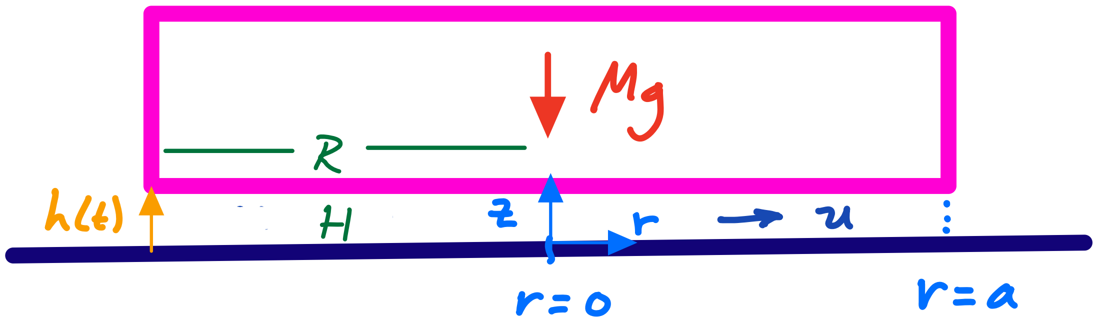
# Lecture 7 : Drops

## The Squeeze Film Problem

Q 1 : why doesn't the drop coalesce ?

- a model for the air film flow induced by drop impact on the bath surface




- a generic "lubrication flow" characterized by disparate vertical and horizontal lengthscales : $H \ll R$
- for such flows, provided $Re \frac{H}{R} = \frac{U H}{\nu} \frac{H}{R} \ll 1$,

nondimensionalizing and scaling Navier Stokes reveals :

1. inertia is negligible throughout the flow
2. there can be no pressure gradient across the thin gap.
3. the dominant viscous term comes from gradients across the thin gap
4. flow is nearly unidirectional since $V/U \sim H/R \ll 1$ by continuity

In this geometry, we thus deduce :

$$\frac{\partial p}{\partial z} = 0 \quad , \qquad 0 = -\frac{\partial p}{\partial r} + \mu \frac{\partial^2 u}{\partial z^2}$$

$\Rightarrow p = p(r)$ only

B.Cs 1. $u = 0$ on $z = 0$ 2. $u = 0$ on $z = h(t)$

**Velocity profile:** $u(r,z) = -\frac{1}{2\mu} \underbrace{\frac{dp}{dr}}_{<0} \underbrace{z(h-z)}_{>0} > 0$

**Volume flux:** $Q(r) = \int_0^h u(r)\, 2\pi r\, dz$

$$Q(r) = \frac{-\pi r}{\mu} \frac{dp}{dr} \int_0^h zh - z^2\, dz = \frac{-\pi r}{6\mu} \frac{dp}{dr} h^3$$

**Conservation of Mass:** $\pi r^2 \dot{h} = Q(r,t) = \frac{\pi r h^3}{6\mu} \frac{dp}{dr}$

$$\Rightarrow \frac{dp}{dr} = 6\mu \frac{\dot{h}}{h} r \Rightarrow p(r) = \frac{3\mu \dot{h}}{h^3}(r^3 - a^3) + p_0$$

**Force Balance:** $Mg = \int_0^a 2\pi r (p(r) - p_0)\, dr$

$$\Rightarrow Mg = \frac{6\pi\mu \dot{h}}{h^3} \int_0^a (r^3 - a^2 r)\, dr = -\frac{3\pi\mu \dot{h}}{2h^3} a^4$$

Integrate: $\int_{h_0}^{h} \frac{dh}{h^3} \left( -\frac{3\pi\mu a^4}{2Mg} \right) = \int_0^t dt$

$$\Rightarrow t = \frac{3\pi\mu a^4}{4Mg} \left( \frac{1}{h^2} - \frac{1}{h_0^2} \right)$$ is time for disc to settle from $h_0$ to $h$

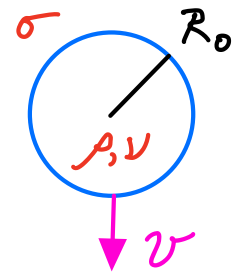
$$\Rightarrow \boxed{h(t) = \left( \frac{1}{h_0^2} + \frac{4}{3} \frac{Mgt}{\pi\mu a^4} \right)^{-\frac{1}{2}}}$$

**Note:**

1. plate takes an infinite time for $h \to 0$
2. in reality, there is generally microscale roughness that ultimately prompts coalescence
3. for liquid-liquid coalescence (e.g. bouncing drops), there is a critical $h_{crit} \sim 100$ Nm below which coalescence is initiated by van der Waals forces.

# Droplets bouncing on a hydrophobic substrate

Richard et al., Nature (2002)
Okomura et al., Europhys. Lett (2003)

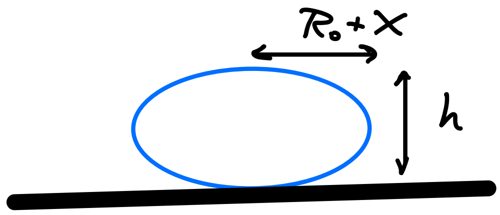



- consider limit of high $Re = \frac{v R_0}{\nu}$, so that viscous effects are negligible
- consider limit of $We = \frac{\rho R_0 v^2}{\sigma} = \frac{v^2}{V_w^2} \leq 1$

$\Rightarrow$ small deformations arise for

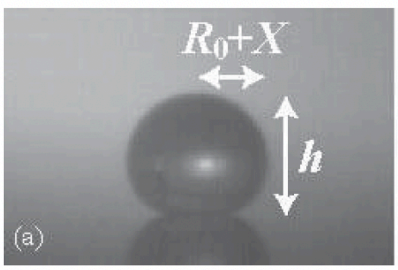
$$v < V_w = \sqrt{\sigma / \rho R_0}$$

CAPILLARY WAVE SPEED

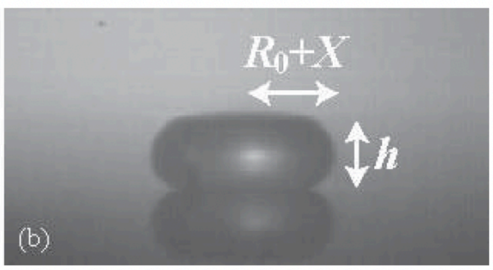



e.g. for millimetric water drop, $V_w \sim 1$ m/s

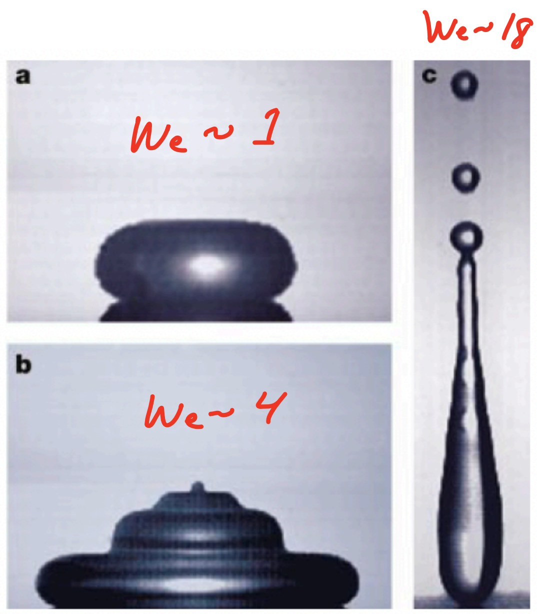
$v < V_w$ ($We < 1$)




$v > V_w$ ($We \gtrsim 1$)




REGIME OF INTEREST




## Scaling for $We \ll 1$

$$\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho \underline{g} \quad \star$$

- small deformation $X \ll R_0$; deformation time $\frac{X}{v}$
- Laplace pressure gradient $\sim \sigma \frac{X}{R_0^3}$

$$\star \Rightarrow \quad \rho v^2 R_0^3 \sim \sigma X^2 - \rho g R_0^3 X$$

K.E. ANOMALOUS SURFACE ENERGY ANOMALOUS G.P.E.

## Static Case ($V = 0$) :

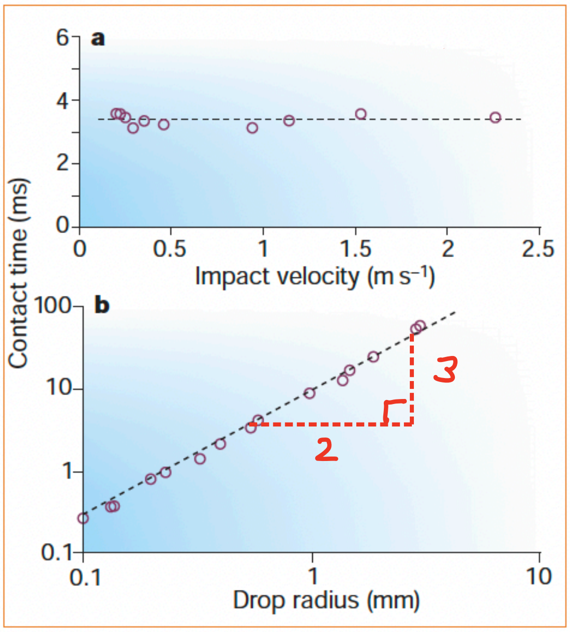
$$\sigma X_s^2 = \rho g R_0^3 X_s$$

$$X_s \sim \frac{\rho g R_0^3}{\sigma} = B_0 \sim R_0^3/\ell_c^2$$

where $\ell_c = \sqrt{\sigma/\rho g}$ is the capillary length

## Dynamic Low Bond Number Case ($B_0 << 1$, $X >> X_s$)

$$\rho V^2 R_0^3 \sim \sigma X^2 \Rightarrow \frac{X}{R_0} \sim \frac{V}{\sqrt{\sigma/\rho R_0}} = \frac{V}{V_w} < 1$$




## Contact Time :

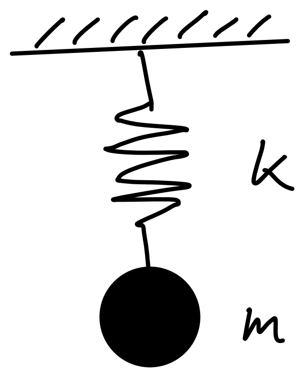
$$\tau_c \sim \frac{X}{V} \sim \frac{R_0}{V_w} \sim \left(\rho \frac{R_0^3}{\sigma}\right)^{\frac{1}{2}}$$

indep of impact velocity

$\Rightarrow$ found to be true with water drops for $0.3 < We < 37$

## Classical Spring

$$\omega = \sqrt{k/m}$$

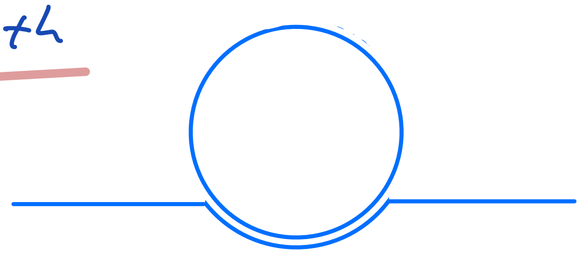



Compare $\omega_{drop} \sim \frac{1}{\tau_c} \sim \sqrt{\frac{\sigma}{m}}$

$\Rightarrow$ when subjected to weak deformations, a drop responds as a linear spring with spring constant $k \sim \sigma$.

$\Rightarrow$ the drop is a linear "water spring"

# Drop Impacting a Liquid Bath



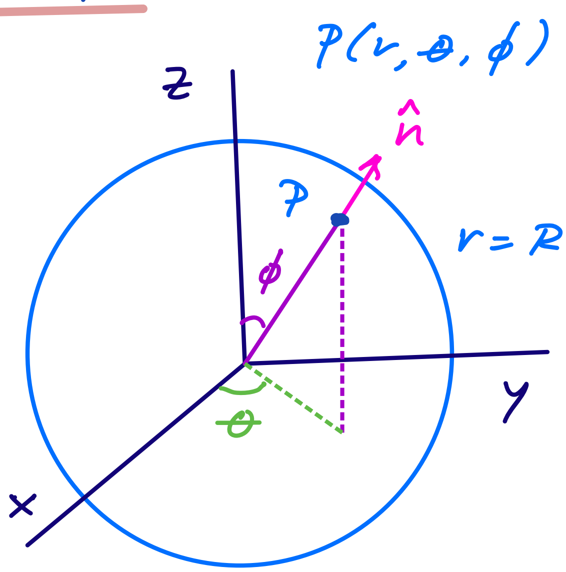
- on PS 1, you are asked to consider an idealized situation
- doing so reveals that, even for a rigid sphere, the bath surface acts as a linear spring with effective spring constant $k \sim \sigma$

$\Rightarrow$ one thus has a linear spring bouncing on a linear spring

$\Rightarrow$ in pilot-wave hydrodynamics, the bath is vibrating, and the system is characterized by a resonance between drop and wave

$\Rightarrow$ resonance effects

# Capillary waves on a spherical drop




Consider inviscid, irrotational flow, for which

$$\underline{u} = \nabla\phi, \quad \text{and} \quad \nabla^2\phi = 0$$

In spherical polar coordinates, we describe a point on the surface as $P(r,\theta,\phi)$, the unperturbed shape by $r=R$, and the perturbed shape by

$$\boxed{r = R(1+\eta(\theta,\phi))}$$

where $\eta \ll R$

In spherical polars,
$$\vec{\nabla}F = \frac{\partial F}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial F}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial F}{\partial\phi}\hat{\phi}$$

and
$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

To find normal to perturbed surface, define functional :
$$\mathcal{F}(r,\theta,\phi) = r - R(1+\eta(\theta,\phi))$$
$$= 0 \quad \text{on} \quad r=R$$

Gradient :
$$\vec{\nabla}\mathcal{F} = \vec{\nabla}[r - R - R\eta(\theta,\phi)]$$
$$= \hat{r} - \frac{R}{r}\eta_\theta\hat{\theta} - \frac{R}{r\sin\theta}\eta_\phi\hat{\phi}$$

Unit normal:
$$\hat{n} = \frac{\vec{\nabla}\mathcal{F}}{|\vec{\nabla}\mathcal{F}|} = \frac{\hat{r} - \frac{R}{r}\eta_\theta\hat{\theta} - \frac{R}{r\sin\theta}\eta_\phi\hat{\phi}}{\sqrt{1+\left(\frac{R}{r}\eta_\theta\right)^2+\left(\frac{R}{r\sin\theta}\eta_\phi\right)^2}}$$

Linearizing in $\varepsilon = \eta/R \ll 1$, one finds

$$\hat{n} = \hat{r} - \eta_\theta \hat{\theta} - \frac{1}{\sin\theta} \eta_\phi \hat{\phi}$$

$$\vec{\nabla} \cdot \hat{n} = \frac{2(1-\eta)}{R} - \frac{1}{R} \nabla^2_{\theta\phi} \eta \qquad \star$$

Boundary Conditions (linearized w.r.t. $\varepsilon = \eta/R \ll 1$)

1. kinematic condition : $u_r = \frac{\partial \phi}{\partial r} = \frac{\partial \eta}{\partial t}$ $\quad \boxtimes$

2. Dynamic condition (Time-dep Bernoulli applied at free surface)

LINEARIZED

$$\rho \frac{\partial \phi}{\partial t} + \cancel{\frac{1}{2} \rho |\nabla \phi|^2} + P_s = f(t)$$

indep of $x$

where $P_s = P_0 + \sigma \vec{\nabla} \cdot \hat{n}$

Plug in $\star \Rightarrow$ $\rho \frac{\partial \phi}{\partial t} = -\frac{2\sigma}{R} + \frac{2\sigma \eta}{R} + \frac{\sigma}{R} \nabla^2_{\theta\phi} \eta$

Diff. w.r.t. $t$ and use $\boxtimes$ to deduce a wave eqn :

$$\rho \frac{\partial^2 \phi}{\partial t^2} = \frac{2\sigma}{R} \frac{\partial \phi}{\partial r} + \frac{\sigma}{R} \nabla^2_{\theta\phi} \frac{\partial \phi}{\partial r} \qquad \odot$$

where $\nabla^2_{\theta\phi} F = \dfrac{F_{\phi\phi} + \sin\theta \frac{\partial}{\partial \theta}(F_\theta)}{\sin^2\theta}$

Seek harmonic functions (sol'ns of $\nabla^2 \phi = 0$) of the form : $\phi = B(t) \left(\frac{r}{R}\right)^\ell Y_\ell^m(\theta, \phi)$

where $Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} \, P_\ell^m(\cos\theta) \, e^{im\phi}$ are spherical harmonics

LEGENDRE POLYNOMIAL

Differentiating $\odot$ w.r.t $r$ thus yields

$$\eta_{tt} = \frac{\sigma}{\rho R^3} \ell \left(2\eta + \nabla^2_{\theta,\phi} \eta\right)$$

Decomposing into spherical harmonics : $\eta = \eta_0(t) Y_\ell^m$

$$\Rightarrow \quad \ddot{\eta}_0 + \frac{\ell(\ell^2-1)(\ell+2)}{(\ell+1)} \frac{\sigma}{\rho R^3} \eta_0 = 0$$

Dispersion Relation : seek $\eta_0(t) = \bar{\eta} e^{i\omega t}$

$$\omega^2_{\ell,m} = (2\pi f_{\ell,m})^2 = \frac{\sigma}{\rho R^3} \ell(\ell+2)(\ell-1)$$

Using $M = \frac{4\pi}{3} R^2 \rho$

$$\Rightarrow \quad \boxed{f^2_{\ell m} = \frac{\sigma}{3\pi M} \ell(\ell+2)(\ell-1)}$$

Summary

$$R = R_0 \left(1 + B\, Y_\ell^m \cos \omega_{\ell m} t\right)$$

$$\phi = -B \frac{\omega_{\ell m} R^2}{\ell} \left(\frac{r}{R}\right)^\ell Y_\ell^m \sin \omega_{\ell m} t$$

$$\omega_{\ell m} = \sqrt{\frac{\sigma}{\rho R^3}} \sqrt{\ell(\ell+2)(\ell-1)}$$

A few points of interest

(1.) Reduces to plane-wave dispersion relation in limit of $R \to \infty$ with $\ell = 2\pi R_0/\lambda$

$$\Rightarrow \quad f^2 = \frac{2\pi\sigma}{\rho \lambda^3}$$

(2.) We can estimate the decay rates of the various inviscid modes arising through influence of viscosity

Viscous dissipation rate per unit volume:

$$\Phi = 2\mu \underline{\underline{E}} : \underline{\underline{E}} \quad \text{where} \quad \underline{\underline{E}} = \frac{1}{2}\left[\nabla\underline{u} + (\nabla\underline{u})^T\right]$$

The total power dissipated within the droplet

$$P = \iiint_{V_{drop}} \Phi \, dV = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{R} \Phi \, r^2 \sin\theta \, dr$$

$$= B^2 \mu \, \omega_{em}^2 R_0^3 \frac{(l-1)(2l+1)(1+\delta_{mo})}{l} \sin^2 \omega_{em} t$$

The total kinetic energy of the drop

$$K = \iiint_{V_{drop}} \rho \, \underline{u}\cdot\underline{u} \, dV = \rho B^2 \omega_{em}^2 R_0^5 \frac{(1+\delta_{mo})}{4l} \sin^2 \omega_{em} t$$

This kinetic energy is diminished in time through the influence of viscous dissipation. Specifically, the total mechanical energy $E$ of the capillary waves

$$E(t) = E_0 e^{-\beta t}$$

$$\text{where} \quad \beta = \frac{P}{K} = \frac{\mu}{R^2} \frac{(2l+1)(l^2-1)}{(l+1)}$$

$\Rightarrow$ the shorter the waves (the larger $l$), the more vigorous the damping

Compare to a planar interface: $\beta = 4\pi^2 \frac{\mu}{\rho \lambda^2}$

3. The fact that shorter waves are most quickly damped by viscosity is important in HQA for both the drops and the waves.