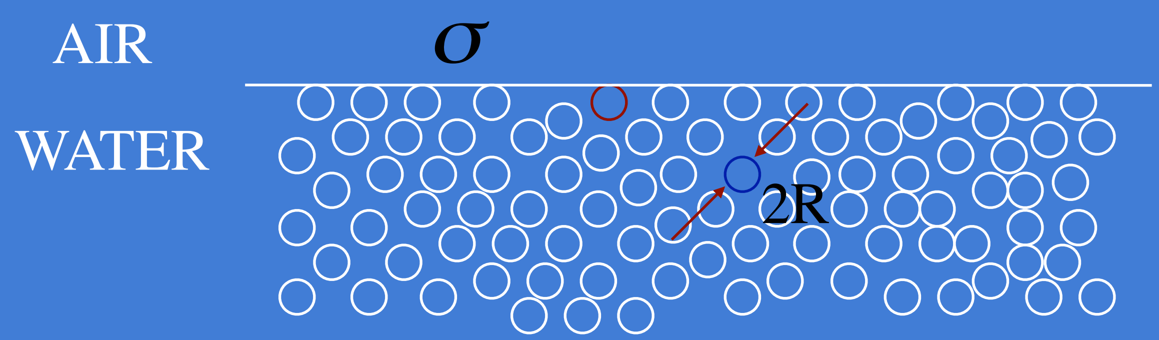
# Lec. 6  Interfacial Fluid Mechanics

- surface tension and interfacial flows

# Surface Tension: molecular origins

- each molecule in a fluid feels a cohesive force with surrounding molecules
- molecules at interface feel half this force; are in an energetically unfavourable state
- the creation of new surface is thus energetically costly




- cohesive energy per molecule of radius R in bulk is U, at surface is U/2
- surface tension is this loss of cohesive energy per unit area:

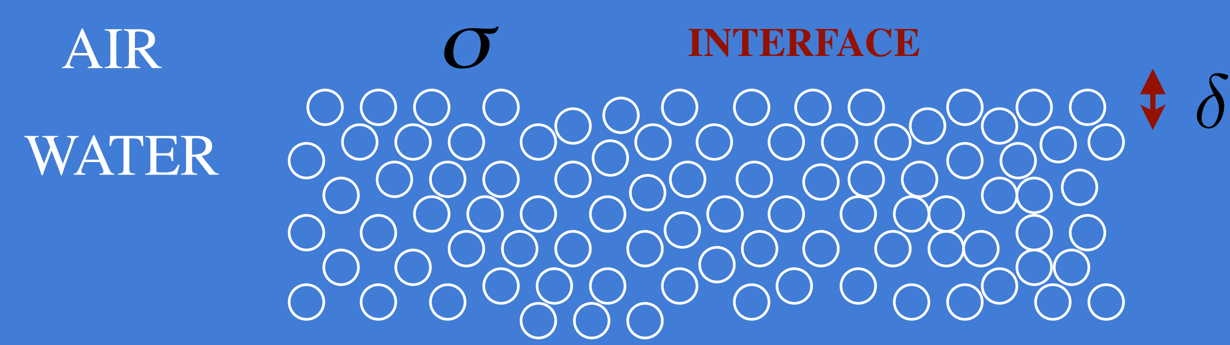
$$\sigma \sim \frac{U}{R^2}$$

Units: $[\sigma] = \frac{\text{ENERGY}}{\text{AREA}} = \frac{\text{FORCE}}{\text{LENGTH}}$

- air-water $\sigma \sim 70$ dyne/cm; oils $\sigma \sim 20$ dyne/cm; liquid metals $\sigma \sim 500$ dyne/cm

# What is an interface?

- an idealized surface between two immiscible fluids; e.g. oil-water, air-water, oil-air
- there is no surface tension between miscible fluids, e.g. water-salt water
- in reality, the interface is rough on a molecular scale




Roughness scale $\delta$

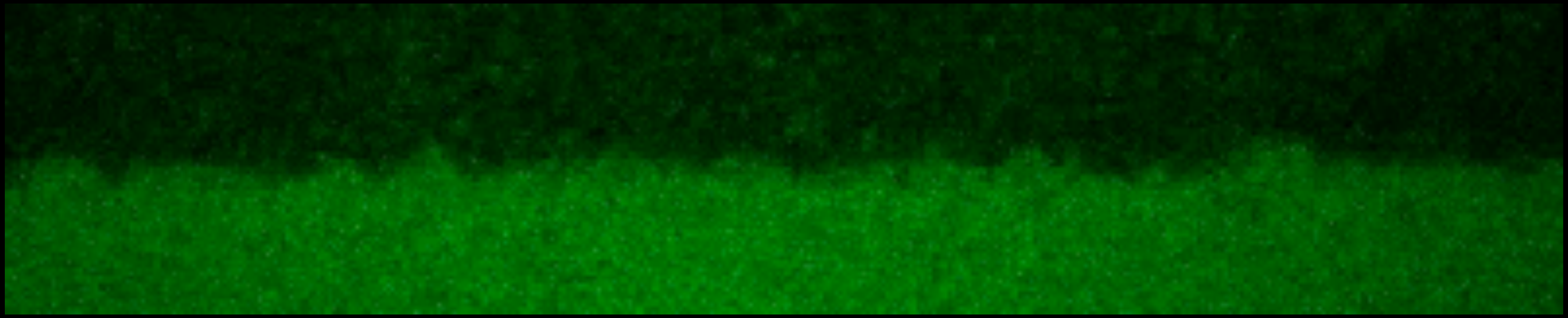
- equate anomalous surface energy with thermal agitation energy

$$\sigma\delta^2 \sim kT \longrightarrow \boxed{\delta \sim (kT/\sigma)^{1/2}}$$

- treating the interface as sharp is consistent with the continuum hypothesis, wherein one assumes fluids are smooth beyond 10 molecular dimensions

(Video)

# Evaporation across a fluid interface



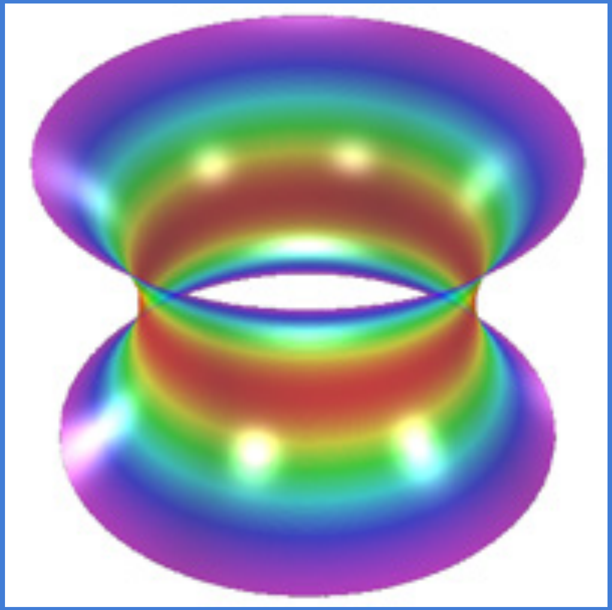
- thermal agitation overcomes interfacial tension

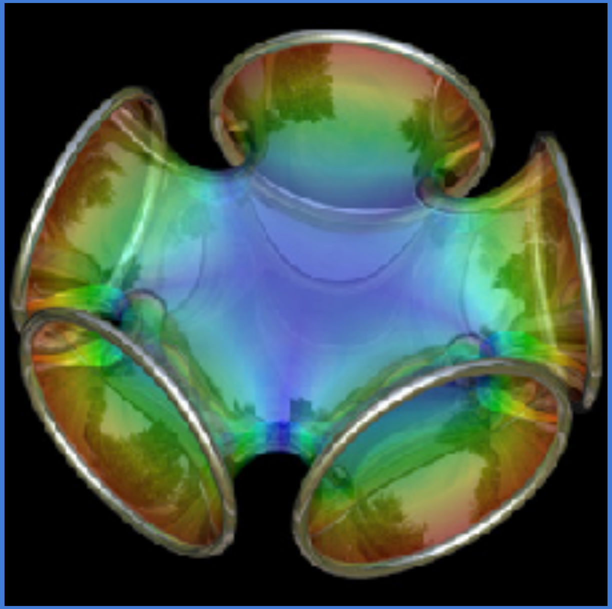
# The creation of surface is energetically costly

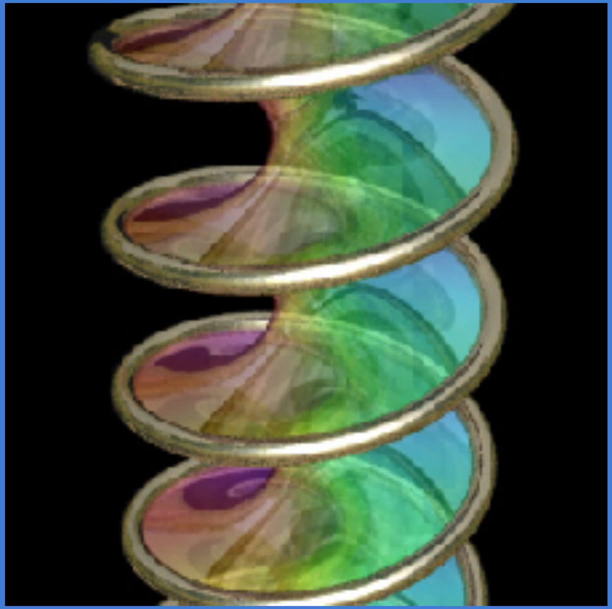
- quasi-static soap films (for which gravity, inertia are negligible) take the form of minimal surfaces

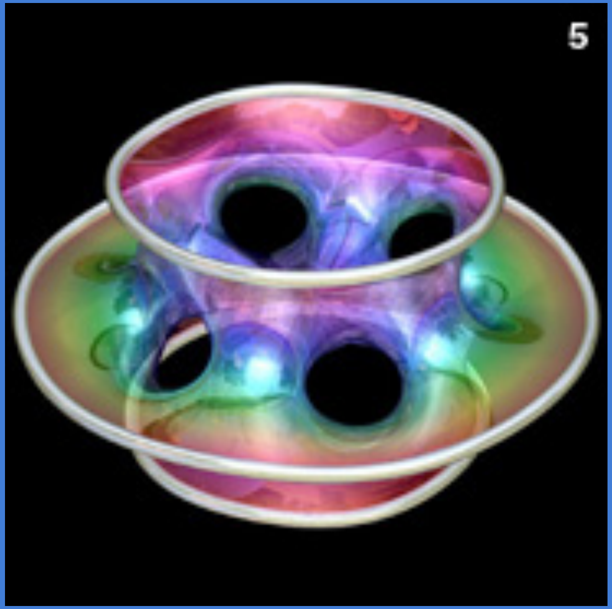
- hence their interest to mathematicians:

*"Find the minimal surface bound by the multiply connected curve C, where C ...."*









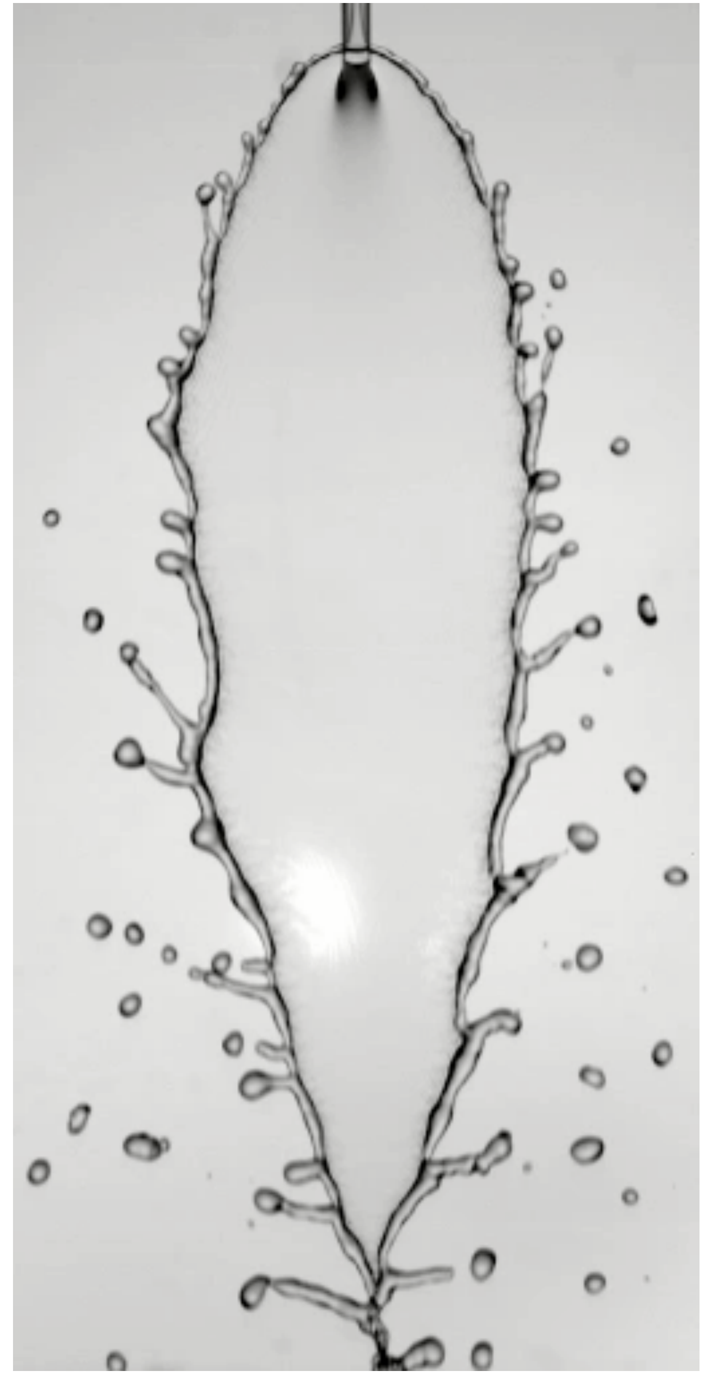
# The minimal surface between a pair of rings

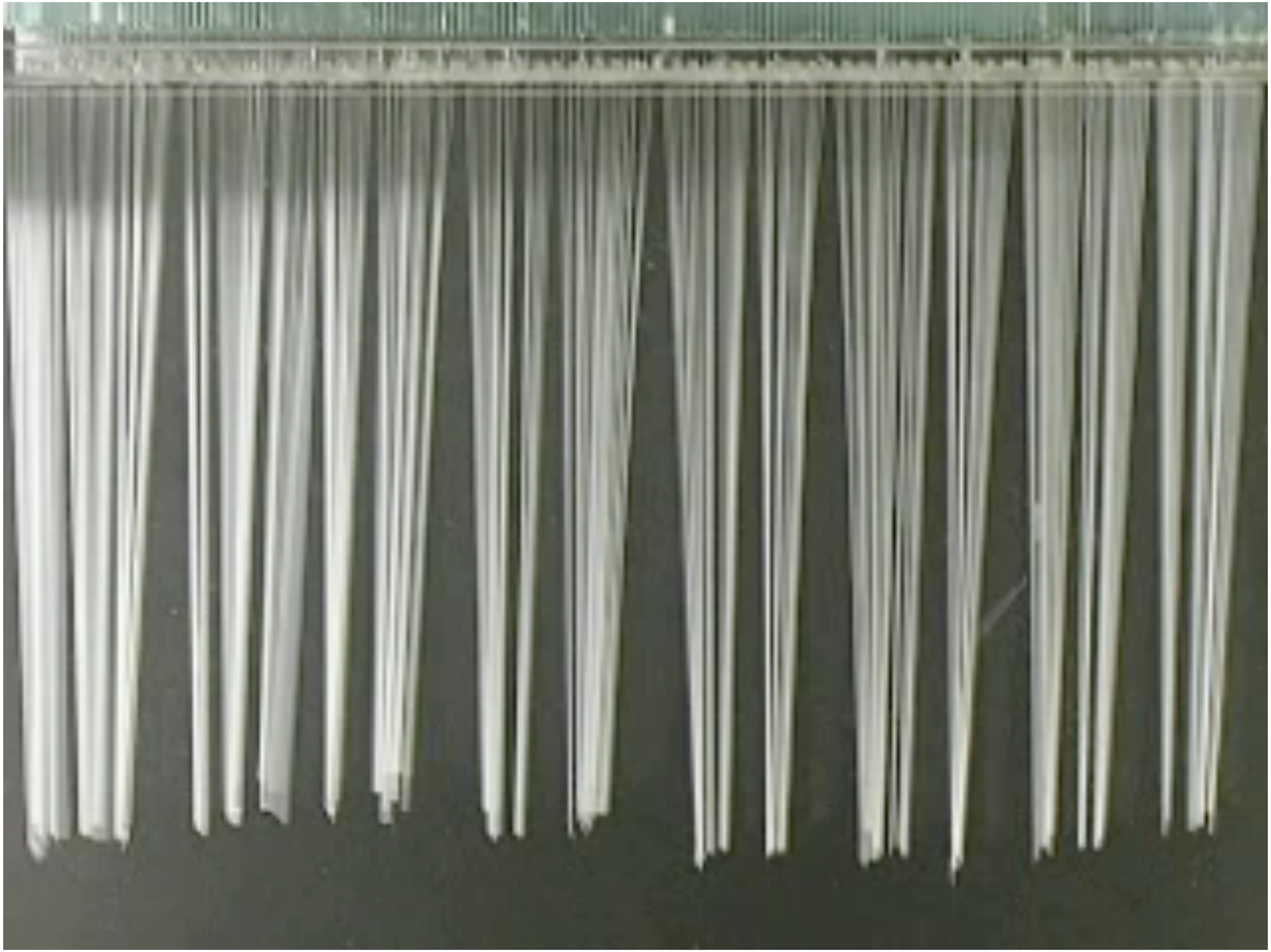
A catenoid when the rings are close,
a pair of circles when they are far apart.

# The creation of surface is energetically costly

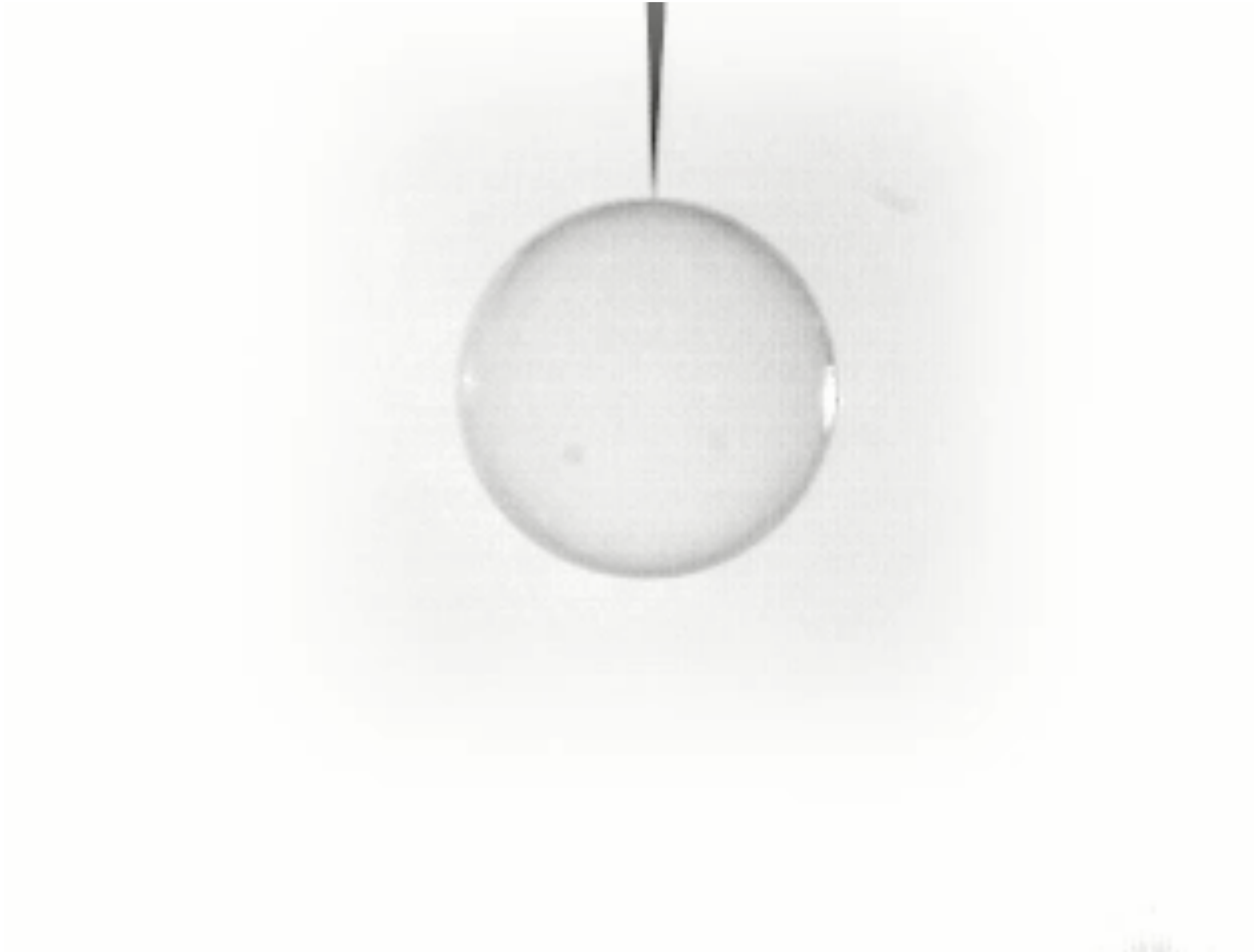
Thus:

- small drops are nearly spherical
- fluid jets pinch off into droplets
- fluid atomization results in spherical drops
- wet hair sticks together: the “wet look”
- bubbles and films are fragile

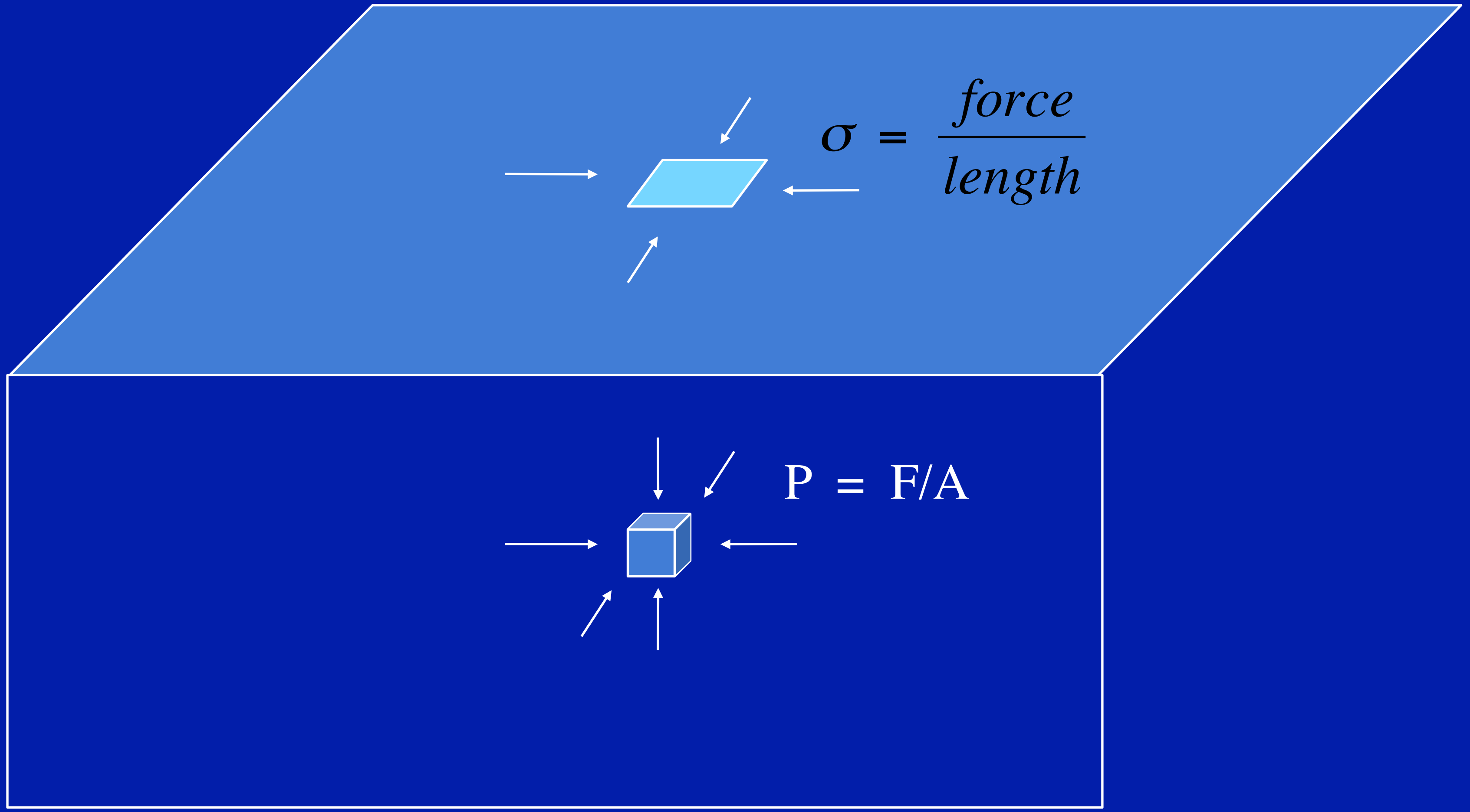


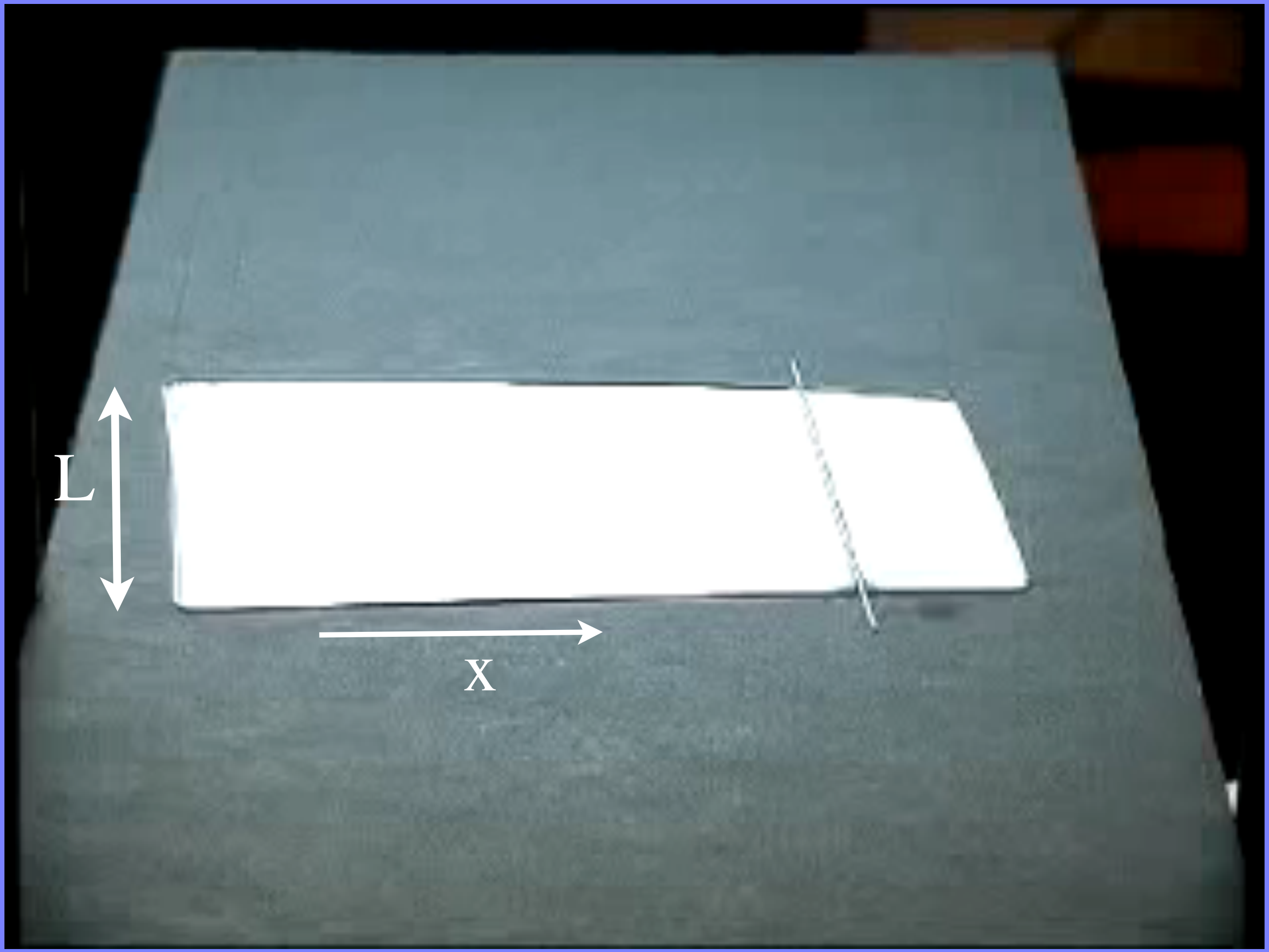
**The wet hair instability: threads clump to minimize surface energy**



**Surface tension:** analogous to a negative surface pressure

- gradients in surface tension necessarily drive surface motion



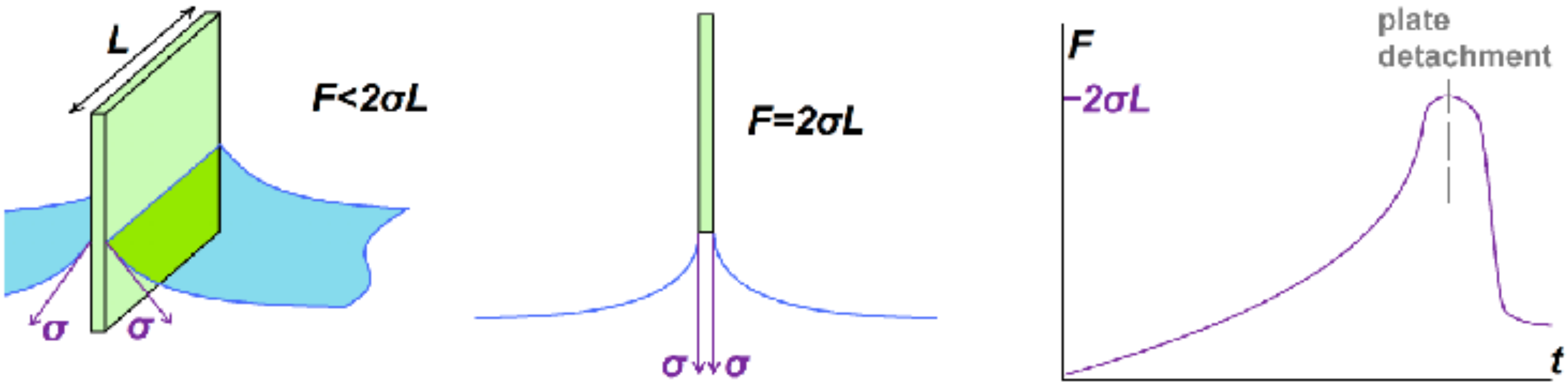



Surface tension: $[\,\sigma\,] = \dfrac{FORCE}{LENGTH} = \dfrac{ENERGY}{AREA}$

Surface energy: $E_\sigma = \int_S \sigma \, dA = 2\,\sigma\,L\,x$

Force acting on rod: $F = \dfrac{dE_\sigma}{dx} = 2\,\sigma\,L$

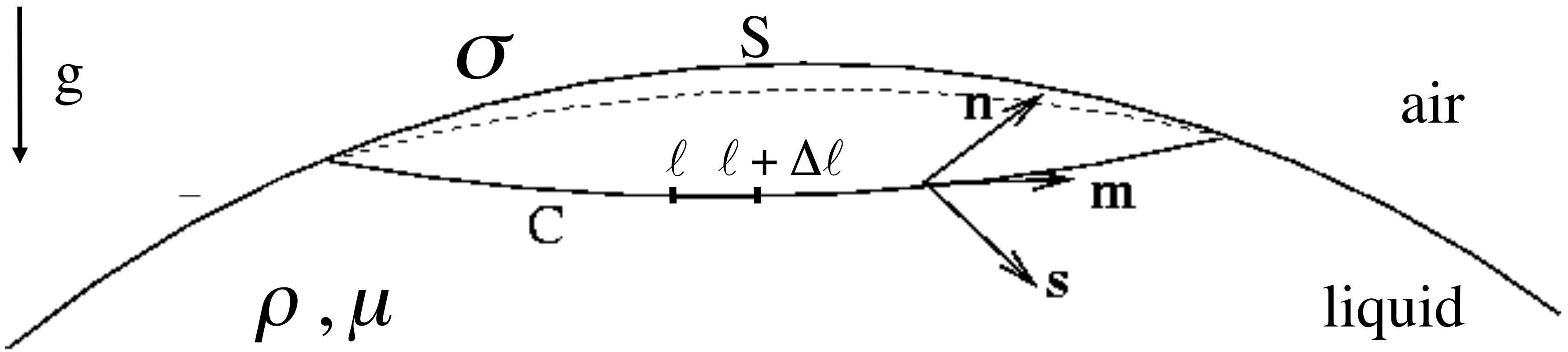
# A simple way to measure surface tension




- measure the force required to withdraw a plate from a free surface

# Surface tension: Geometry

Along a contour C bounding a surface S there is a tensile force per unit length $\sigma$ acting in the $\mathbf{s}$ direction




Net force on S:

$$\oint_C \sigma \, \mathbf{s} \, d\ell = \iint_S \sigma \, (\nabla \cdot \mathbf{n}) \, \mathbf{n} \, dS + \iint_S \nabla\sigma \, dS$$

curvature pressure — Marangoni stress

1) normal curvature pressure $\sigma \nabla \cdot \mathbf{n}$ resists surface deformation

2) tangential Marangoni stresses may arise from $\nabla\sigma$

# Interfacial fluid mechanics: Governing Equations

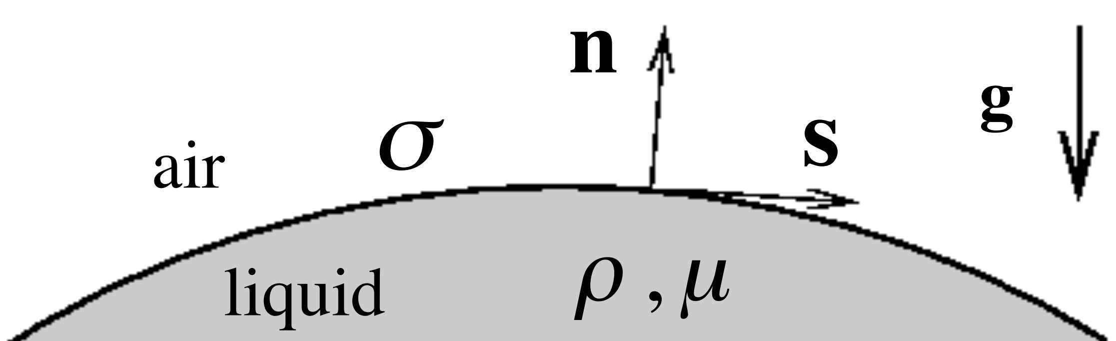
## Navier-Stokes equations:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{u} \quad , \qquad \nabla\cdot\mathbf{u} = 0$$

## Boundary Conditions

Normal stress: $\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{n}\,| = \sigma\,\nabla\cdot\mathbf{n}$

Tangential stress: $\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{s}\,| = \nabla_s\sigma$

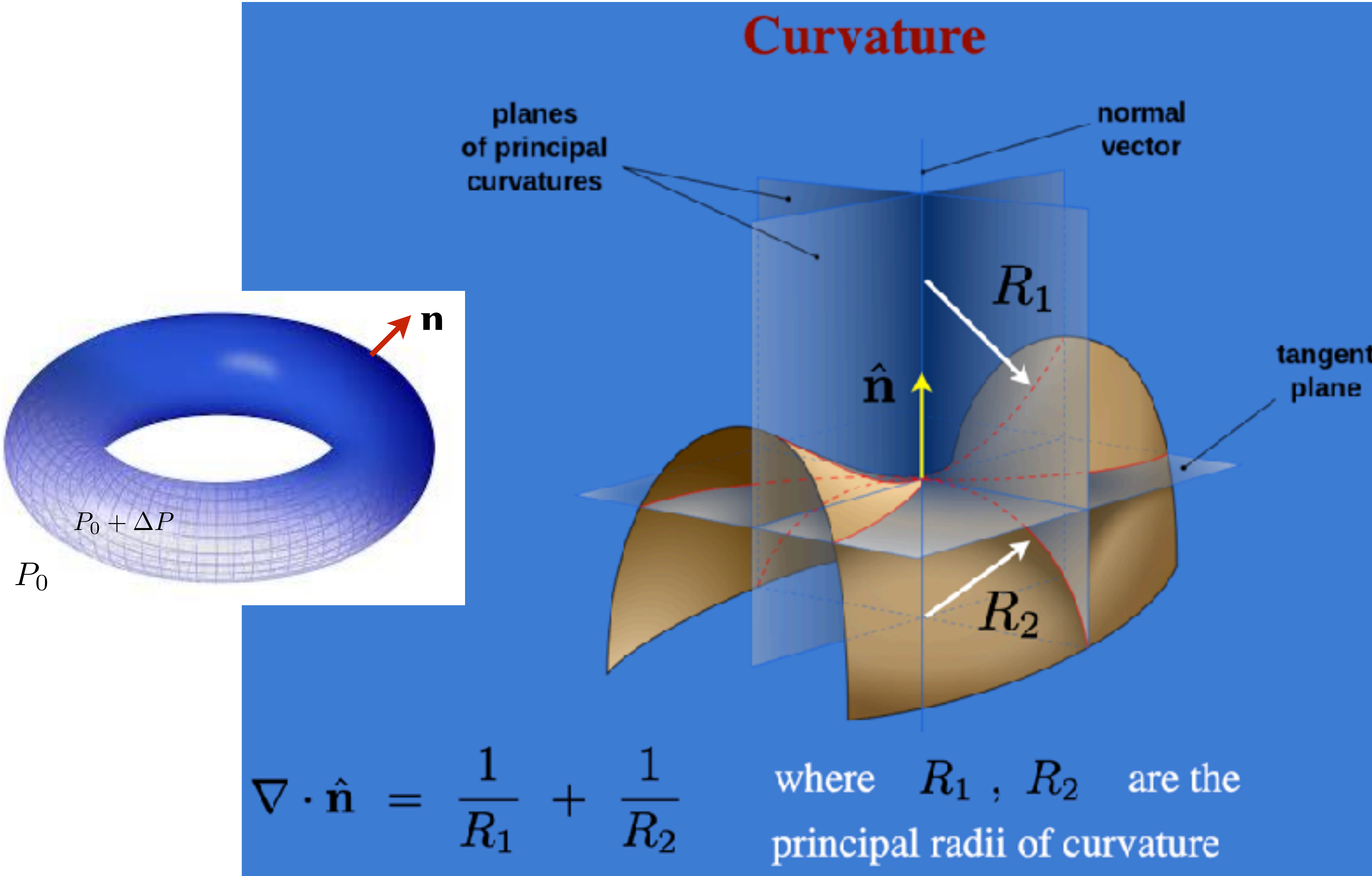



Stress tensor

$$\mathbf{T} = -p\mathbf{I} + \mu\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$$

# Laplace pressure

- at a curved interface, there is a pressure jump: $\Delta P = \sigma \nabla \cdot \mathbf{n}$

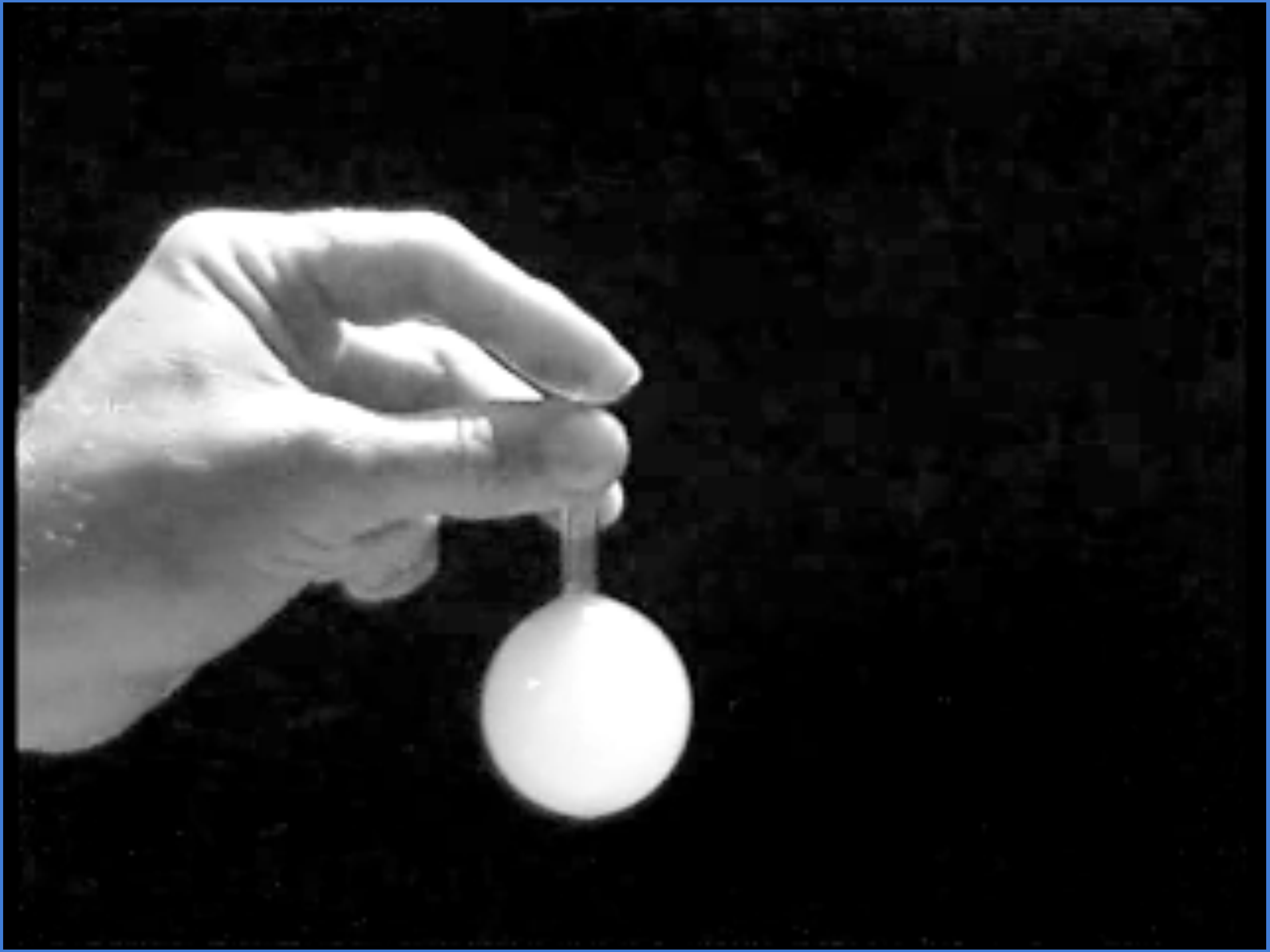


Curvature pressures, $\sigma \nabla \cdot \mathbf{n}$, make the surface behave as a trampoline.

Curvature: $\nabla \cdot \hat{\mathbf{n}} = \frac{1}{R_1} + \frac{1}{R_2}$ = 2/R for a sphere

Capillary pressure: $\Delta p = \frac{4\sigma}{R}$

Which way does the air go?

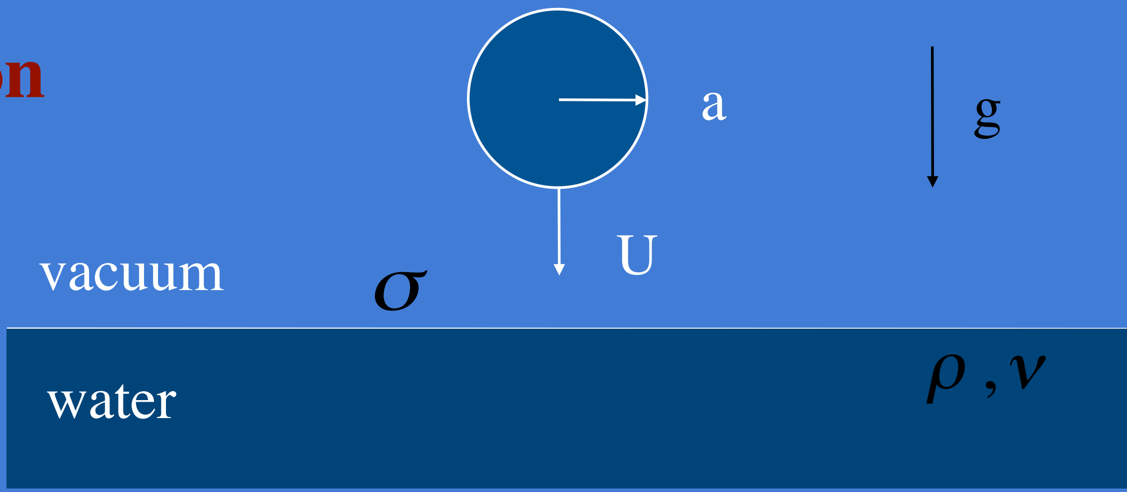


**Laplace pressure drives vigorous flow**

# Who cares about surface tension ?

When is it important ?

# The scaling of surface tension




$$W_e = \frac{\rho U^2 a}{\sigma} = \frac{\text{INERTIA}}{\text{CURVATURE}} = \text{Weber number}$$

$$C_a = \frac{\rho \nu U}{\sigma} = \frac{\text{VISCOSITY}}{\text{CURVATURE}} = \text{Capillary number}$$

$$B_o = \frac{\rho g a^2}{\sigma} = \frac{\text{GRAVITY}}{\text{CURVATURE}} = \text{Bond number}$$

Note: $\sigma$ is dominant relative to gravity when $B_o < 1$

i.e. $\mathrm{a} < \left(\frac{\sigma}{\rho \mathrm{g}}\right)^{1/2} = \ell_c =$ capillary length ~ 2mm for air-water

# When is surface tension important relative to gravity?

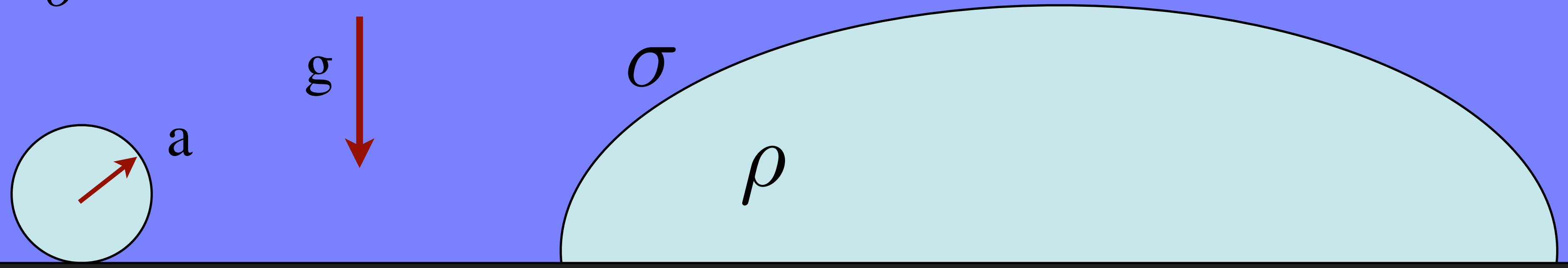
- when curvature pressures are large relative to hydrostatic:

Bond number: $$B_o = \frac{\rho g a}{\sigma / a} = \frac{\rho g a^2}{\sigma} < 1$$

i.e. for drops small relative to the capillary length:

$$\boxed{a < l_c = \left(\frac{\sigma}{\rho g}\right)^{1/2}} \sim$$ 2 mm for air-water ($\sigma = 70$ dynes/cm)

$B_o \ll 1$ $\quad$ $B_o > 1$




Surface tension dominates the world of insects - and of microfluidics.

# Falling rain drops

Force balance:

$$\rho_a U^2 a^2 \sim M g = \tfrac{4}{3}\pi a^3 \rho g$$

Fall speed: $U \sim \sqrt{\frac{\rho g a}{\rho_a}}$

Drop integrity requires:

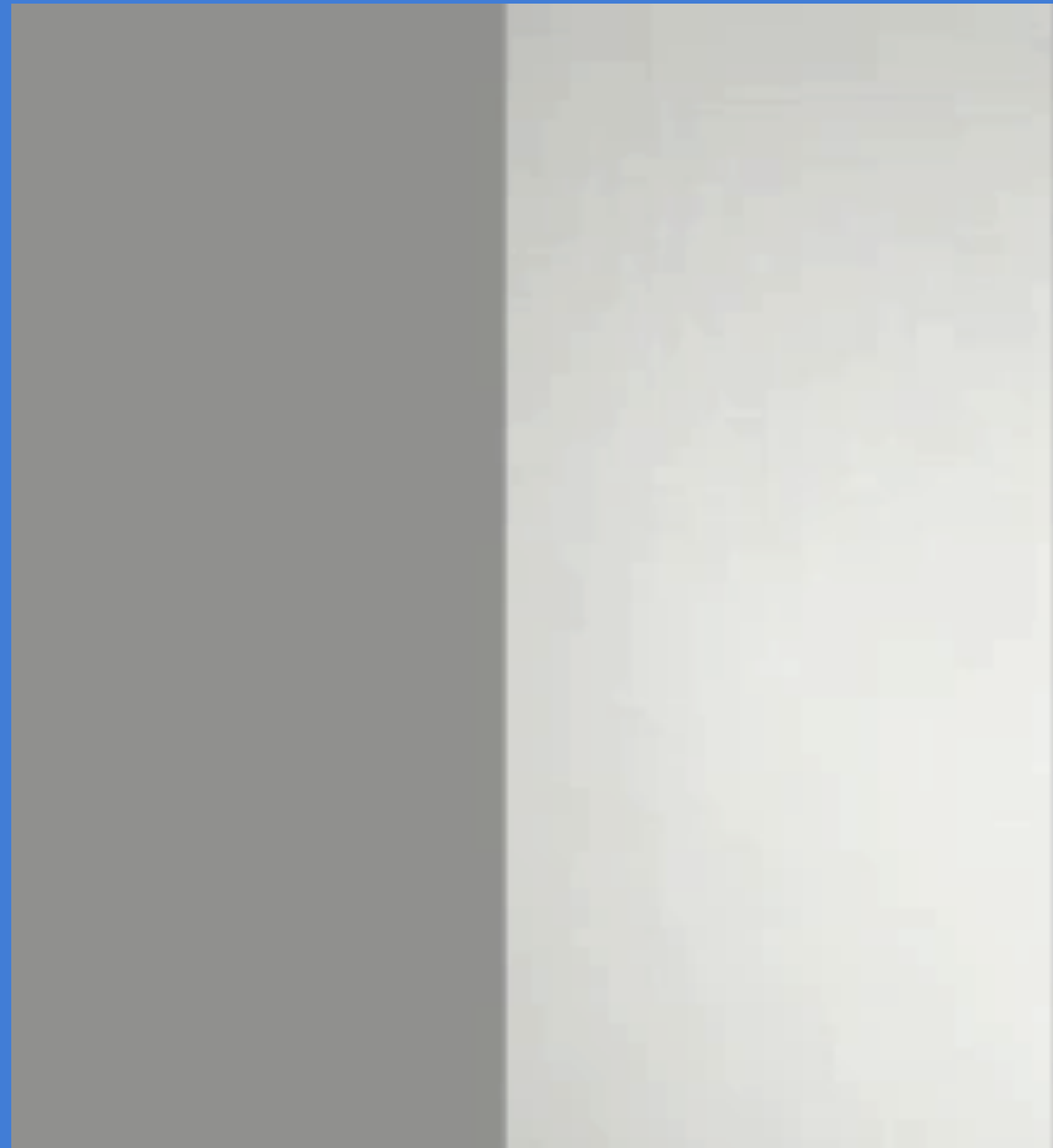
$$\rho_a U^2 \sim \rho g a < \sigma / a$$

# Small drops

If a drop is small relative to the capillary length

$$a < \ell_c = \sqrt{\sigma / \rho g} \approx 2\text{mm} \quad ,$$

$\sigma$ maintains it against the destabilizing influence of aerodynamic stresses.



# Big drops

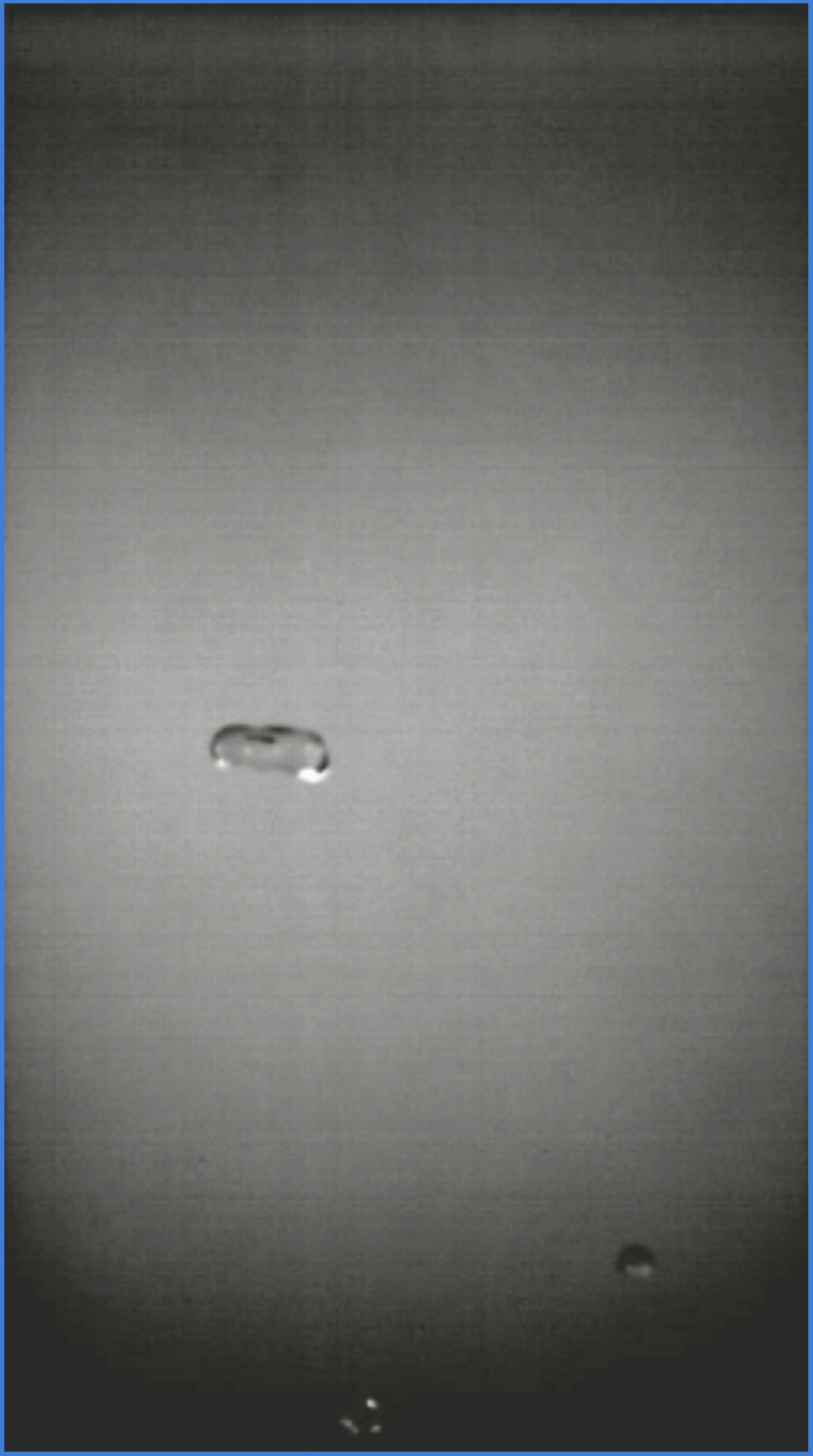
Drops larger than the capillary length ,

$$a > \ell_c \approx 2\text{mm}$$

break up under the influence of aerodynamic stresses.

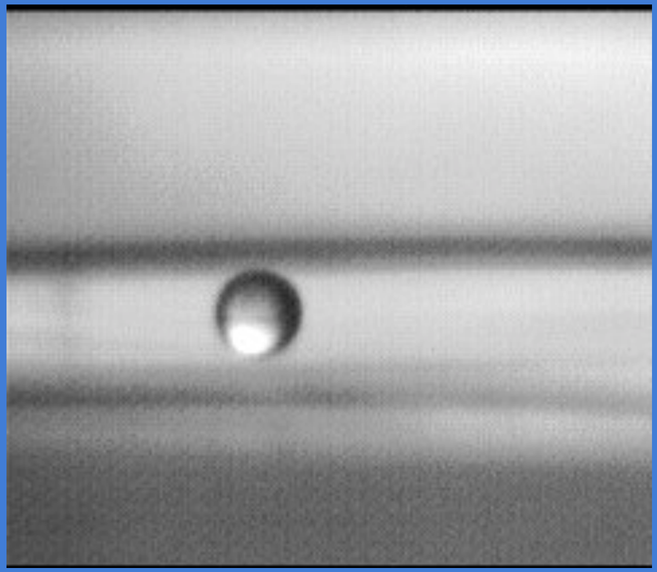
The break-up yields drops with size of order:

$$\ell_c \approx 2\text{mm}$$

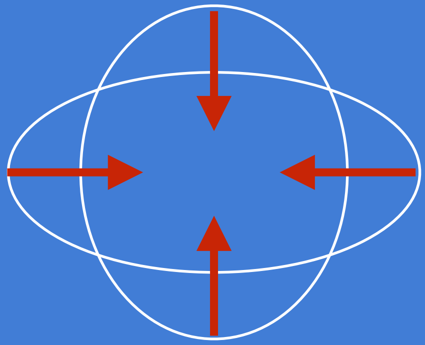


# Dimensional analysis and oscillating drops

If there are three physical constants in the Universe, $\rho$ and $\sigma$, what is the natural frequency of oscillation of a drop of radius $a$?



$$\omega_d = \sqrt{\frac{\sigma}{\rho a^3}}$$



- the characteristic frequency of inertial-capillary waves