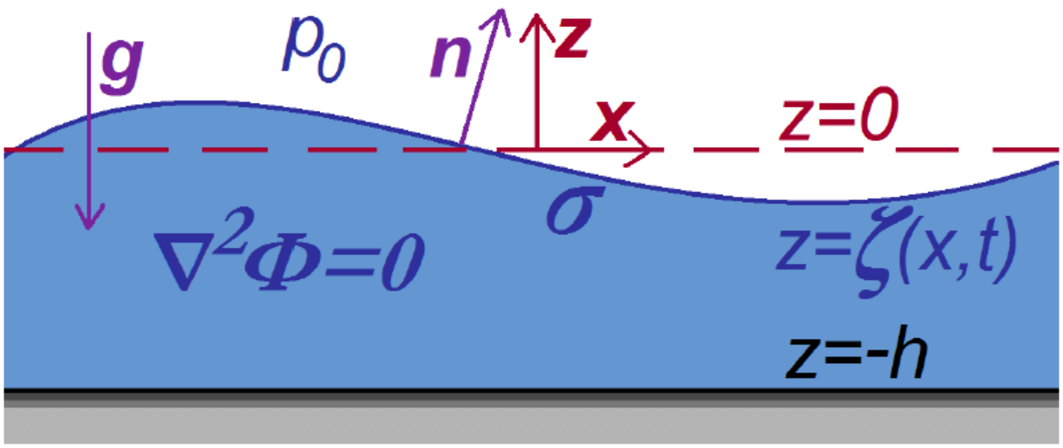
Lecture 8: Water Waves

Ref. Whitham's "Linear + Nonlinear Waves" textbook

Assume $Re \gg 1$, so that...



- motion of fluid may be described to leading order as inviscid and irrotational.
- must deduce a sol'n for the velocity potential ϕ satisfying (where $\underline{u} = \underline{\nabla}\phi$):

$$\boxed{\nabla^2 \phi = 0}$$

subject to kinematic + dynamic boundary conditions

B.C.s

1. $$\frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = -h$$

2. Kinematic BC: $$\frac{D\zeta}{Dt} = u_z \quad \text{on } z = \zeta$$

$$\Rightarrow \frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} = \frac{\partial \phi}{\partial z} \quad \text{on } z = \zeta$$

3. Dynamic B.C. (Time-dep Bernoulli applied at free surface)

$$\rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho |\underline{\nabla}\phi|^2 + \rho g \zeta + P_s = \underbrace{f(t)}_{\text{indep of } x}$$

where $P_s = P_0 + \sigma\, \underline{\nabla} \cdot \underline{n}$

Here $\underline{n} = \frac{(-\zeta_x, 1)}{(1+\zeta_x^2)^{\frac{1}{2}}}$ is the unit normal

$$\underline{\nabla}\cdot\underline{n} = \frac{-\zeta_{xx}}{(1+\zeta_x^2)^{3/2}} \quad \text{is the curvature}$$

$$p_s = p_0 - \sigma\frac{\zeta_{xx}}{(1+\zeta_x^2)^{3/2}}$$

Now consider small-amplitude waves and linearize the system of eqns and BCs (i.e. assume ϕ, ζ small, so neglect any terms involving ϕ^2, ζ^2, $\phi\zeta$ or their derivatives)

$$\nabla^2\phi = 0 \quad \text{in} \quad -h \le z \le 0$$

B.C.s $\underline{1}$. $\frac{\partial\phi}{\partial z} = 0$ on $z = -h$

$\underline{2}$. $\frac{\partial\zeta}{\partial t} = \frac{\partial\phi}{\partial z}$ on $z = 0$

$\underline{3}$. $\rho\frac{\partial\phi}{\partial t} + \rho g\zeta + p_0 - \sigma\zeta_{xx} = f(t)$ on $z=0$

Seek normal modes, sol'ns of the form:

$$\left.\begin{aligned}\zeta &= \hat{\zeta}\, e^{ik(x-ct)} \\ \phi &= \hat{\phi}_z(z)\, e^{ik(x-ct)}\end{aligned}\right\}$$

travelling waves in x-direction with phase speed $c = \omega/k$ and wavelength $\lambda = 2\pi/k$.

Subing into harmonic equation:

$$\hat{\phi}_{zz} - k^2\hat{\phi} = 0$$

$\Rightarrow$ solns are $\hat{\phi}(z) = e^{kz}, e^{-kz}$ <u>or</u> $\sinh kz, \cosh kz$

One may satisfy B.C. 1, $\dfrac{\partial \hat{\phi}}{\partial z} = 0$ on $z = -h$ by choosing:

$$\hat{\phi}(z) = A \cosh k(z+h)$$

$\leftarrow$ a constant

Now, <u>B.C. 2</u> $\Rightarrow -ikc\hat{\zeta} = Ak \sinh kh$ ☆

<u>B.C. 3</u> $\Rightarrow \left(-ikc\rho A \cosh kh + \rho g\hat{\zeta} + k^2\sigma\hat{\zeta}\right)e^{ik(x-ct)}$

$= f(t)$ indep of x

i.e. $-ikc\rho A \cosh kh + \rho g\hat{\zeta} + k^2\sigma\hat{\zeta} = 0$ ⊠

☆ $\Rightarrow A = \dfrac{-ic\hat{\zeta}}{\sinh kh} \Rightarrow$ sub into ⊠

$$\boxed{c^2 = \left(\frac{g}{k} + \frac{\sigma k}{\rho}\right)\tanh kh}$$

where $c = \dfrac{\omega}{k}$ is the phase speed

Since $c = \omega/k$, this yields the

<u>Dispersion Relation</u>:

$$\boxed{\omega^2 = \left(gk + \frac{\sigma k^3}{\rho}\right)\tanh kh}$$

Note: as $h \to \infty$, $\tanh h \to 1$ and we obtain the deep-water dispersion relation $\omega^2 = gk + \frac{\sigma k^3}{\rho}$

Physical Interpretation

- the relative importance of surface and gravity is prescribed by the Bond number

$$B_o = \frac{\rho g}{\sigma k^2} = \frac{\rho g \lambda^2}{4\pi^2 \sigma} = \frac{\lambda^2}{\lambda_c^2}$$

- for air-water, $B_o \sim 1$ for $\lambda_c \sim 1.7\,\text{cm}$ CAP LENGTH
- for $B_o \gg 1$ ($\lambda \gg \lambda_c$), surface tension negligible
  $\Longrightarrow$ GRAVITY WAVES
- for $B_o \ll 1$ ($\lambda \ll \lambda_c$), gravity negligible
  $\Rightarrow$ CAPILLARY WAVES

Special Cases

Recall: Shallow ($kh \ll 1$): $\tanh kh \simeq kh - \frac{1}{3}k^3h^3 + \dots$

Deep ($kh \gg 1$): $\tanh kh \simeq 1$

$\boxed{A.}$ Gravity Waves : $B_o \gg 1$, $c^2 = \frac{g}{k}\tanh kh$

a.) Shallow Water ($kh \ll 1$) $\Rightarrow$ $c = \sqrt{gh}$

- all wavelengths travel at same speed
  i.e. NON-DISPERSIVE

$\Rightarrow$ one can only surf in shallow water

b.) Deep Water ($kh \gg 1$) $\Rightarrow$ $c = \sqrt{g/k}$

- long waves travel fastest

e.g. drop large stone in a pond

B. Capillary Waves: $B_o \ll 1$, $c^2 = \frac{\sigma k}{\rho} \tanh kh$

a.) Deep water: $kh \gg 1 \Rightarrow c = \sqrt{\frac{\sigma k}{\rho}}$

- short waves travel fastest

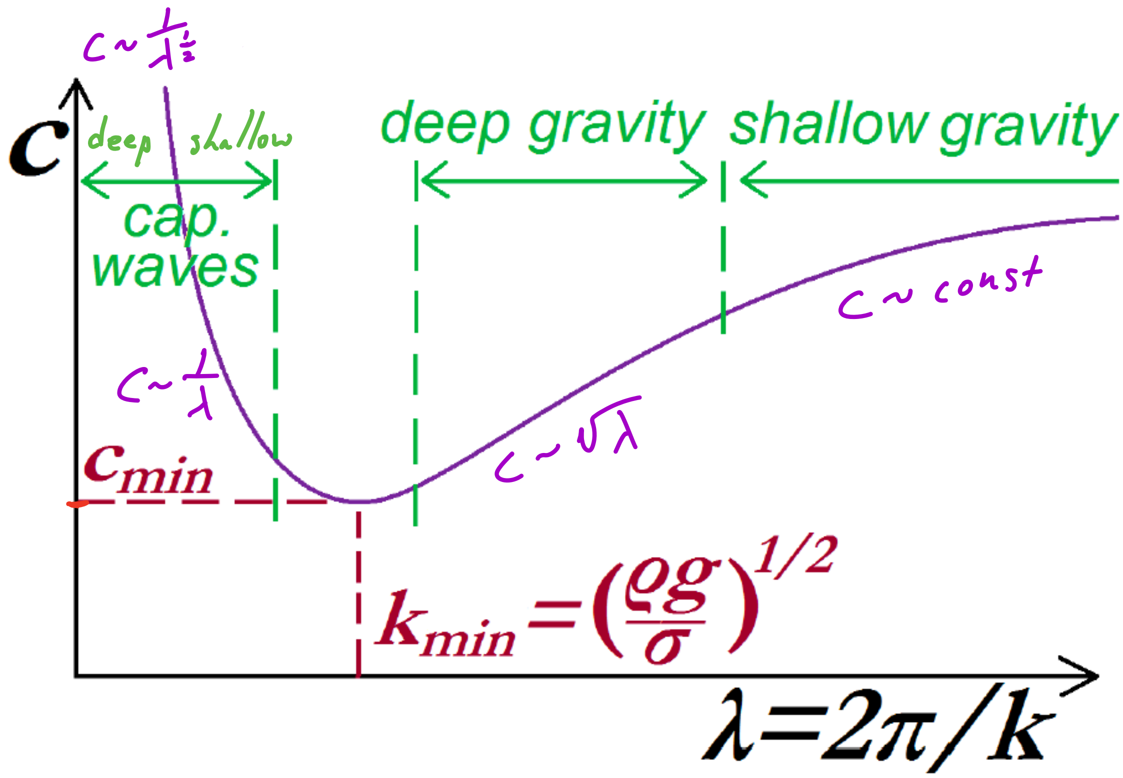
e.g. raindrop hits a pond

- short waves also quickly damped by viscosity

b.) Shallow water: $kh \ll 1$

$$\Rightarrow c = \sqrt{\frac{\sigma h k^2}{\rho}}$$

Summary



Note: 1. Four distinct scalings for $C(\lambda)$.

2. When $\frac{dc}{dk} = 0$, we have

$$C_{min} = \left(\frac{4 g \sigma}{\rho}\right)^{\frac{1}{4}} \quad \text{for} \quad k = \left(\frac{\rho g}{\sigma}\right)^{\frac{1}{2}}$$

3. Group Velocity: when $c = c(\lambda)$, a wave is called DISPERSIVE since the different (Fourier) wave components (corresponding to different λ, k) separate or disperse

e.g. deep-water gravity: $c \sim \sqrt{\lambda}$

- in a dispersive system, the energy of a wave component does not propagate at the phase speed $c = \omega/k$, but rather at the...

GROUP VELOCITY : $c_g = \frac{d\omega}{dk} = \frac{d}{dk}(ck)$

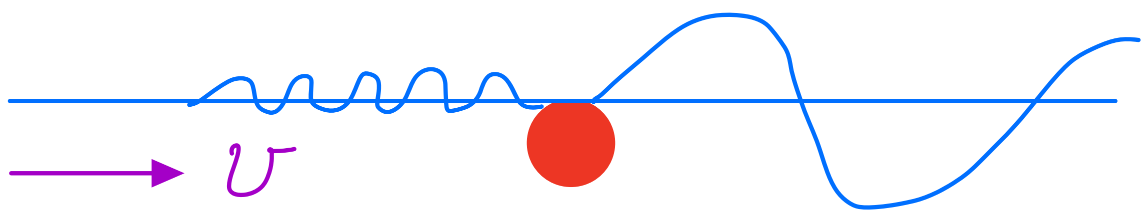
e.g. DEEP GRAVITY WAVES : $c_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}c$

DEEP CAPILLARY WAVES : $c_g = \frac{3}{2}c$

4. Flow past an obstacle

Note c_{min} : if $U < c_{min}$, no steady waves generated by the obstacle

- if $U > c_{min}$: there are 2 k-values for which $c = U$



i) the smaller represents a gravity wave with

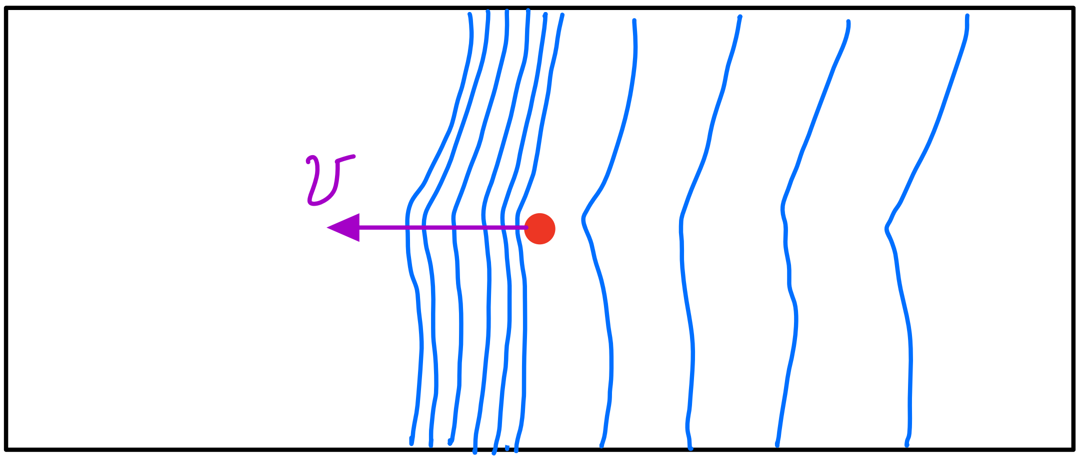
$c_g = c/2 < c$ $\Longrightarrow$ energy swept downstream

ii) the larger k represent a capillary wave with

$c_g = \frac{3}{2}c > c$ $\Longrightarrow$ energy swept upstream

(but quickly dissipated due to small λ)

e.g. waves generated by a fishing line



5. Gravity – Capillary Waves

- neither g nor σ effects negligible
  e.g. in PWH, $\lambda \sim \ell_c$
- consider short waves ($kh \gg 1$), for which

$$\omega^2 = gk + \frac{\sigma}{\rho} k^3$$

$\ell_c = \left(\frac{\sigma}{\rho g}\right)^{\frac{1}{2}}$

$$\Rightarrow C(k) = \left(\frac{g}{k} + \frac{\sigma}{\rho} k\right)^{\frac{1}{2}} \text{ and } C_g(k) \overset{*}{=} \frac{C}{2} \frac{1 + 3\ell_c^2 k^2}{1 + \ell_c^2 k^2}$$

Phase velocity has MIN at $k_{min} = \frac{1}{\ell_c}$, where $C_g = C = 23.2 \frac{cm}{s}$

Gravity branch: $\lambda > \lambda_m$, $C_g < C$

Capillary branch: $\lambda < \lambda_m$, $C_g > C$

For any given $C > C_{min}$, there are 2 possible λ.

MIN $C_g(k) = 0.77\, C_{min} = 17.9$ cm/s arises at $\lambda = 2.54\, \lambda_m$

Note: in lab modelling of shallow water waves

In $kh \ll 1$ limit, dispersion relation yields

$$c^2 \simeq \left(\frac{g}{k} + \frac{\sigma k}{\rho}\right)\left[kh - \frac{k^3h^3}{3} + O(kh)^5\right]$$

$$= gh + \left(\frac{\sigma h}{\rho} - \frac{1}{3}gh^3\right)k^2 + O(kh)^4 gh$$

$\Rightarrow$ can get a good approx to nondispersive waves for $h_c = \left(\frac{3\sigma}{\rho g}\right)^{\frac{1}{2}}$

Eg. in water, $h_c = \left(\frac{3 \cdot 70}{10^3}\right)^{\frac{1}{2}} \sim 0.5$ cm

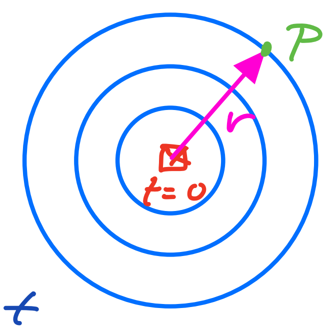
in Si oil, $h_c \sim \left(\frac{3 \cdot 30}{10^3}\right)^{\frac{1}{2}} \sim 0.2$ cm

5. Dispersion from an Instaneous Point Source

e.g. an impactor of scale L

- waves from a point source spread out isotropically
- spectrum depends on I.C.s, but typically one expects $0 \leq \lambda \leq L$ with a peak at $\lambda \sim L$
- different k values propagate out with corresponding group velocities $c_g(k)$
- at time t, any particular k value will be found at $r = c_g(k)t$. Hence $k(r,t)$ is the sol'n of

$$c_g(k) = \frac{r}{t}$$

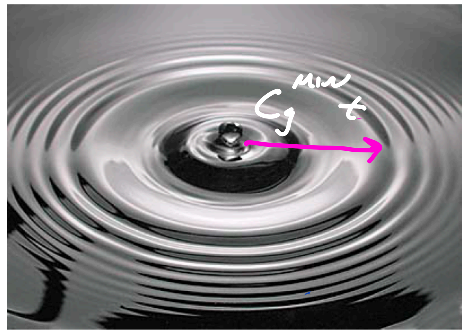


Eg. 1 Deep water gravity waves

$$c_g(k) = \frac{1}{2}\sqrt{g/k} \Rightarrow k = \frac{gt^2}{4r^2}, \quad \omega = \frac{gt}{2r}$$

$\Rightarrow$ confirmed for ocean waves produced by storms in the South Pacific

Eg. 2 Gravity capillary waves



Recall $$C_g(k) = \frac{c}{2}\,\frac{1+3l_c^2k^2}{1+l_c^2k^2}$$

- since $C_g(k)$ has a MIN value of 18 cm/s, there is a quiescent circle propagating out from source
- beyond $r = C_g^{MIN} t$, there are two values of k for each r/t, one on gravity branch, one on capillary branch $\Rightarrow$ two superposed wave trains

  $\Rightarrow$ shorter wavelengths more quickly damped