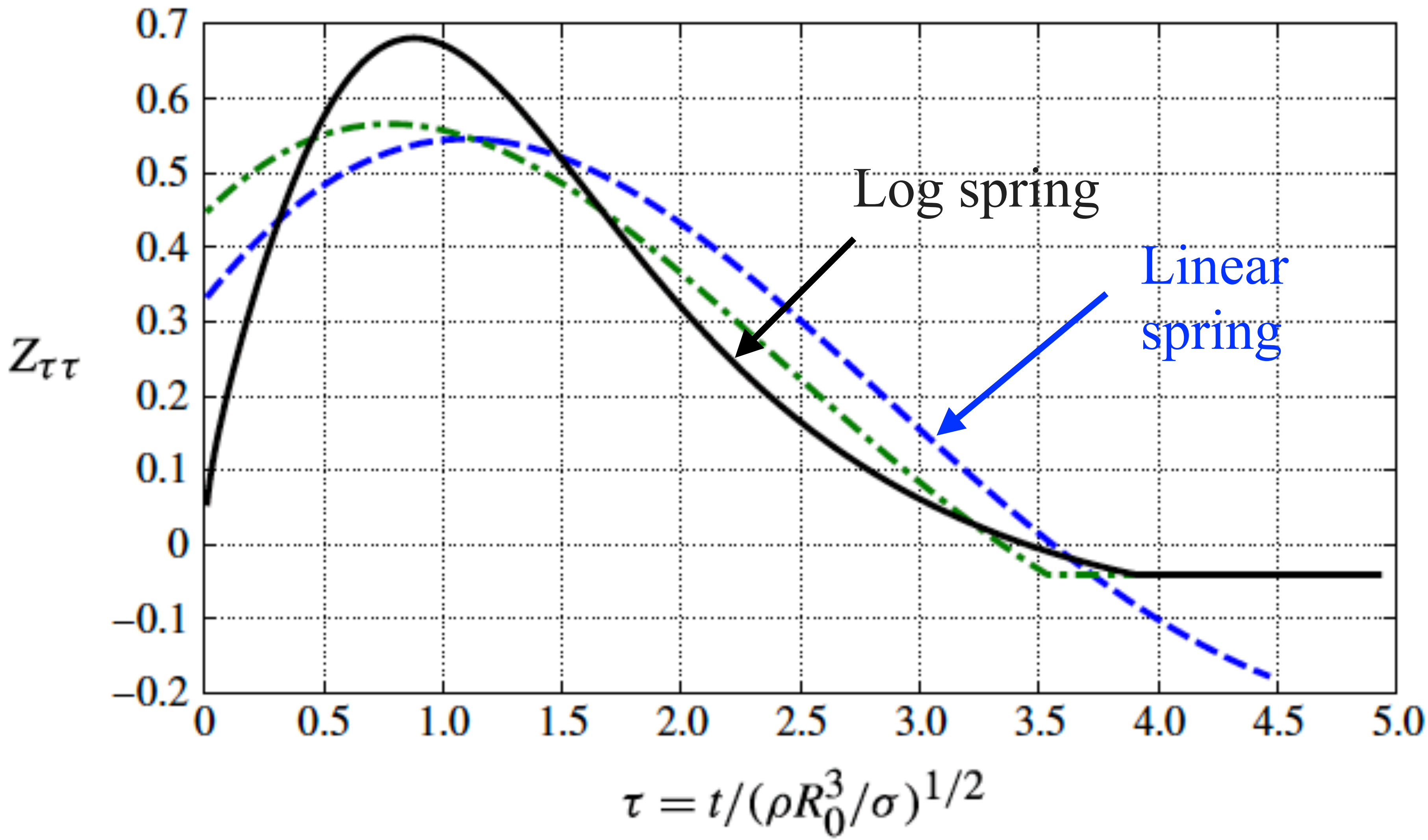
- evolution well described by linear spring, slightly better by log spring

$|Z| = |z|/R_0 We^{1/2}$

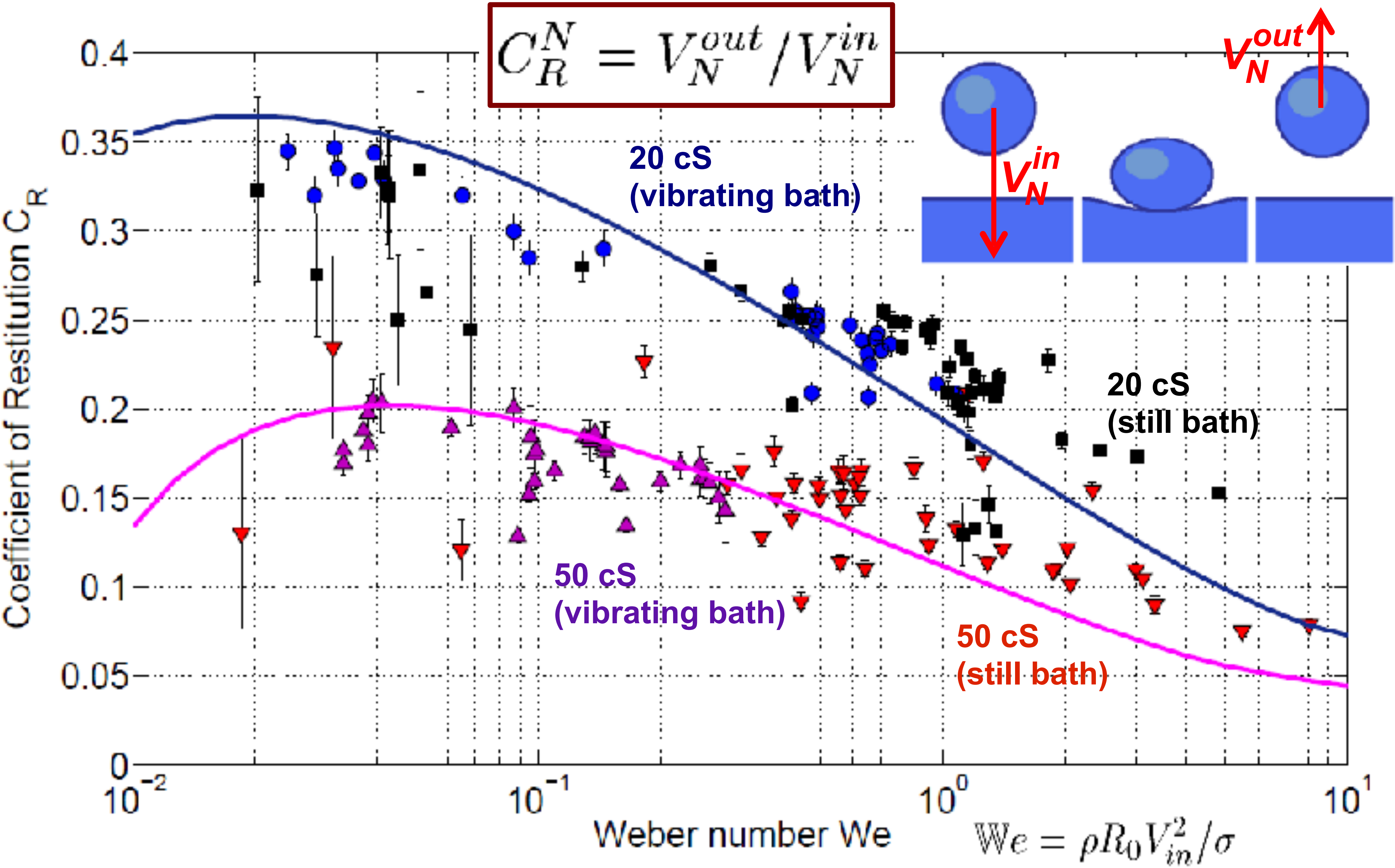


Drop acceleration

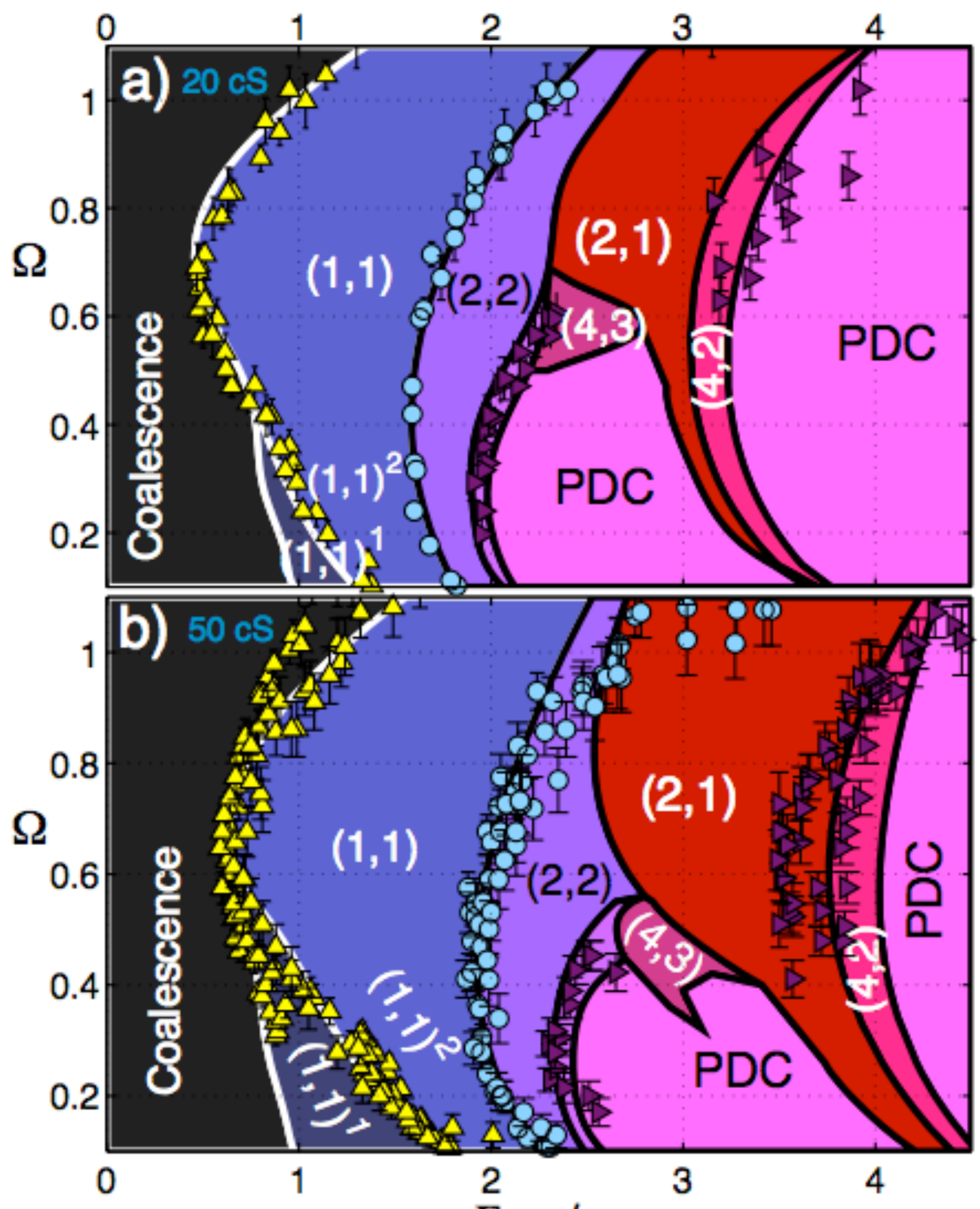
- linear and log spring yield slightly different results
- acceleration of drop center of mass: $Z_{\tau\tau} = (\mathrm{d}^2 z/\mathrm{d}t^2)(\rho R_0^3/\sigma V_{in}^2)^{1/2}$



Logarithmic Spring Model: Predictions



Regime diagrams: f = 80Hz

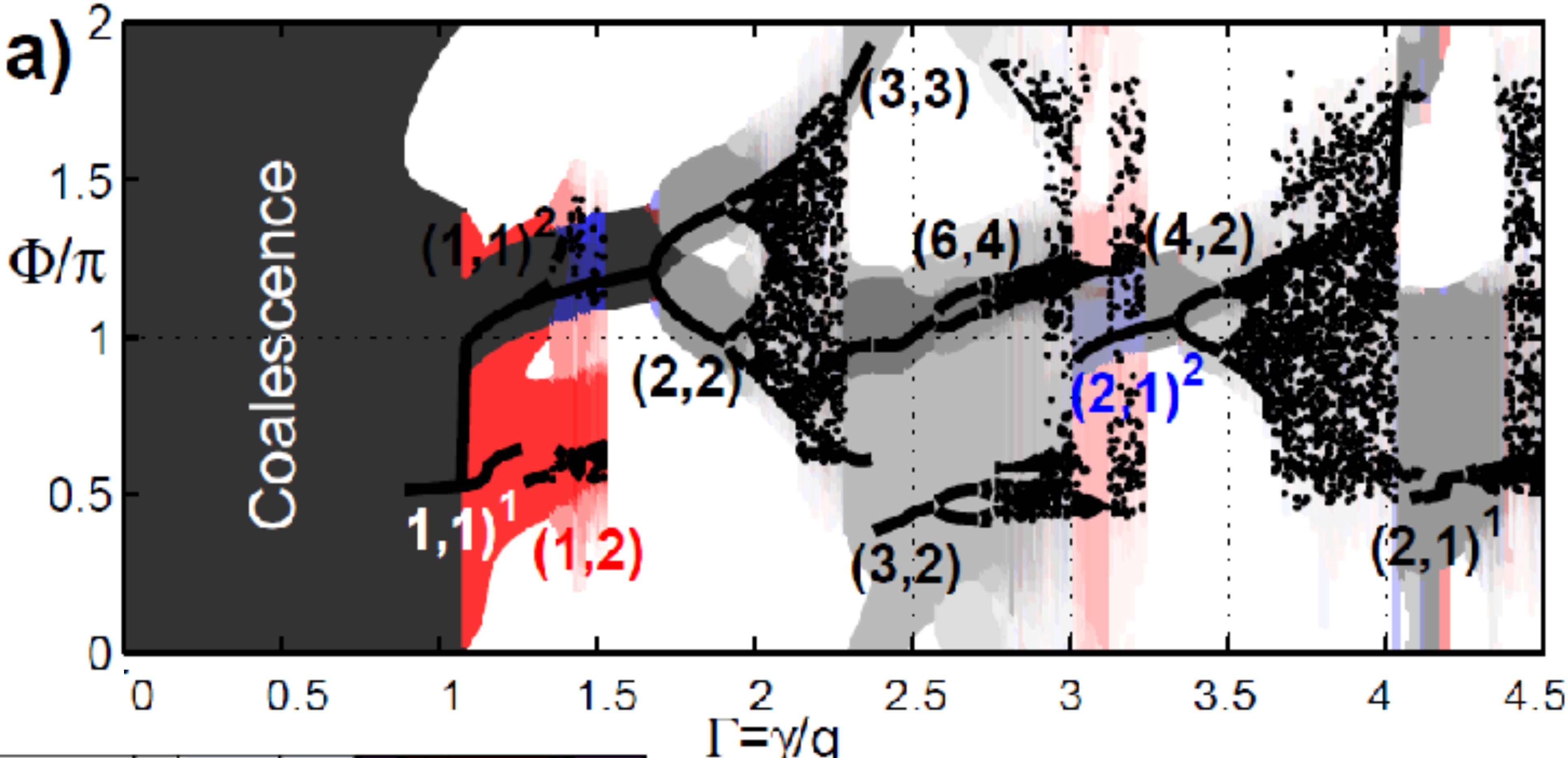


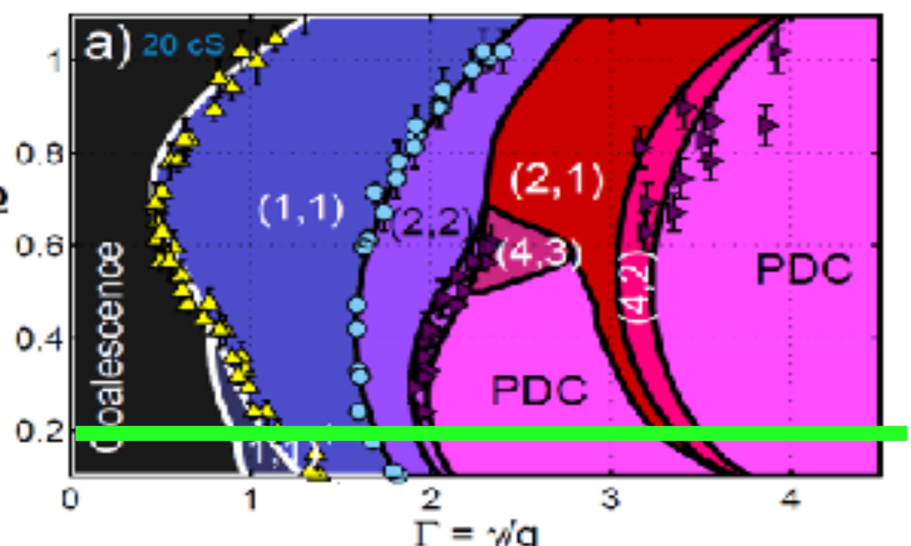
$$\Omega = 2\pi f(\rho R_0^3/\sigma)^{1/2}$$

- model captures bouncing threshold, period-doubling transitions

Molacek & Bush (2013a)

Phase of impact





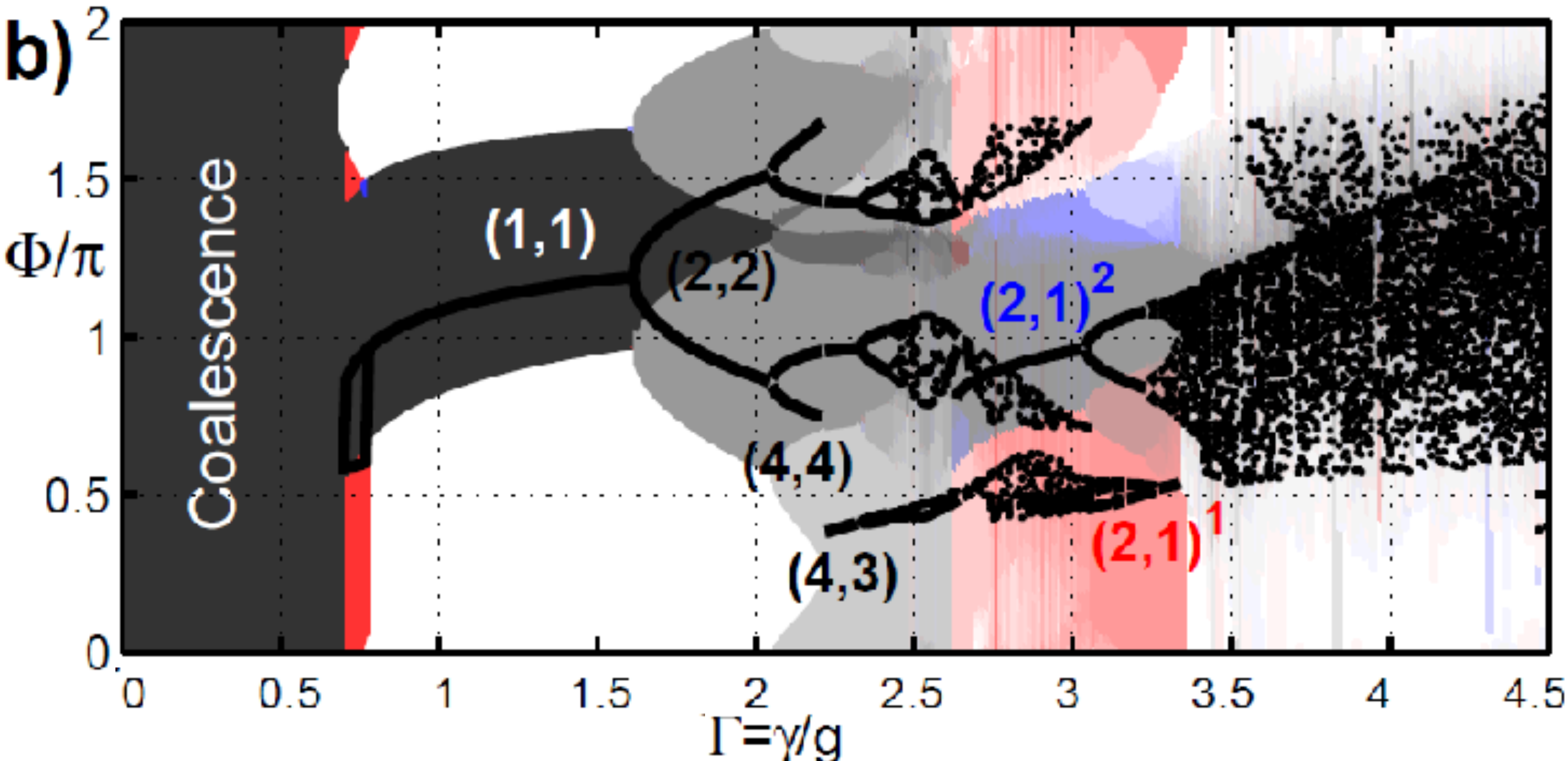
Note: bath displacement in lab frame

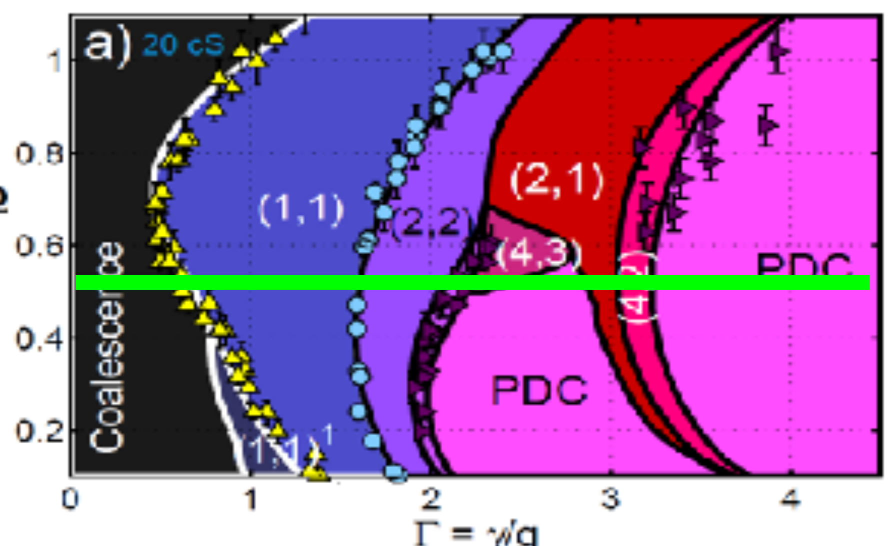
$$-(Bo/\Omega)\cos(\Omega\tau)$$

Bath transfers maximal energy at $\Phi = \pi$

More energetic bouncing states near $\Phi = \pi$

Phase of impact





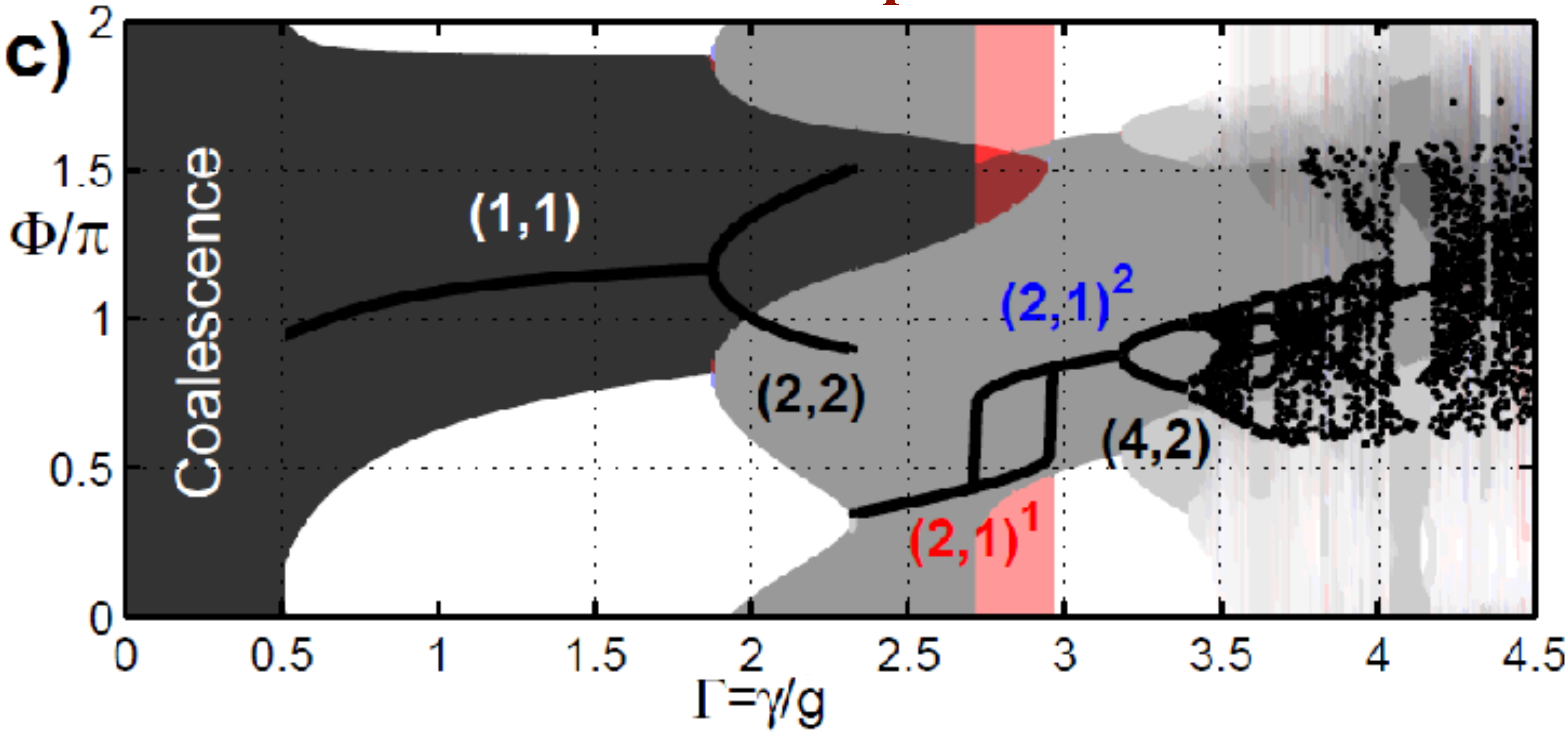
Note: bath displacement in lab frame

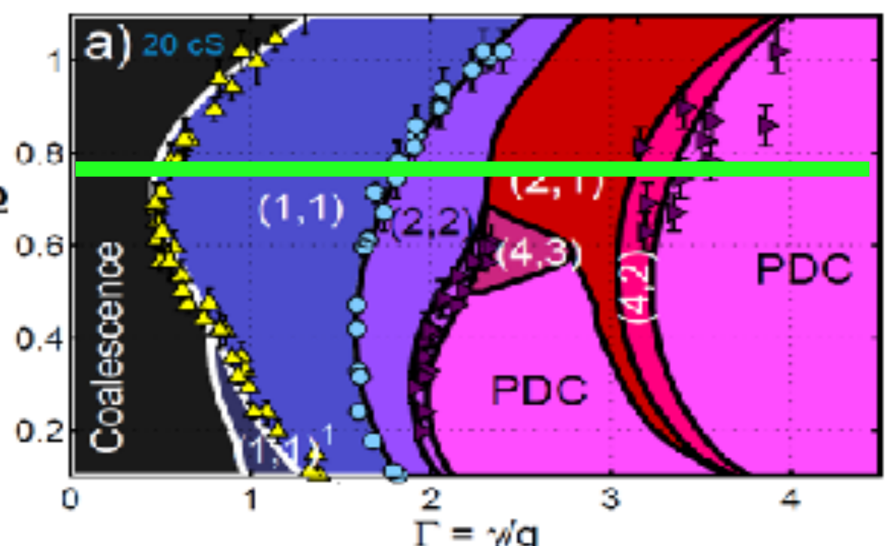
$$-(Bo/\Omega)\cos(\Omega\tau)$$

Bath transfers maximal energy at $\Phi = \pi$

More energetic bouncing states near $\Phi = \pi$

Phase of impact





Note: bath displacement in lab frame

$$-(Bo/\Omega)\cos(\Omega\tau)$$

Bath transfers maximal energy at $\Phi = \pi$

More energetic bouncing states near $\Phi = \pi$

Linear versus logarithmic spring models

- both models assume bath restored to flat before next impact
- linear spring captures bouncing threshold, first period-doubling transitions
- linear spring does not capture behavior arising at high memory
- linear spring's shortcomings at high Me motivated development of log spring
- these shortcomings later found to be attributable to wave persistence

  - Couchman et al., JFM (2019 Couchman & Bush, JFM (2020)

- current theoretical models have reverted to the linear spring

Half-time

Walking dynamics

- requires consideration of horizontal drop dynamics, wave dynamics

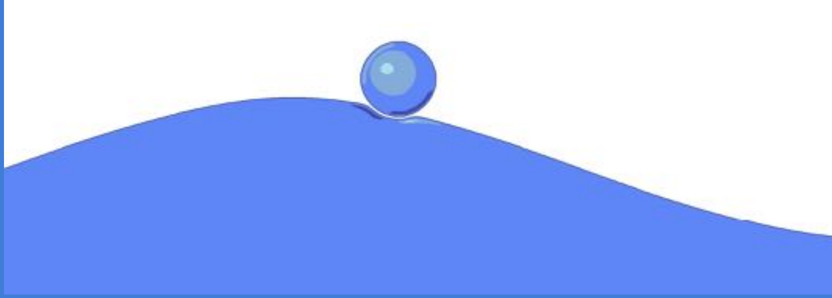
J. Fluid Mech. (2013), *vol.* 727, *pp.* 612–647.
doi:10.1017/jfm.2013.280

Drops walking on a vibrating bath: towards a hydrodynamic pilot-wave theory

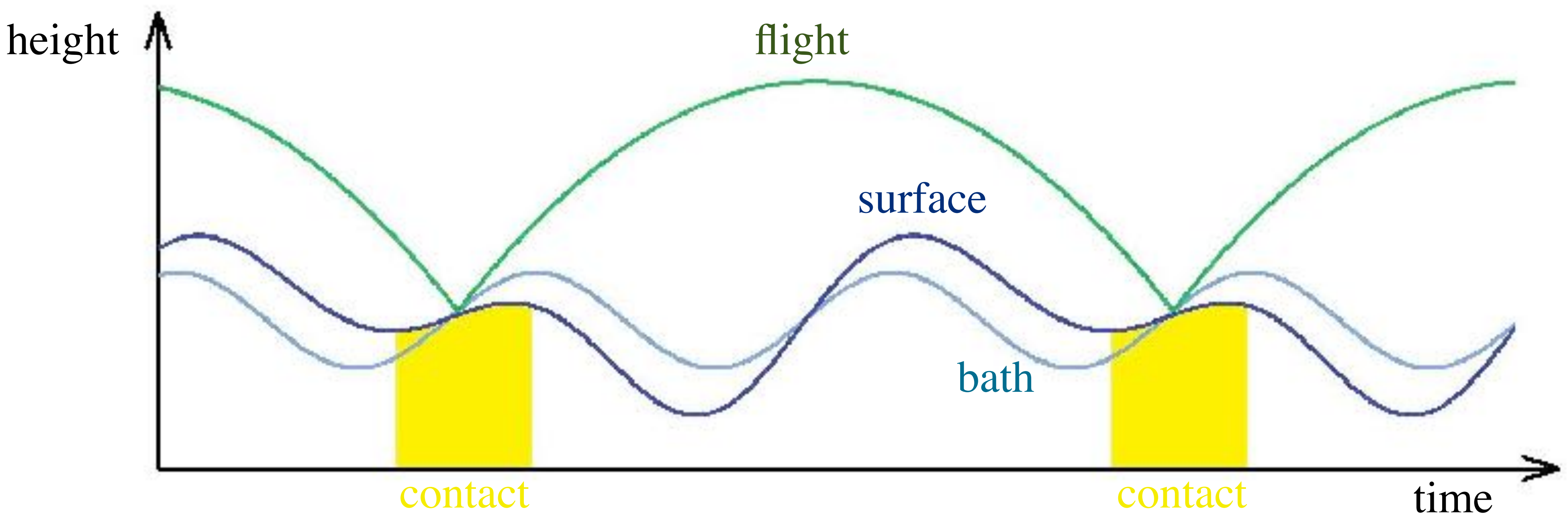
Jan Moláček and John W. M. Bush†

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA

Walking dynamics



- requires resonance between drop and wave
- walking arises when the drop's period matches that of its Faraday waves
- time of flight plus contact time equals Faraday period
- drop lands on front of wave, thus receiving a horizontal kick

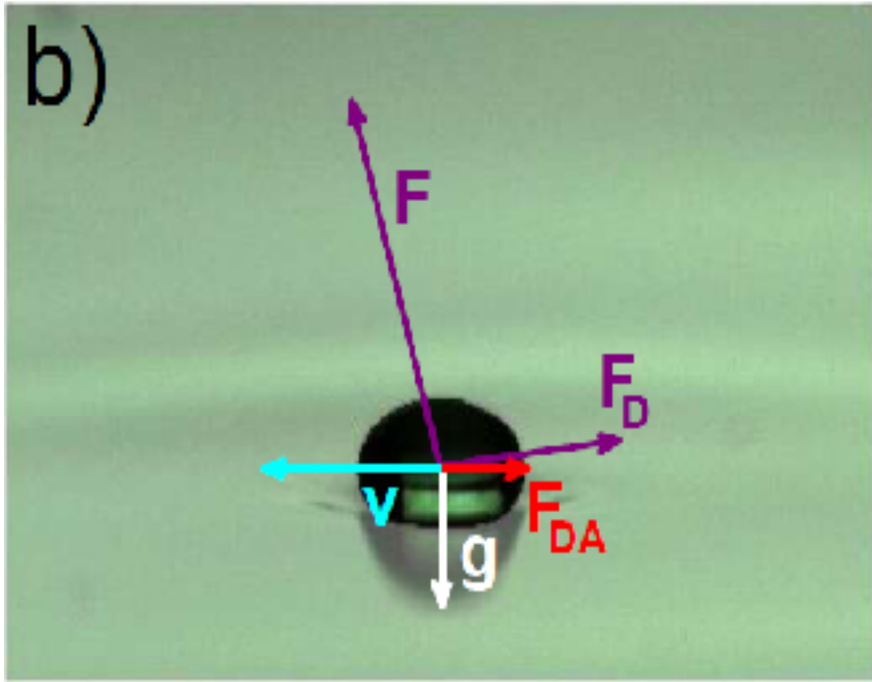


Walking dynamics

Model for ***bouncers*** describes interaction between drop and a quiescent bath.

To describe ***walkers***, we also need to understand:

- spatio-temporal profile of the standing waves created by drop impacts
- forces acting on the drop in the tangential direction: drag & kick

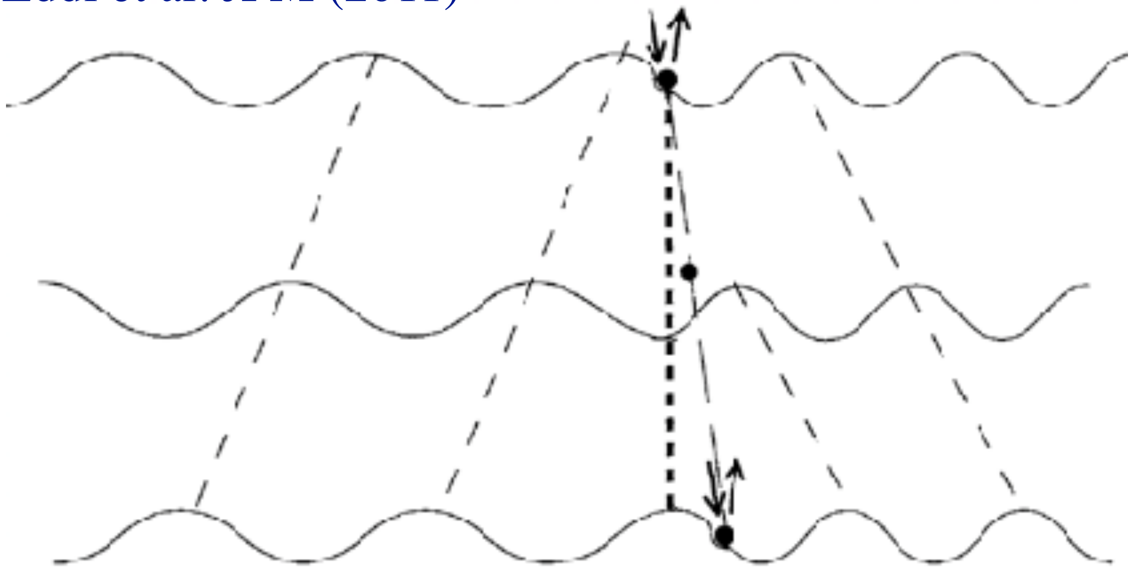


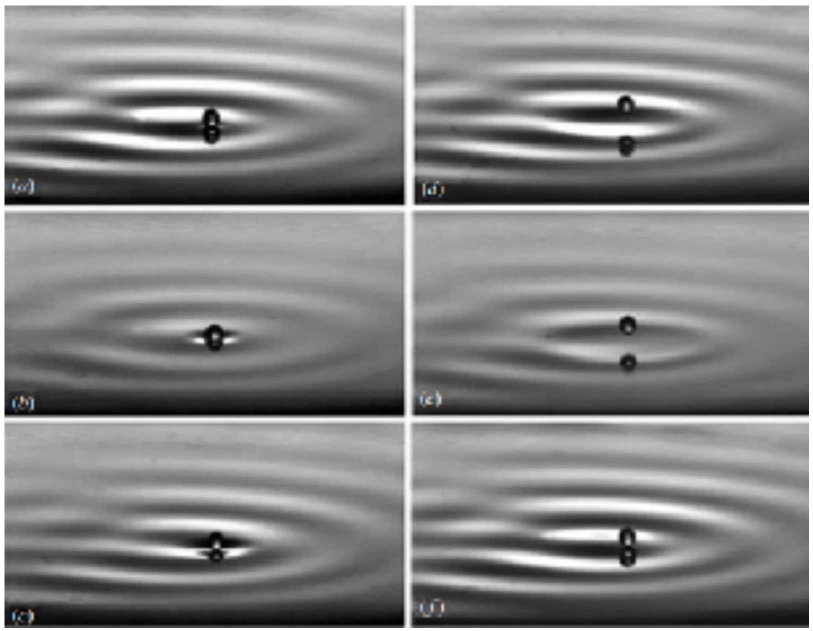
Couder's model

$$\boxed{m\frac{d^2x}{dt^2} = F^b \sin\left(2\pi\frac{dx/dt}{V_F^{\varphi}}\right) - f^v dx/dt}$$

inertia wave drag

Eddi et al. JFM (2011)





Wave forms

Initially: $\xi(\vec{\mathbf{r}}, t_i) = \sum_{n=-\infty}^{i-1} \mathrm{Re}\left[\frac{A}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_n|^{1/2}} \exp -\left(\frac{t_i - t_n}{\tau}\right) \exp -\left(\frac{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_n|}{\delta}\right) \exp i\left(\frac{2\pi|\vec{\mathbf{r}} - \vec{\mathbf{r}}_n|}{\lambda_F} + \phi\right)\right]$

Subsequently: $\xi(r,t) = \sum_{m=1}^{\infty} a_m(t) J_0\left(\frac{\alpha_m}{r_c} r\right)$ where coefficients satisfy viscous Matthieu eqn

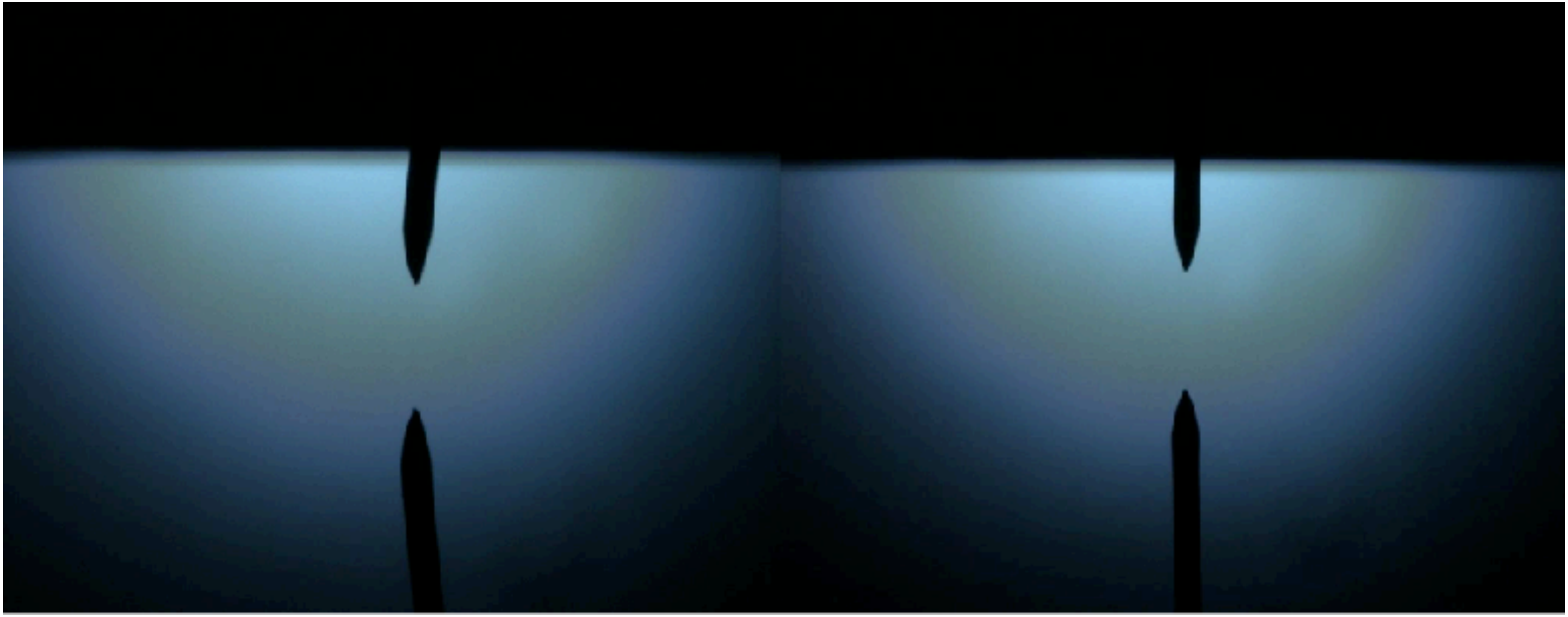
$$\frac{d^2 a_m}{dt^2} + 2\nu_{phen} k_m^2 \frac{da_m}{dt} + \left(\omega^2(k_m) - k_m \tanh(k_m h_0)\gamma\cos(\omega_0 t)\right) a_m = 0$$

Disturbance of forced and unforced interfaces

- withdraw millimetric needle from interface

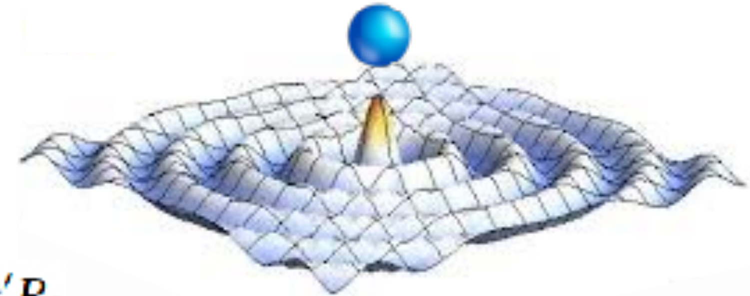
No forcing

Vibrational forcing



- waves quickly disperse
- field of Faraday waves persist

- vibration predisposes bath to monochromatic wave field with Faraday wavelength



Nondimensionalization:

$$h = h'/R_0, \quad r = r'/R_0, \quad \tau = t\omega_D, \quad Z = z/R_0, \quad k = k'R_0$$

Wave field from a

Surface potential energy: $\Delta\mathscr{S}\mathscr{E} = \sigma R_0^2 \int_0^\infty 2\pi r \left| \sqrt{1 + h'^2(r)} - 1 \right| \mathrm{d}r \approx \pi\sigma R_0^2 \int_0^\infty r h'^2(r)\, \mathrm{d}r,$

Gravitational potential energy: $\Delta\mathscr{P}\mathscr{E} = \rho g R_0^4 \int_0^\infty 2\pi r \frac{1}{2} h^2(r)\, \mathrm{d}r = \pi\rho g R_0^4 \int_0^\infty r h^2(r)\, \mathrm{d}r$

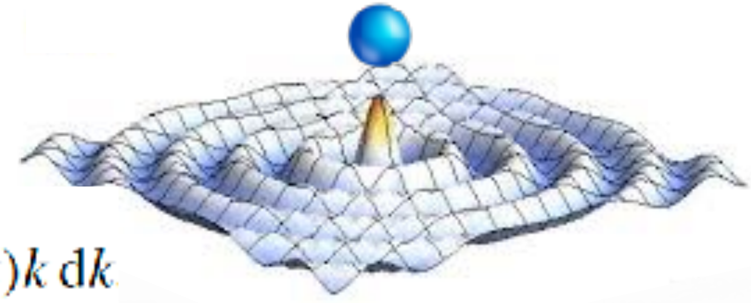
Pressure potential energy: $\Delta\mathscr{P}\mathscr{E}_P = p(\tau) R_0^3 \int_0^w 2\pi r h(r)\, \mathrm{d}r = 2\pi R_0^3 p(\tau) \int_0^w r h(r)\, \mathrm{d}r.$

Hankel transform: $H(k) = \int_0^\infty h(r) J_0(kr) r \mathrm{d}r$ so that $h(r) = \int_0^\infty H(k) J_0(kr) k \mathrm{d}k$

Plancherel Theorem: for functions f(r), g(r) and their Hankel transforms F(k), G(k)

$$\boxed{\int_0^\infty f(r) g(r) r\, \mathrm{d}r = \int_0^\infty F(k) G(k) k\, \mathrm{d}k}$$

Wave field from a single impact



Surface potential energy: $$\Delta\mathscr{P}\mathscr{E} = \pi g R_0^4 \int_0^\infty H^2(k)k\,\mathrm{d}k$$

Gravitational potential energy: $$\Delta\mathscr{S}\mathscr{E} = \pi\sigma R_0^2 \int_0^\infty H^2(k)k^3\,\mathrm{d}k$$

Pressure potential energy: $$\Delta\mathscr{P}\mathscr{E}_p = 2\pi R_0^3 p(\tau)\int_0^\infty H(k)k\int_0^w \mathrm{J}_0(kr)r\,\mathrm{d}r\,\mathrm{d}k$$
$$= 2\pi R_0^3 p(\tau)\int_0^\infty H(k)\mathrm{J}_1(kw)w\,\mathrm{d}k.$$

Potential flow: $\nabla^2\Phi = 0$

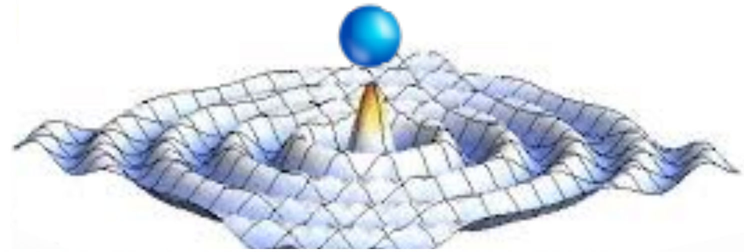
General solution decaying as $r\to\infty$: $$\Phi(r,z,t) = \int_0^\infty \varphi(k,\tau)\mathrm{J}_0(kr)\mathrm{e}^{kz}k\,\mathrm{d}k$$

Kinematic BC at surface: $$R_0\,\partial h(r,t)/\partial t = u_z(x,0,t) = (1/R_0)\,\partial\Phi/\partial z|_{z=0}$$

$$R_0^2\int_0^\infty \dot{H}(k,t)\mathrm{J}_0(kx)k\,\mathrm{d}k = \int_0^\infty \varphi(k,t)\mathrm{J}_0(kx)k^2\,\mathrm{d}k \longrightarrow \varphi(k,t) = R_0^2\dot{H}(k,t)/k$$

Velocity potential: $$\Phi(r,z,\tau) = R_0^2\int_0^\infty \dot{H}(k,\tau)\mathrm{J}_0(kx)\mathrm{e}^{kz}\,\mathrm{d}k.$$

Wave field from a single impact



Bath kinetic energy:

$$\frac{\mathscr{K}\mathscr{E}}{\rho} = \frac{1}{2}\int_V \nabla\Phi\cdot\nabla\Phi\,\mathrm{d}V = \frac{1}{2}\int_V \nabla\cdot(\Phi\nabla\Phi)\,\mathrm{d}V$$

$$= \frac{1}{2}\int_S \Phi\nabla\Phi\cdot\mathrm{d}\mathbf{S} = \frac{R_0}{2}\int_0^\infty \Phi\frac{\partial\Phi}{\partial z}2\pi x\,\mathrm{d}x = \pi R_0^5\int_0^\infty \dot{H}^2(k,t)\,\mathrm{d}k,$$

Dissipation in bath: $\mathscr{D} = 8\pi\mu R_0^3\int_0^\infty \dot{H}^2(k,t)k^2\,\mathrm{d}k.$

Euler-Lagrange equation with dissipation: $\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathfrak{L}}{\partial\dot{H}} + \frac{1}{2}\frac{\partial\mathscr{D}}{\partial\dot{H}} = \frac{\partial\mathfrak{L}}{\partial H}}$

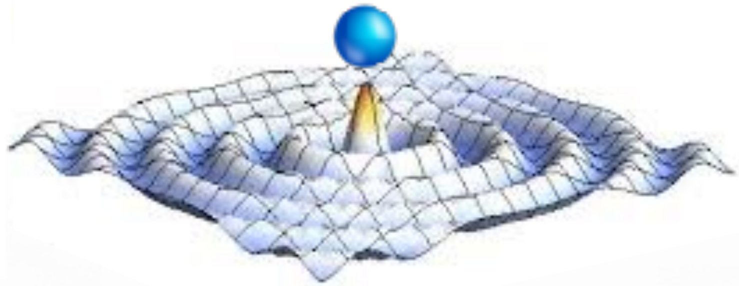
where the Lagrangian: $\mathfrak{L} = \mathscr{K}\mathscr{E} - \Delta\mathscr{S}\mathscr{E} - \Delta\mathscr{P}\mathscr{E} - \mathscr{P}\mathscr{E}_P$

Substitution: $H_{\tau\tau} + 2Ohk^2H_\tau + [k^3 + kBo]H + \frac{R_0 pw}{\sigma}\mathrm{J}_1(kw) = 0$

Use $F_R = R_0^2\int_0^w 2\pi rp\,\mathrm{d}r = \pi w^2R_0^2p,$ and $m\ddot{z} = F_R - mg$

$\rightarrow$ $Z_{\tau\tau} = \frac{3}{4}R_0w^2p/\sigma - Bo \equiv F - Bo$

Wave field from a single impact



Equation for Hankel transform H(k):

$$H_{\tau\tau} + 2Oh_e k^2 H_\tau + [k^3 + kBo^*(\tau)]H = -\frac{4}{3}F\frac{J_1(wk)}{w}.$$

where $\quad Bo^*(\tau) = Bo(1 + \Gamma \sin \Omega\tau)$

Point force approximation: use small argument expansion $\quad J_1(wk)/w = k/2$

Equation for Hankel transform H(k):

$$\boxed{H_{\tau\tau} + 2Oh_e k^2 H_\tau + H(k^3 + kBo^*(\tau)) = -\tfrac{2}{3}kF(\tau)}$$

dissipation

surface energy

gravitational energy

forcing by the drop

Standing wave field

Hankel transform: $H(k) = \int_0^\infty h(r) J_0(kr) r \mathrm{d}r$ so that $h(r) = \int_0^\infty H(k) J_0(kr) k \mathrm{d}k$

Equation for H(k):

$$\boxed{H_{\tau\tau} + 2\mathcal{O}h_e k^2 H_\tau + H\left(k^3 + k\mathcal{B}o^*(\tau)\right) = -\frac{2}{3}kF(\tau)}$$

Solve $H(\tau) \approx \left[\int -\frac{2}{3}kF(\tau')G_1(\tau')\mathrm{d}\tau'\right] e^{-\beta\tau} H(\tau)$, $\Upsilon(k) = \mathcal{O}h_e(k) \cdot k^2$

where $G_1(u) - \dfrac{\exp(\beta u)\bar{H}\left(\frac{\pi}{\Omega} - u\right)}{\bar{H}_\tau(0)\bar{H}\left(\frac{\pi}{\Omega}\right) + \bar{H}(0)\bar{H}_\tau\left(\frac{\pi}{\Omega}\right) + 2\left(\Upsilon - \beta\right)\bar{H}(0)\bar{H}\left(\frac{\pi}{\Omega}\right)}$

Alternatively, $H(k,\tau) \approx A(k)e^{\beta(k)\tau}\cos\left(\frac{\Omega}{2}\tau - \frac{\epsilon}{2} + \frac{k - k_C}{2\epsilon k_F}\right)$

where $\beta(k) = (\Gamma/\Gamma_F - 1)\dfrac{1}{\tau_D} - \beta_1(k - k_C)^2$

Transform back:

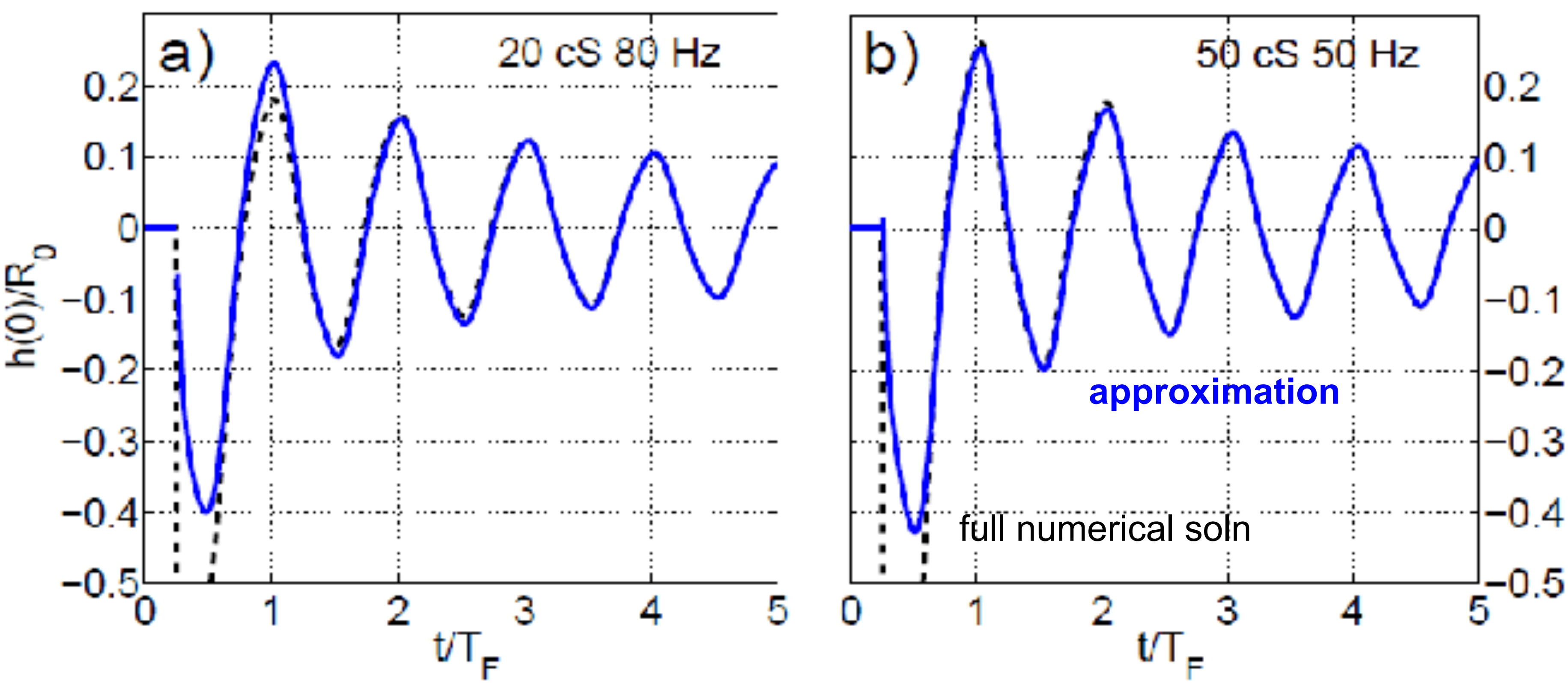
$$\boxed{h(r,\tau) \approx \cos\frac{\Omega\tau}{2}A(k_C)\frac{\exp\left\{(\Gamma/\Gamma_F - 1)\tau/\tau_d\right\}}{\exp\left\{1/16\epsilon^2 k_F^2\beta_1\tau\right\}}\sqrt{\frac{\pi}{\beta_1\tau}}J_0(k_C r)k_C\left[1 + O\left(\frac{r^2}{4\beta_1\tau}\right)\right]}$$

Standing wave field

- standing wave created by a single drop impact:

$$h(r,\tau) \approx \frac{4\sqrt{2\pi}}{3\sqrt{\tau}} \frac{k_C^2 k_F \mathrm{Oh}_e^{1/2}}{3k_F^2 + \mathbb{B}o} \left[\int F(u) \sin \frac{\Omega u}{2} \mathrm{d}u \right] \cos \frac{\Omega \tau}{2} \exp \left\{ \left(\frac{\Gamma}{\Gamma_F} - 1 \right) \frac{\tau}{\tau_d} \right\} J_0(k_C r) .$$

- most unstable mode $k_C \approx k_F$ where $k_F^3 + \mathbb{B}o \cdot k_F = \frac{1}{4}\Omega^2$

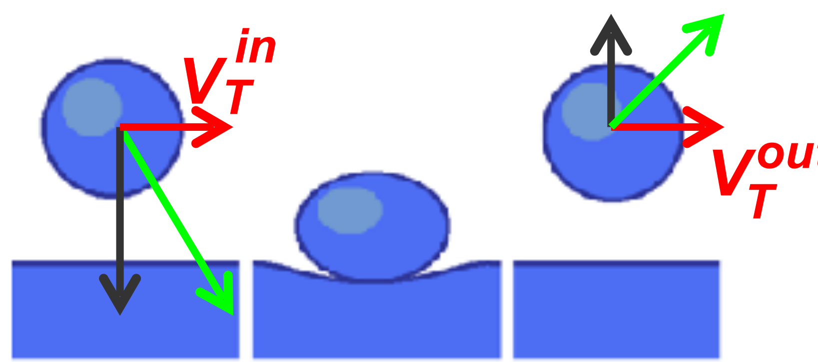


Drag suffered at impact

- infer horizontal drag imparted at impact via experiment

Tangential coefficient of restitution

$$\boxed{C_R^T = V_T^{out} / V_T^{in}}$$



Horizontal drag over impact

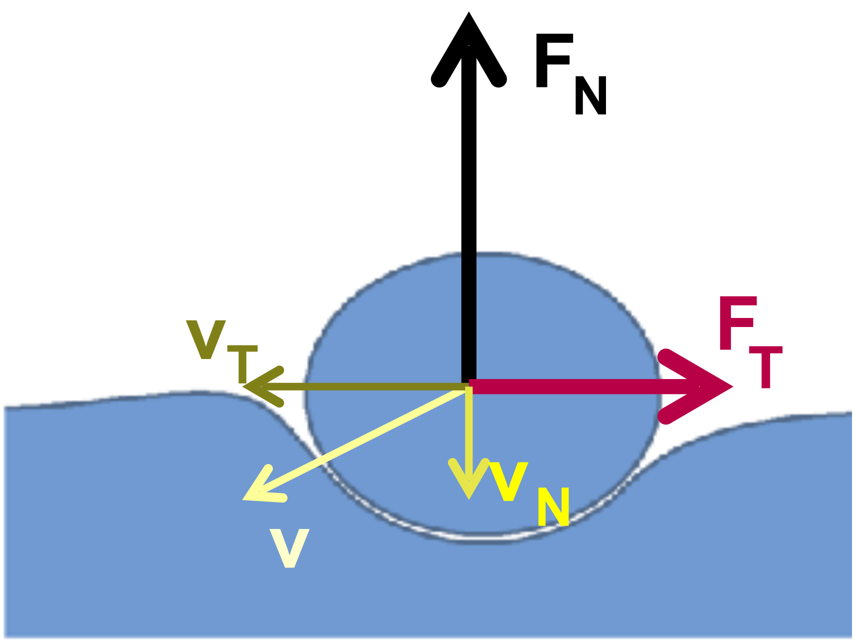
- infer horizontal drag imparted at impact via experiment

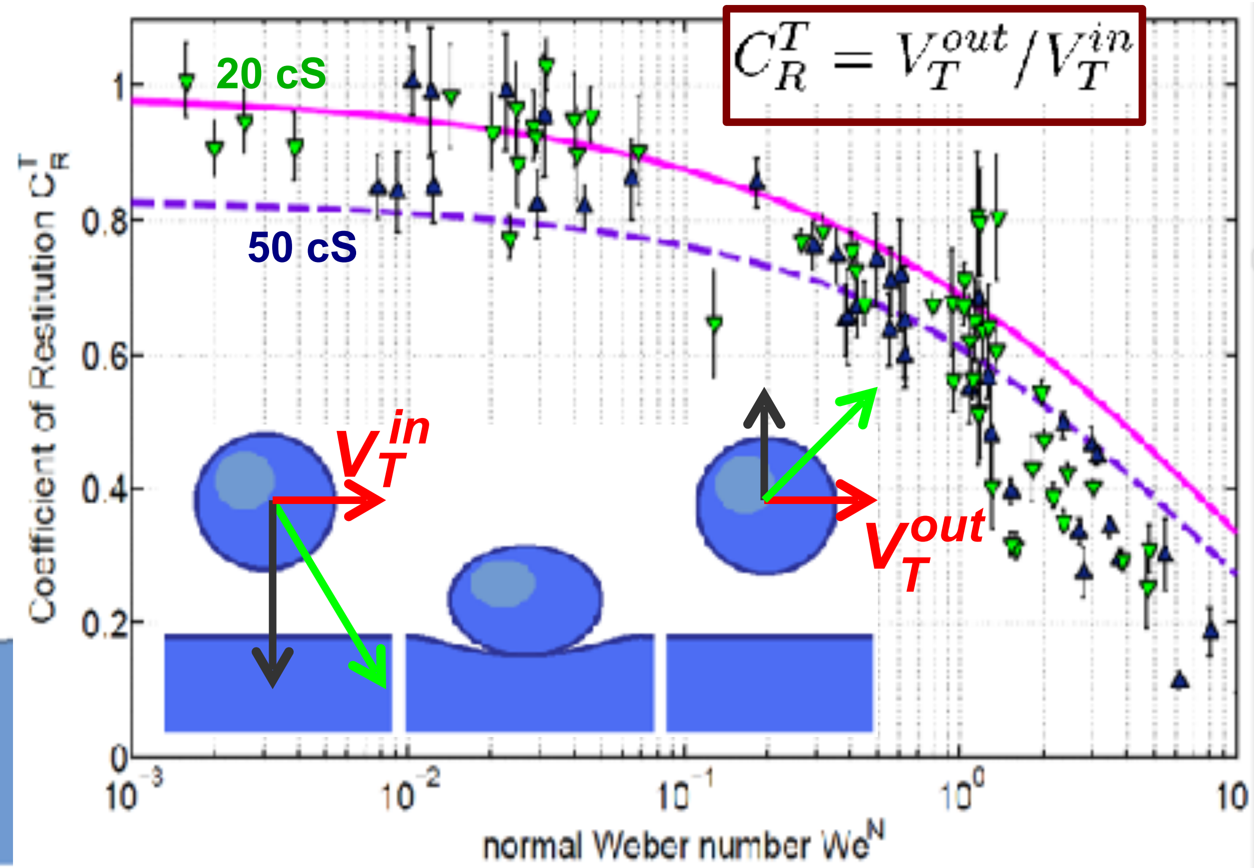
$$\boxed{F_T \approx C \cdot F_N \cdot \mathbb{W}e_T^{1/2}} = C \cdot F_N \cdot V_T^{in} \sqrt{\frac{\rho R_0}{\sigma}}$$

tangential reaction force (drag)

normal reaction force

tangential Weber number





Model summary

Vertical dynamics

$$m_D\ddot{Z} = -m_D g\,(1+\Gamma\sin\omega t) \quad \text{when} \quad Z \geq 0$$

$$m_D\ddot{Z}\left(1+\frac{c_3}{Q^2(Z)}\right)+\frac{\mu R_0 c_2}{Q(Z)}\dot{Z}+\frac{3\sigma Z}{2Q(Z)} = -m_D g\,(1+\Gamma\sin\omega t) \quad \text{otherwise}$$

Standing wave evolution

$$h(X,\tau)=\sum_{n=1}^{N} h_0(X,X_n,\tau,\tau_n)$$

$$h_0(X,X_n,\tau,\tau_n)\approx \frac{4\sqrt{2\pi}}{3}\frac{k_C^2 k_F \Omega h_c^{1/2}}{3k_F^2+\mathrm{Bo}}\left[\int F(u)\sin\frac{\Omega u}{2}\,\mathrm{d}u\right]\frac{\bar{H}(\tau)}{\sqrt{\tau-\tau_n}}\exp\left\{(\Gamma/\Gamma_F-1)\frac{\tau-\tau_n}{\tau_d}\right\}J_0\left(k_C(X-X_n)\right)$$

$$X_n \quad \int_{\tau_c} F(u)X(u)\,\mathrm{d}u \Big/ \int_{\tau_c} F(u)\,\mathrm{d}u \quad , \quad \tau_n \quad \int_{\tau_c} F(u)u\,\mathrm{d}u \Big/ \int_{\tau_c} F(u)\,\mathrm{d}u$$

Horizontal dynamics

$$m_D\ddot{X}+\left[CF_N\sqrt{\rho R_0/\sigma}+6\pi\mu_a R_0\right]\dot{X} = -\frac{\partial h}{\partial X}F_N$$

contact drag

air drag

sloped interface

Trajectory equation: resonant walker

$$\boxed{m\ddot{\mathbf{x}}_p + D\dot{\mathbf{x}}_p = -mg\nabla h(\mathbf{x}_p, t)}$$

Drag coefficient: $D = 6\pi\mu_a R + Cmg \cdot \sqrt{\frac{\rho R}{\sigma}}$

Wave field: $h(\mathbf{x}, t) = A \sum_{k=-\infty}^{\lfloor t/T_F \rfloor} J_0(k_F |\mathbf{x} - \mathbf{x}_p(kT_F)|) e^{-(t-kT_F)/(T_F M_e)}$

Wave amplitude: $A = \frac{4\sqrt{2\pi}}{3} \frac{R^4 k_F^3 \mathbb{O}h_e^{1/2}}{3R^2 k_F^2 + \mathbb{B}o} \cdot \frac{\mathbb{B}o T_F}{\sqrt{\rho R^3/\sigma}} \sin \Phi_I$

Memory parameter: $M_e = \frac{T_d}{T_F(1 - \gamma/\gamma_F)}$ Impact phase: Φ_I

Bond number: $\mathbb{B}o = \frac{\rho g R^2}{\sigma}$ Ohnesorge number: $\mathbb{O}h_e \approx \frac{\mu}{\sqrt{\rho\sigma R}}$

Walking threshold

$$m\ddot{x} + \bar{D}\dot{x} = -mg\nabla h = -\frac{1}{2}Amg\sin\Phi_i\,\nabla\sum_{n=1}^{N}\frac{e^{-n/\mathcal{M}_e}}{\sqrt{nT_F}}J_0(k_F(\boldsymbol{x}-\boldsymbol{x}_n))$$

Seek walking solution with steady speed v: $x(t+T_F) - x(t) = vT_F$

Speed satisfies: $$v = \frac{5}{2}\sqrt{\frac{\sigma}{\rho R_0}}A\sin\Phi_i\,k_F T_F^{\ \ 1/2}\sum_{n=1}^{\infty}e^{\ \ n/\mathcal{M}_e}n^{\ \ 1/2}J_1(nk_F T_F v)$$

where $0.25 < \sin\Phi_i < 0.65.$

$$v = 0 \quad \text{or} \quad 1 = \frac{5}{4}\sqrt{\frac{\sigma}{\rho R_0}}A\sin\Phi_i\,k_F^2 T_F^{1/2}\sum_{n=1}^{\infty}e^{-n/\mathcal{M}_e}n^{1/2}.$$

Use $\sum_{n=1}^{\infty}e^{\ \ n/\mathcal{M}_e}n^{1/2} = \int_0^{\infty}e^{\ \ x/\mathcal{M}_e}x^{1/2}\,\mathrm{d}x(1+O(\mathcal{M}_e^{\ \ 1})) \approx \Gamma\left(\frac{3}{2}\right)\mathcal{M}_e^{3/2}$ to deduce

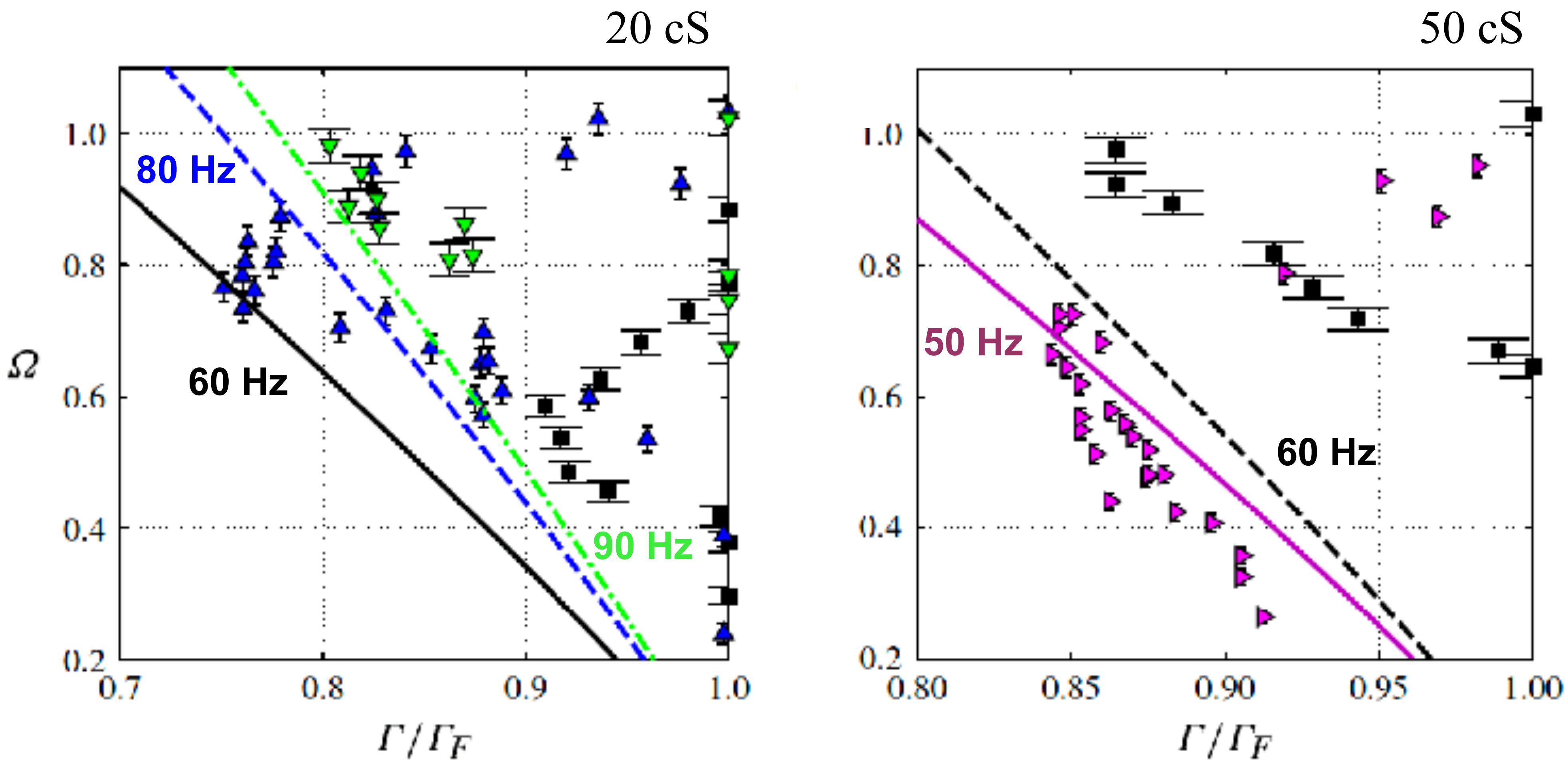
Critical memory for walking:

$$\boxed{\mathcal{M}_e^c \approx \left[\frac{\sqrt{\pi}}{2}\frac{5}{4}A\sin\Phi_i\,k_F^2\sqrt{\frac{\sigma T_F}{\rho R_0}}\right]^{-2/3} = \left[\frac{5\sqrt{2}\,\pi\sin\Phi_i(k_F R_0)^5}{6(3k_F^2R_0^2+Bo)}\sqrt{\frac{\mu_e g^2 T_F^3}{\sigma R_0}}\right]^{2/3}}$$

Predicted walking thresholds

$$\mathcal{M}_e = \frac{T_d}{T_F(1 - \Gamma/\Gamma_F)},$$

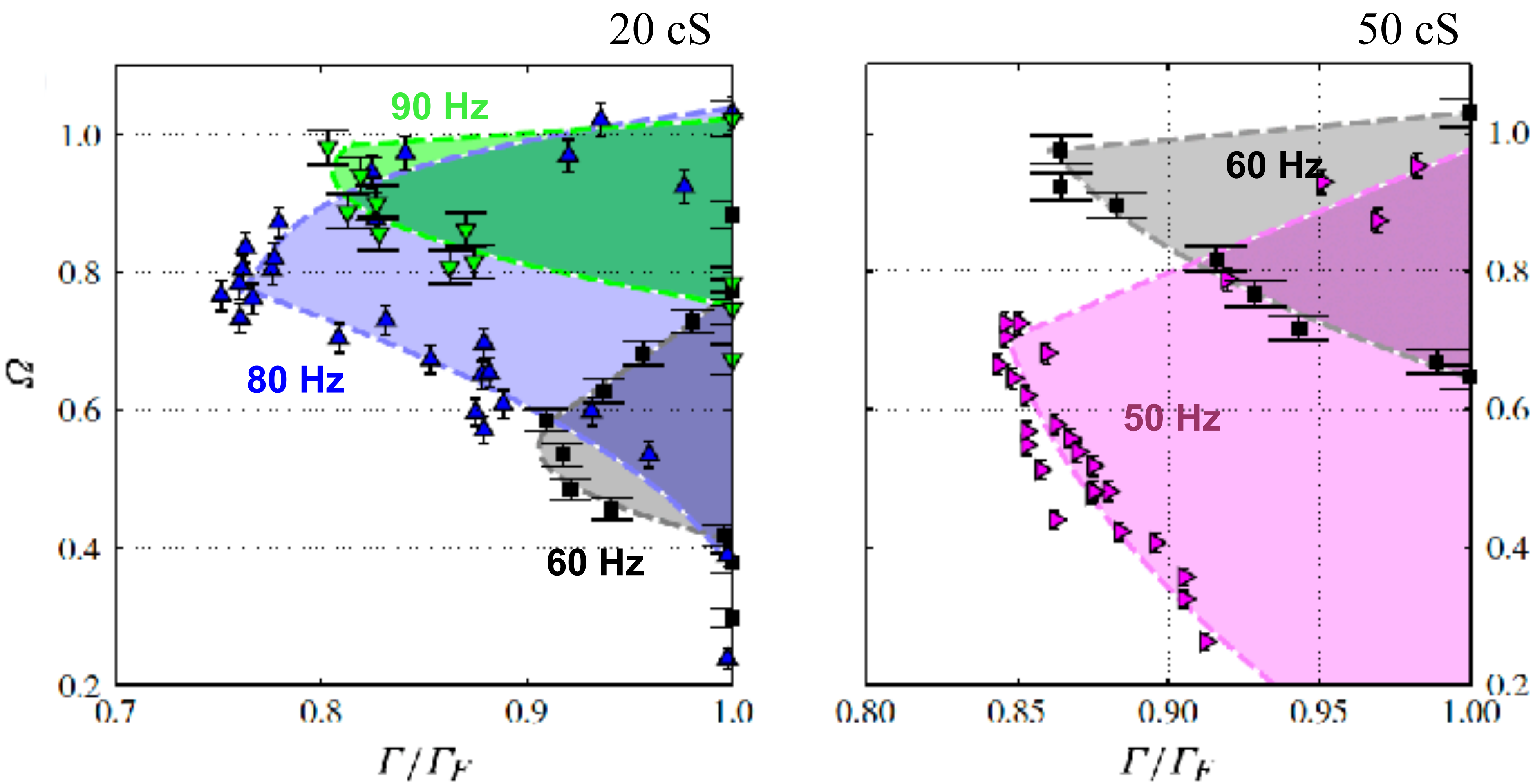
- the model prescribes the extent of walking region



$$\Omega = \omega/\omega_D = 2\pi f\sqrt{\rho R_0^3/\sigma}$$

Measured walking thresholds

$$\mathscr{M}_e = \frac{T_d}{T_F(1 - \Gamma/\Gamma_F)},$$



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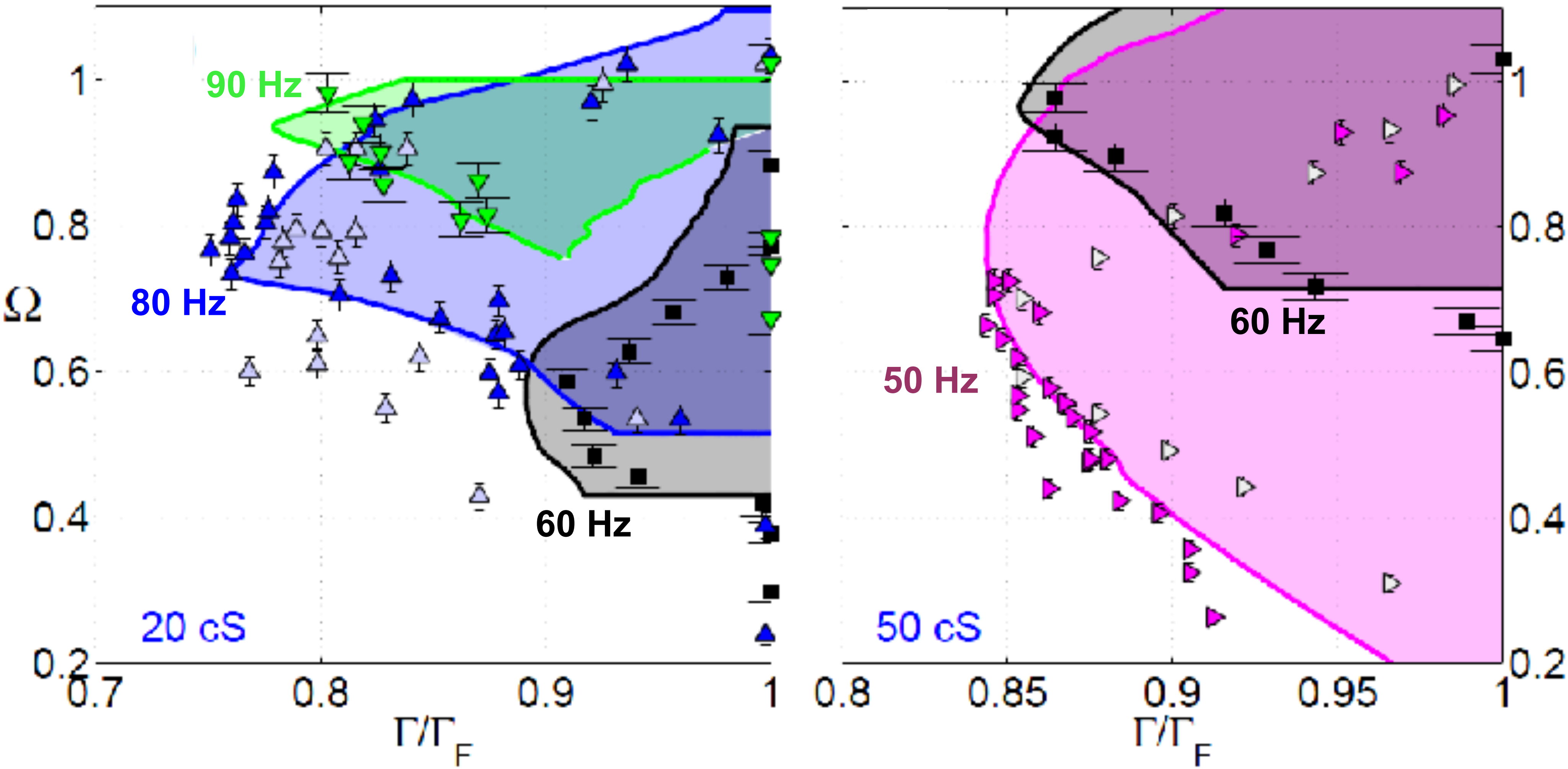
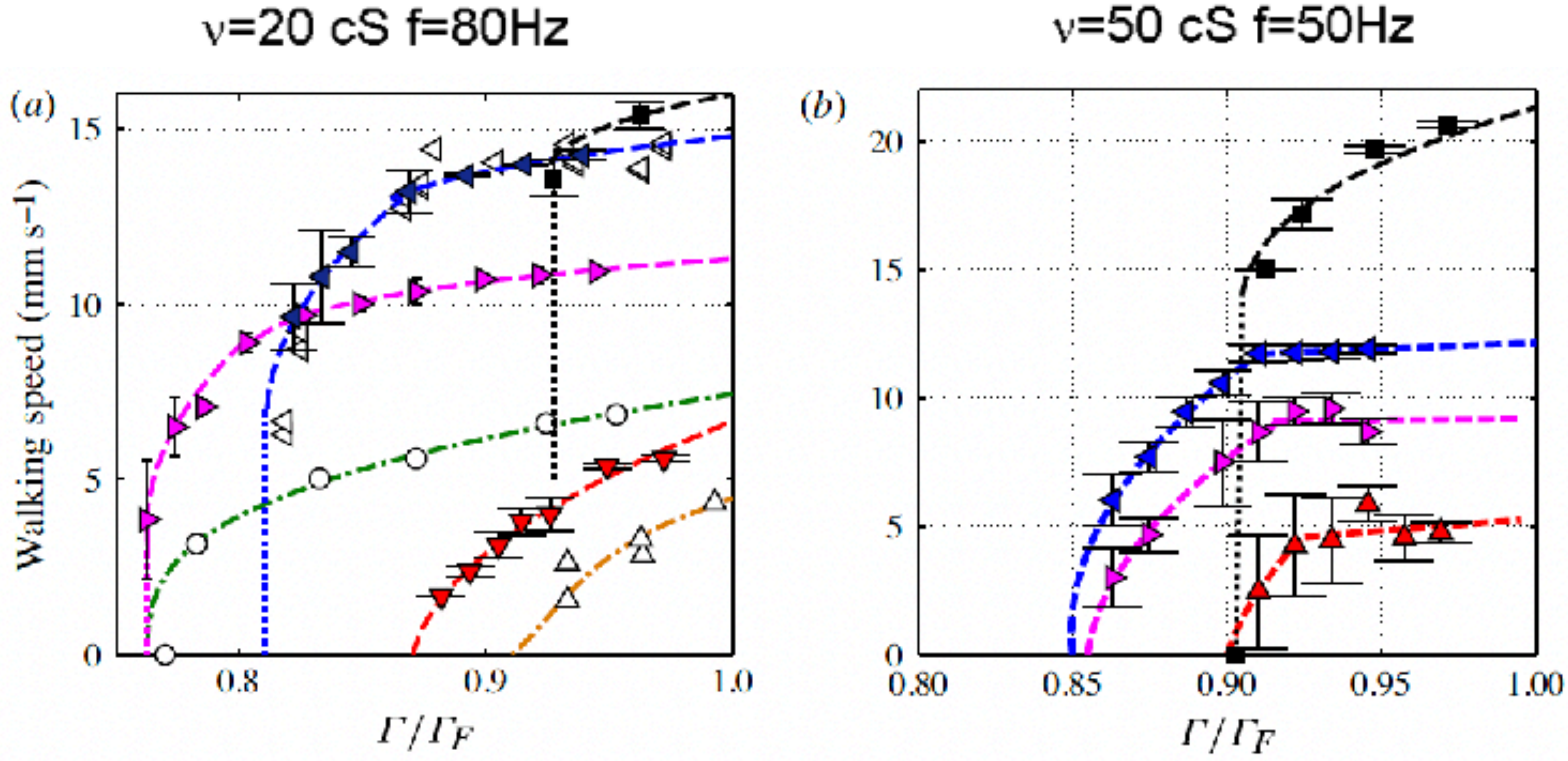


$$\Omega = \omega/\omega_D = 2\pi f\sqrt{\rho R_0^3/\sigma}$$

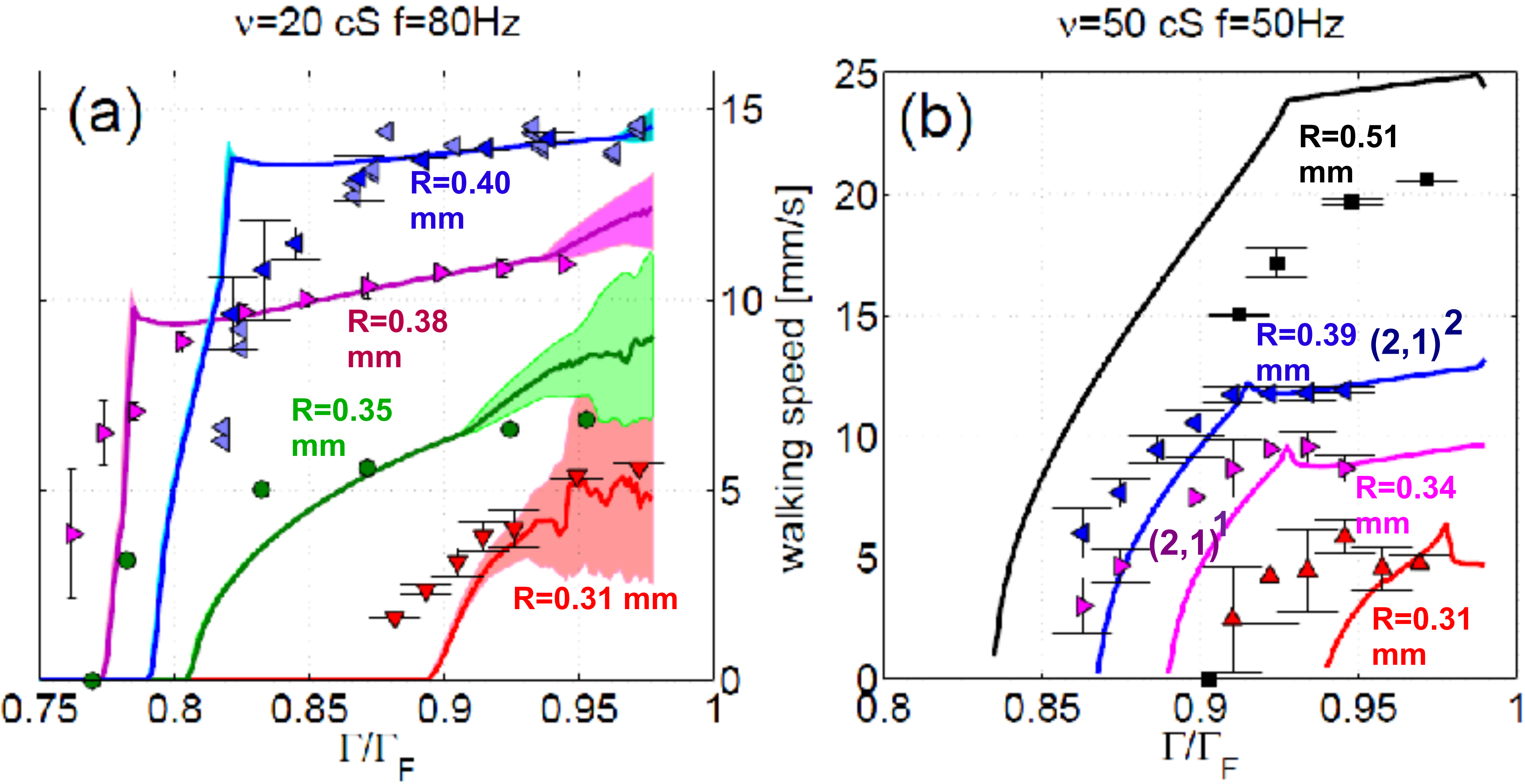
Walking speed dependence: experiment



(a) $R_0 = 0.31$ mm (▲), 0.35 mm (◯), 0.38 mm (▶) and 0.40 mm (◀, ◁)

(b) $R_0 = 0.25$ mm (▲), 0.34 mm (▶), 0.39 mm (◀) and 0.51 mm (■)

Predicted walking speeds



- discontinuities associated with transition to more energetic walking state:

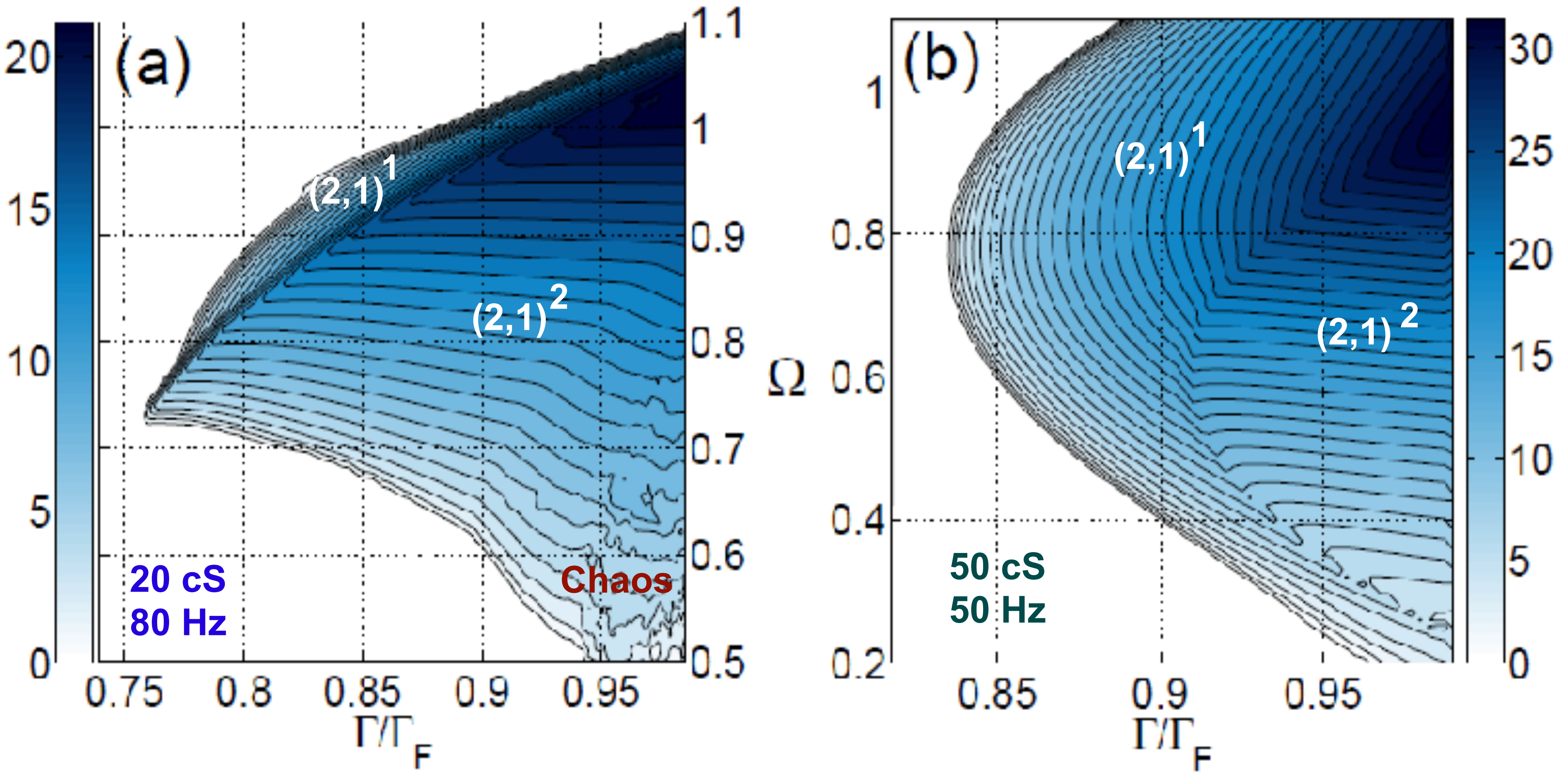
$(2,1)^1 \longrightarrow (2,1)^2$

Walking speed predictions

$$\Omega = \omega/\omega_D$$

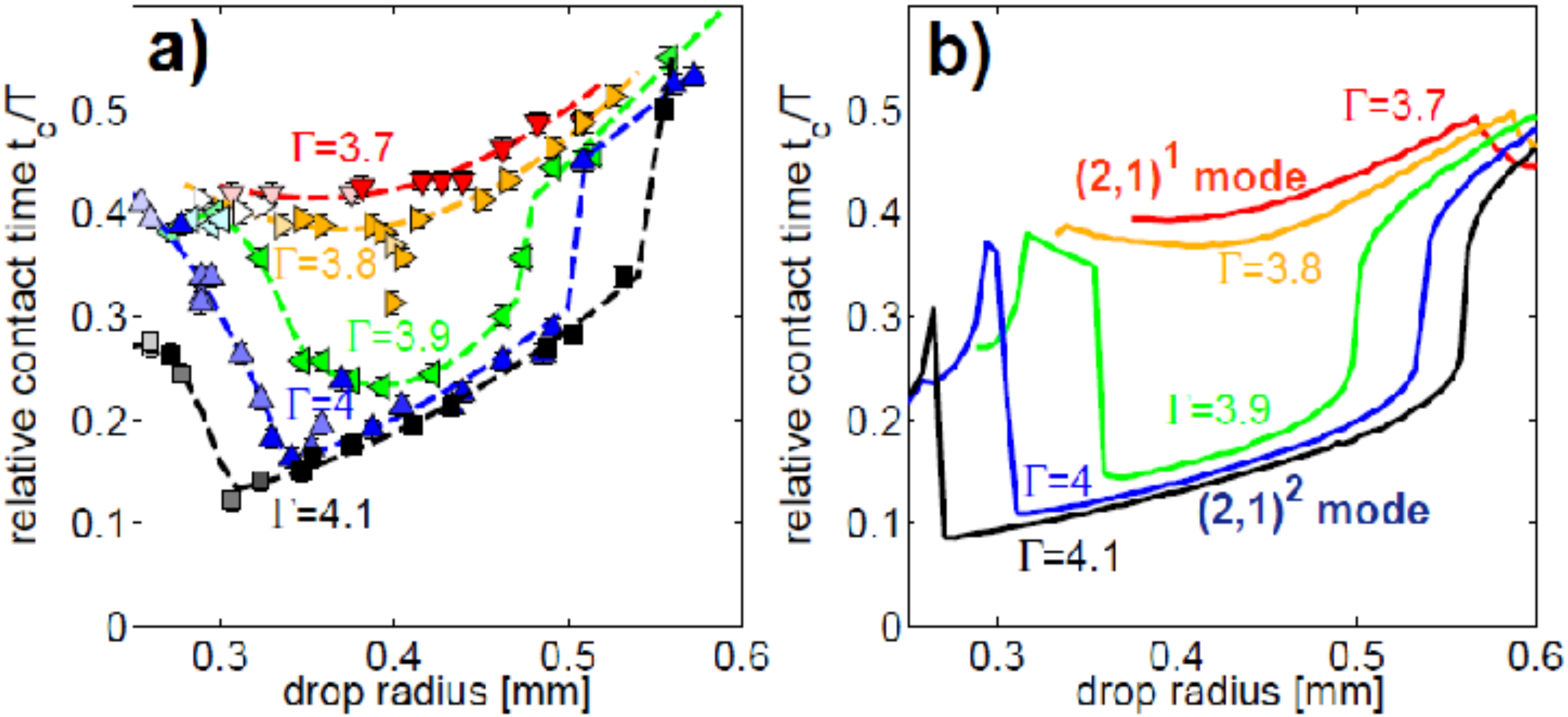
Walking speed [mm/s] depends on driving and drop size.

Mode switchings clearly visible in discontinuities.



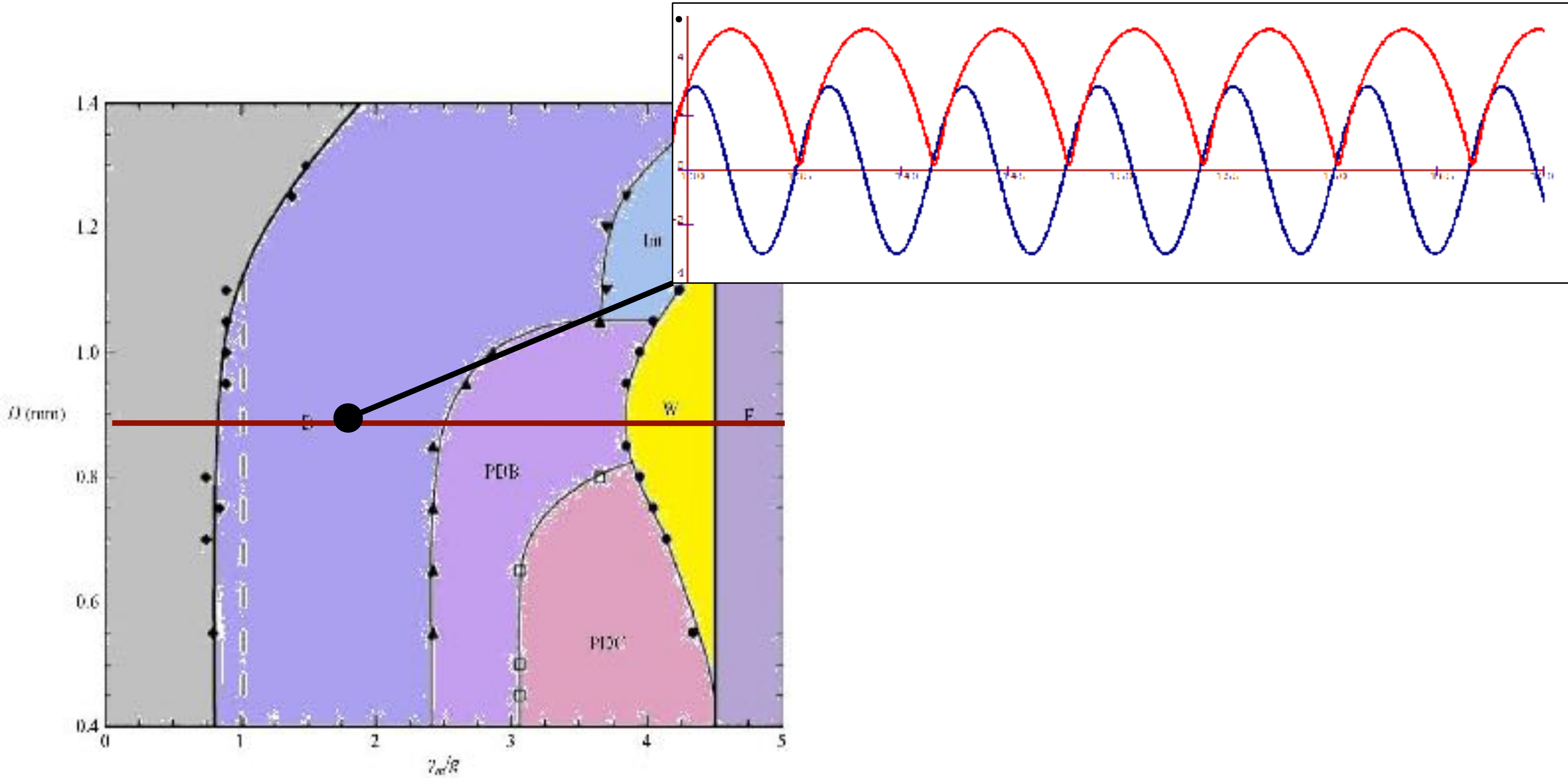
Contact time

Depends strongly on bouncing mode, minimal in the $(2,1)^2$



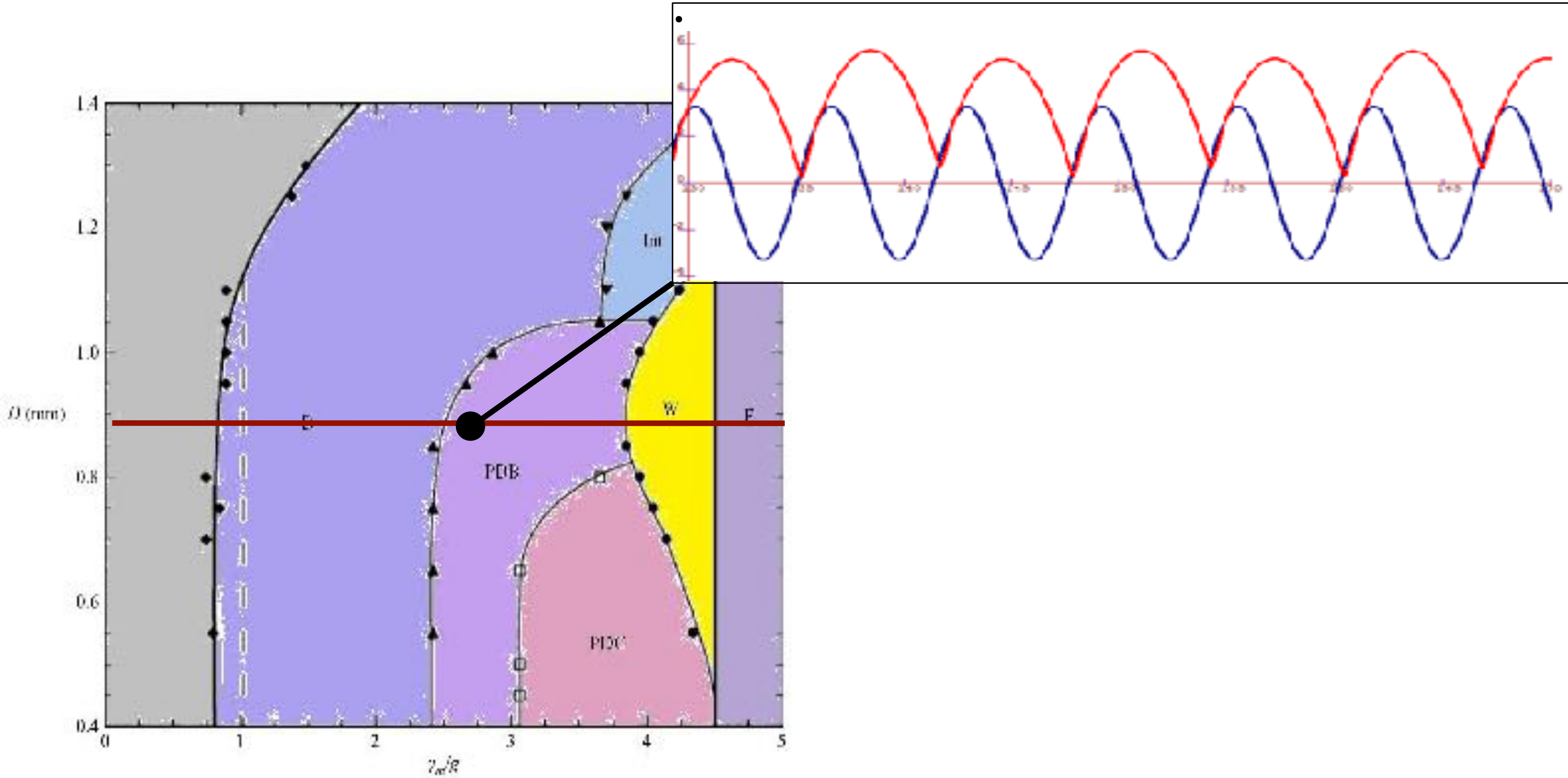
Regime diagram

- can trace evolution of vertical dynamics from bouncing to walking



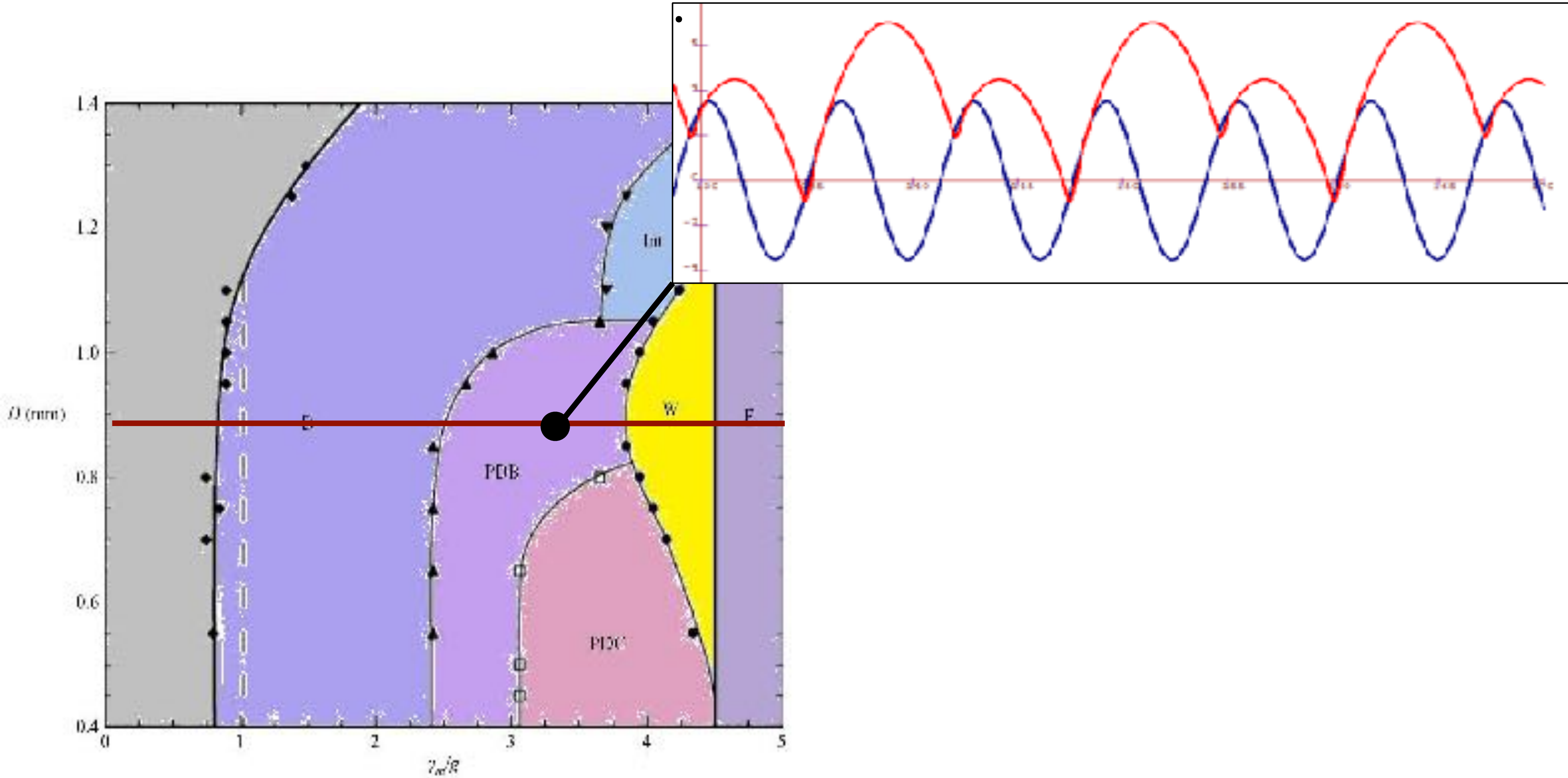
Regime diagram

- can trace evolution of vertical dynamics from bouncing to walking



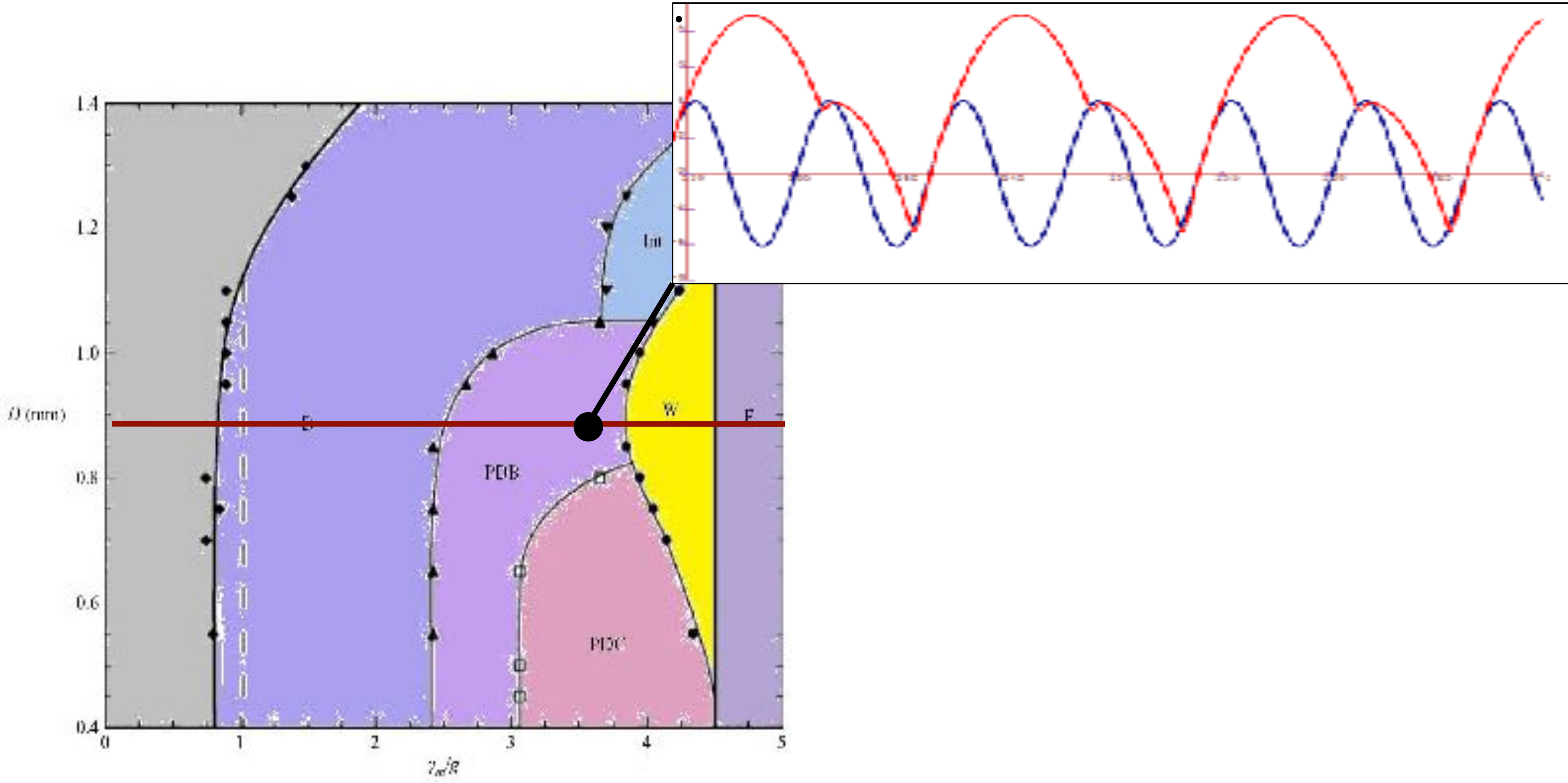
Regime diagram

- vertical dynamics in the period-doubled bouncing state



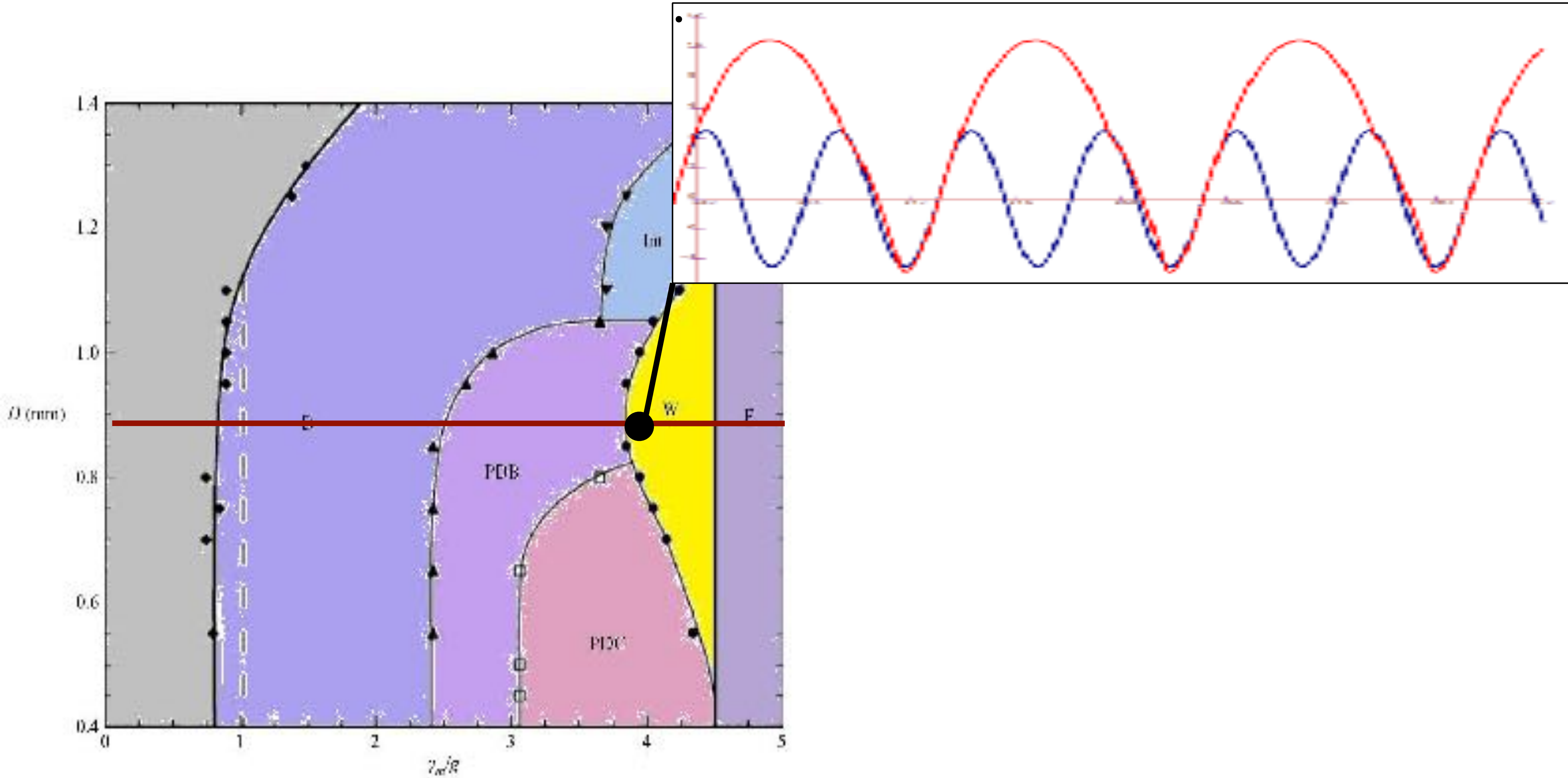
Regime diagram

- vertical dynamics depends on both driving acceleration



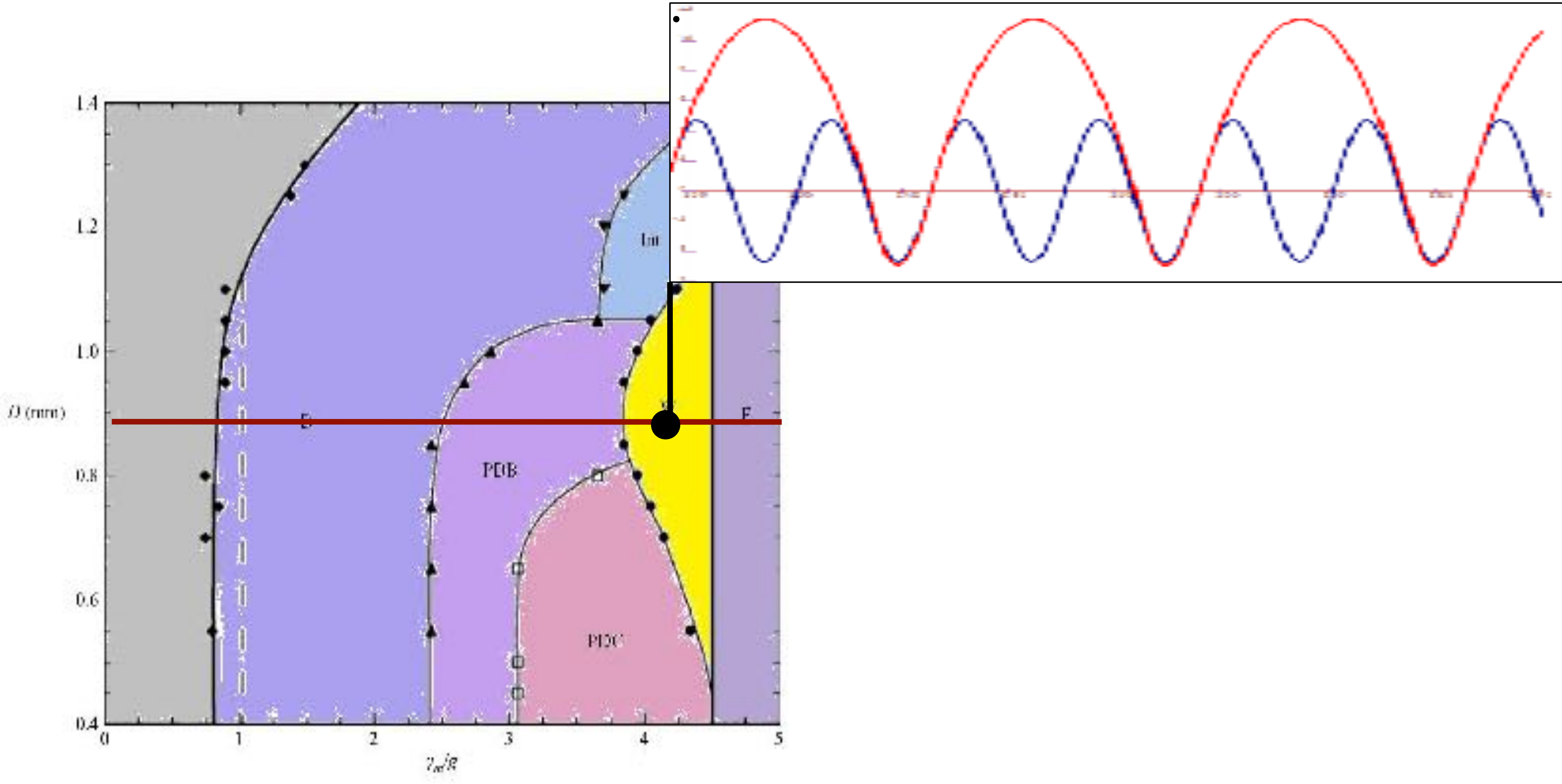
Regime diagram

- vertical dynamics of the low energy walking state **(2,1)[1]**



Regime diagram

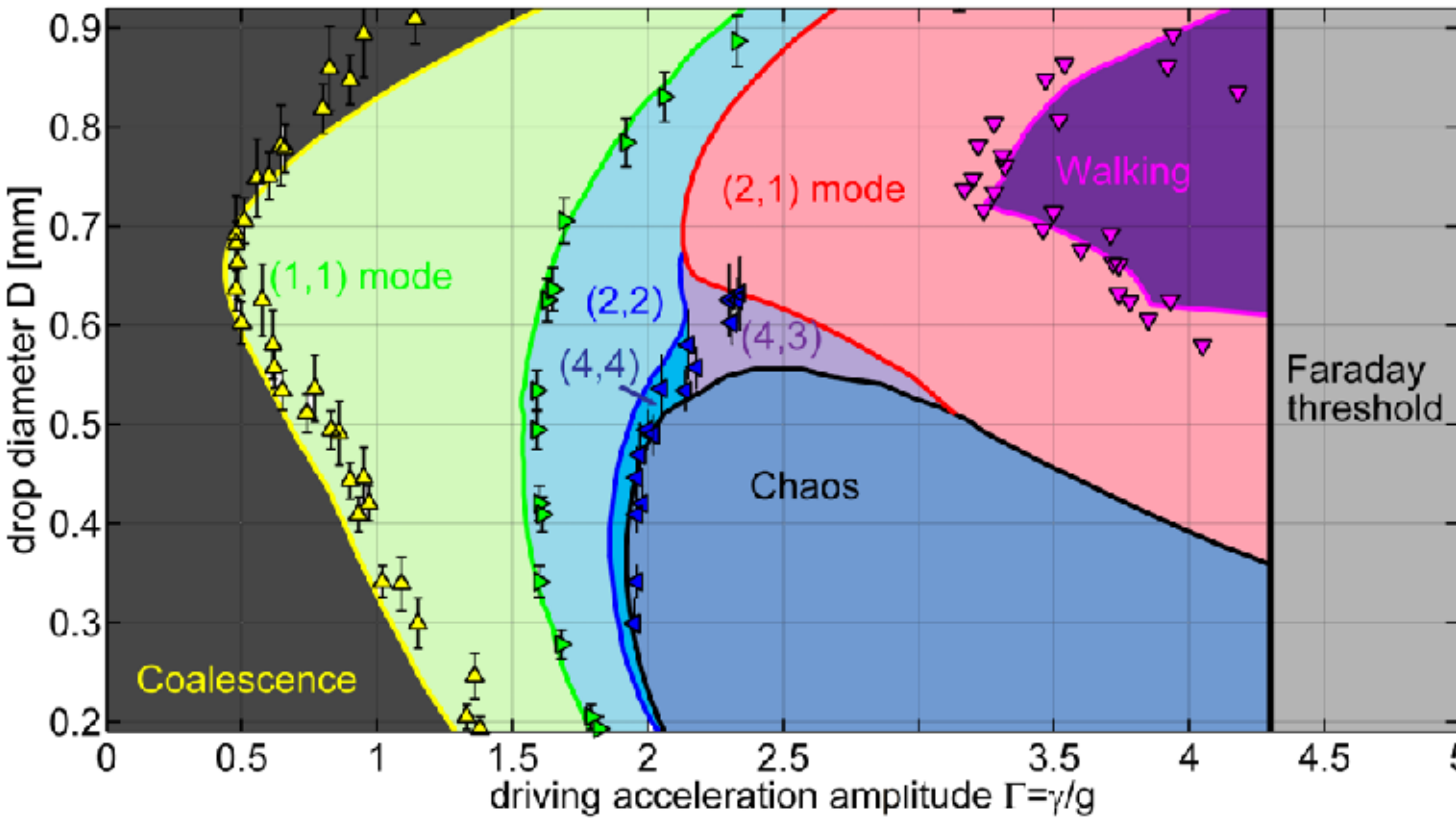
- vertical dynamics of the high energy walking state **(2,1)^2**



Regime diagram

20 cS silicone oil, 80Hz

- predicted on the basis of the nonlinear spring model for the interface



Molacek & Bush (2012b)

Predicted regime diagrams

20 cS, 80 Hz

$$\Omega = 2\pi f(\rho R_0^3/\sigma)^{1/2}$$