Lec. 6 : Interfacial flows

- the study of fluid systems dominated by the influence of surface tension at interfaces

Preliminaries : what is surface tension, and what is an interface?

$\Rightarrow$ SLIDES 1–13

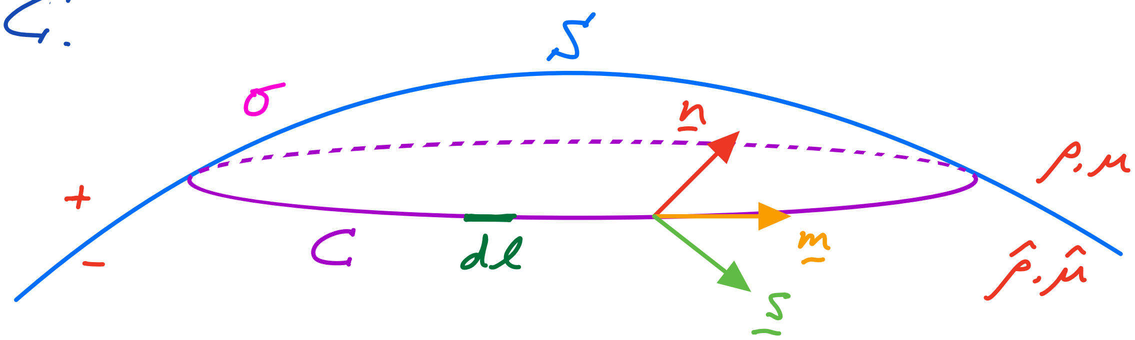
Surface tension : an energy/area at the interface between 2 fluids (liquid–liquid or liquid–gas), of molecular origins that serves to resist the creation of new interface

$\Rightarrow$ imparts a tensile force per unit length to the interface

Deduction of the interfacial stress B.C.s requires consideration of the influence of σ on curved interfaces.

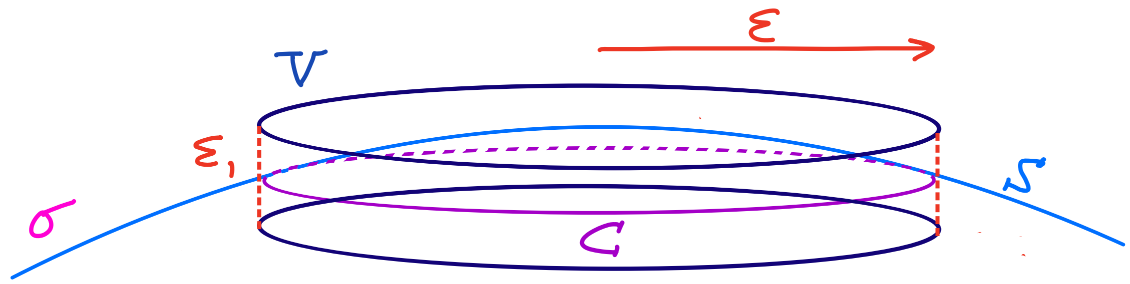
Interfacial Boundary Conditions

Consider an interfacial surface bound by a closed curve C.



where $\underline{n}$ is unit normal to S, C
$\underline{s}$ is unit normal to C, tangent to S
$\underline{m}$ is unit tangent to S, C

Consider an infinitesimal cylindrical pillbox V of radius ε and height ε_1 (s.t. $\varepsilon_1 \ll \varepsilon$) that intersects C.



For a force balance on V, we must consider:

$\underline{t}(\underline{n}) = \underline{n} \cdot \underline{\underline{T}}$ is the force/area exerted by upper fluid (+)

$\hat{\underline{t}}(\hat{\underline{n}}) = \hat{\underline{n}} \cdot \hat{\underline{\underline{T}}}$ is " " lower fluid (−)

Force balance :

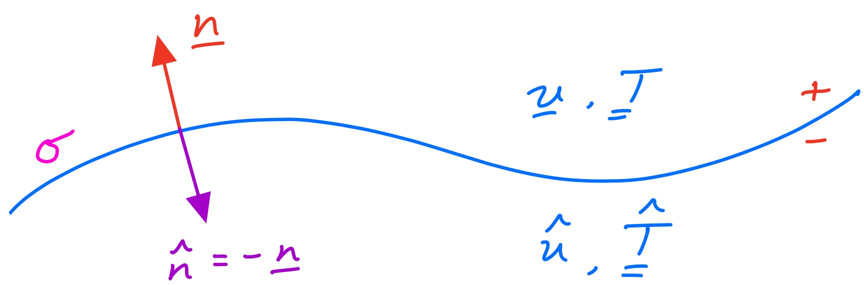
$$\underbrace{\int_V \rho \frac{D\underline{u}}{Dt}\, dV}_{\text{inertial force}} = \underbrace{\int_V \underline{f}\, dV}_{\text{body forces}} + \underbrace{\int_S \left[\underline{t}(\underline{n}) + \hat{t}(\hat{n})\right] dS}_{\text{hydrodynamic forces acting on upper + lower surfaces}} + \underbrace{\int_C \sigma \underline{s}\, dl}_{\text{surface tension}}$$

Now, the inertial + body forces must scale as $\varepsilon^2 \varepsilon_1$, while the surface forces scale as ε^2. Hence, in the limit of $\varepsilon_1 \to 0$, surface forces must balance:

$$\int_S \underline{t}(\underline{n}) + \hat{t}(\hat{\underline{n}})\, dS + \int_C \sigma \underline{s}\, dl = 0$$

Now, we have $\underline{t}(\underline{n}) = \underline{n} \cdot \underline{\underline{T}}$, $\quad \hat{t}(\hat{\underline{n}}) = \hat{n} \cdot \hat{\underline{\underline{T}}}$

$$= -\underline{n} \cdot \hat{\underline{\underline{T}}}$$



Moreover, application of Stokes Thm yields

$$\int_C \sigma \underline{s}\, dl = \int_S \underline{\nabla}_s \sigma - \sigma \underline{n} (\underline{\nabla}_s \cdot \underline{n})\, dS$$

where $\underline{\nabla}_s \equiv (\underline{\underline{I}} - \underline{n}\,\underline{n}) \cdot \underline{\nabla} = \underline{\nabla} - n \frac{\partial}{\partial n}$

is the tangential gradient operator,

required because σ and $\underline{n}$ are only defined on the interface. We proceed by dropping the subscript, $\underline{\nabla}_s \to \underline{\nabla}$, with this understanding.

The surface force balance:

$$\int_S \left(\underline{n} \cdot \underline{\underline{T}} - \underline{n} \cdot \underline{\underline{\hat{T}}} \right) dS = \int_S \sigma \underline{n} (\underline{\nabla} \cdot \underline{n}) - \underline{\nabla}\sigma \, dS$$

Now since the surface element S is arbitrary, the integrand must vanish identically.

$\Rightarrow$ Interfacial Stress Balance Equation

$$\boxed{\underbrace{\underline{n} \cdot \underline{\underline{T}} - \underline{n} \cdot \underline{\underline{\hat{T}}}}_{\text{STRESS JUMP AT INTERFACE}} = \underbrace{\sigma \underline{n} \, \underline{\nabla} \cdot \underline{n}}_{\text{NORMAL LAPLACE/CURVATURE PRESSURE}} - \underbrace{\underline{\nabla} \sigma}_{\text{MARANGONI STRESS}}}$$

Normal Stress Balance

Taking $\underline{n} \cdot$ ✩ yields

$$\underbrace{\underline{n} \cdot \underline{\underline{T}} \cdot \underline{n} - \underline{n} \cdot \underline{\underline{\hat{T}}} \cdot \underline{n}}_{\text{JUMP IN NORMAL STRESS}} = \underbrace{\sigma \, \underline{\nabla} \cdot \underline{n}}_{\text{LAPLACE PRESSURE}}$$

$\Rightarrow$ DEPENDS ON BOTH $\underline{u}$, p

Tangential Stress Balance

Taking $\underline{t} \cdot \star$, where $\underline{t}$ is ANY vector tangent to the interface:

$$\underbrace{\underline{n} \cdot \underline{\underline{T}} \cdot \underline{t} - \underline{n} \cdot \hat{\underline{\underline{T}}} \cdot \underline{t}}_{\text{JUMP IN TANGENTIAL STRESS}} = \underbrace{\underline{\nabla} \sigma \cdot \underline{t}}_{\text{MARANGONI STRESS ASSOC. w } \underline{\nabla}\sigma}$$

Note: 1. $\underline{\nabla}\sigma$ may arise since $\sigma(C, T)$

chemistry ↗ ↖ temperature

⇒ Marangoni stresses arise from $\vec{\nabla} C$ or $\vec{\nabla} T$

⇒ have not yet proven to be significant in HQA

⇒ but can resist drop coalescence

⟹ coffee drop

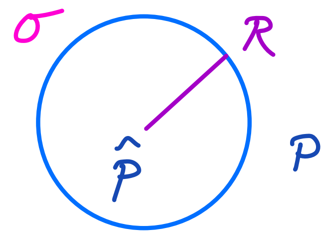
Fluid Statics

- the stress tensor reduces to $\underline{\underline{T}} = -p\underline{\underline{I}}$, so that $\underline{n} \cdot \underline{\underline{T}} \cdot \underline{n} = -p$ and the normal stress balance:

$$\underbrace{\hat{p} - p}_{\Delta p} = \underbrace{\sigma \, \underline{\nabla} \cdot \underline{n}}_{\text{LAPLACE PRESSURE}}$$

- tangential stress must take the form $\underline{\nabla}\underline{\sigma} = 0$.

I. Stationary Bubble



Recall in spherical coords

$$\bar{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 F_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, F_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}F_\phi$$

Here $\vec{F} = \underline{n} = (1, 0, 0)$ in (r, θ, ϕ) coords

$$\Rightarrow \quad \underline{\nabla}\cdot\underline{n} = \frac{1}{r^2}\frac{d}{dr}(r^2) = \frac{2}{R}$$

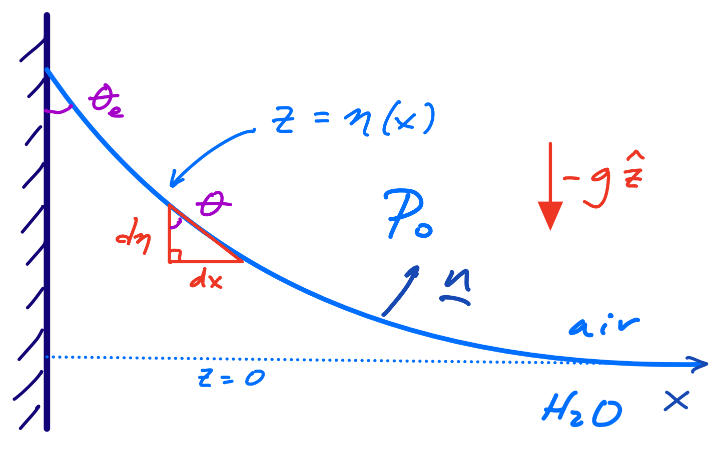
$$\Rightarrow \quad \Delta p = \hat{p} - p = \frac{2\sigma}{R}$$

$\Rightarrow$ smaller bubbles louder

II. The Static Meniscus

Pressure varies according to $p(z) = P_0 + \rho g z$.

Normal stress balance:



$$P_0 - \rho g \eta(x) = P_0 + \sigma\, \underline{\nabla}\cdot\underline{n}$$

$$\Rightarrow \quad -\rho g \eta(x) = \sigma\, \underline{\nabla}\cdot\underline{n}$$

Now $\underline{\nabla}\cdot\underline{n} = \dfrac{-\eta_{xx}}{(1+\eta_x^2)^{3/2}}$

We can integrate directly

$$\rho g \eta \eta_x = \sigma \frac{\eta_x \eta_{xx}}{(1+\eta_x^2)^{3/2}}$$

$$\frac{d}{dx}\frac{\eta^2}{2} = \ell_c^2 \frac{d}{dx}\frac{1}{(1+\eta_x^2)^{\frac{1}{2}}}$$

Integrate from x to ∞ :

$$LHS = \int_x^\infty \frac{d}{dx}\frac{\eta^2}{2}\,dx = \frac{1}{2}\left[\eta^2(\infty) - \eta^2(x)\right] = -\frac{\eta^2}{2}$$

$$RHS = \int_x^\infty \frac{d}{dx}\frac{1}{(1+\eta_x^2)^{\frac{1}{2}}}\,dx = \frac{1}{(1+0)^{\frac{1}{2}}} - \frac{1}{(1+\eta_x^2)^{\frac{1}{2}}} = 1-\sin\theta$$

$$\Rightarrow \boxed{\sigma \sin\theta + \frac{1}{2}\rho g \eta^2 = \sigma} \quad \star$$

Note: at $z = h$, $\theta = \theta_e$ $\Rightarrow$ $\frac{1}{2}\rho g h^2 = \sigma(1-\sin\theta_e)$

$$\Rightarrow \boxed{h = \sqrt{2}\,\ell_c\,(1-\sin\theta_e)^{\frac{1}{2}}}$$

is the maximum rise height

When is surface tension important?

Indicated by scaling: We, Bo, Ca

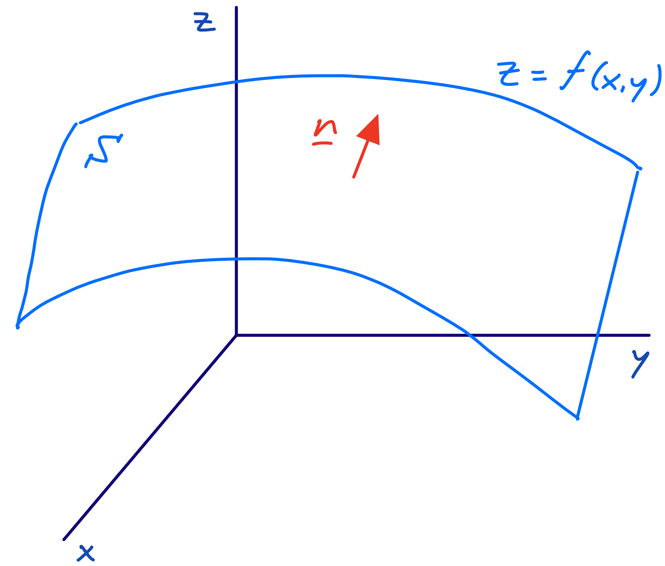
$\Longrightarrow$ SLIDES

Bonus / Supplementary Material not in lecture 6 ...

SKIP

Quick Review on Computing Normals

Normal to $z = f(x,y)$



Define functional:

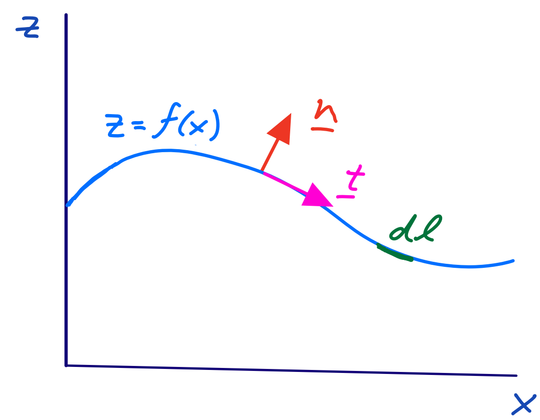
$$g(x,y,z) = z - f(x,y)$$
$$= 0 \text{ on } S$$

$$\therefore \quad \vec{\nabla} g = \vec{\nabla}(z - f(x,y))$$
$$= -f_x \hat{i} - f_y \hat{j} + \hat{k}$$

and $$\hat{n} = \frac{\vec{\nabla} g}{|\vec{\nabla} g|} = \frac{\hat{k} - f_x \hat{i} - f_y \hat{j}}{\sqrt{1 + f_x^2 + f_y^2}}$$

and curvature deduced from $\underline{\nabla} \cdot \underline{n}$.

Special Case: normal to curve $z = f(x)$



$$\Rightarrow \underline{n} = \frac{\hat{k} - f_x \hat{i}}{\sqrt{1 + f_x^2}}$$

$$\underline{\nabla} \cdot \underline{n} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\hat{k} - f_x \hat{i}}{\sqrt{1 + f_x^2}}\right)$$
$$= \frac{-f_{xx}}{(1 + f_x^2)^{3/2}}$$

Floating Bodies

"Heavy things sink. Light things float."

$\Rightarrow$ Not really true for small objects.

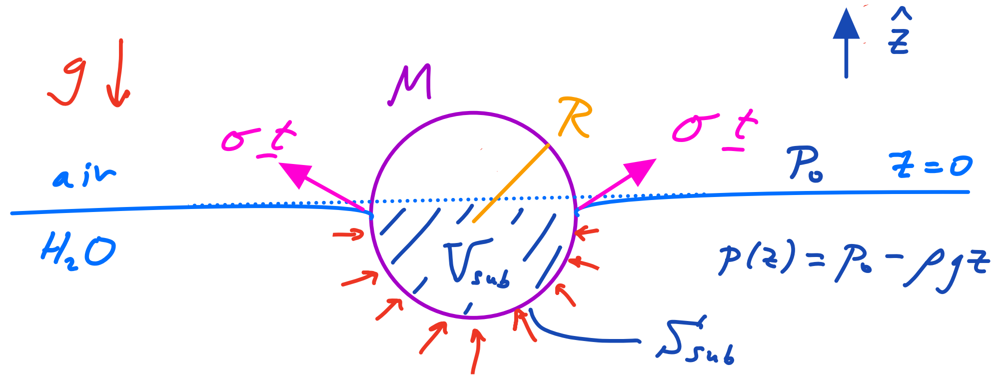
Archimedes Principle: buoyant force is equal to the wt of the displaced fluid

Generalization to Floating Bodies

Case 1: Large bodies ($R \gg \ell_c$)

Force balance $\quad Mg = \int_{S_{sub}} \rho g \, \underline{n} \cdot \hat{z} \, dS + 2\pi R \sigma \underline{t} \cdot \hat{z}$

BUOYANCY $\qquad$ SURFACE TENSION



$$\Rightarrow Mg = \rho g V_{sub} + 2\pi R \sigma \underline{t} \cdot \hat{z}$$

$$\frac{\text{BUOYANCY}}{\text{CAPILLARITY}} \sim \frac{\rho g V_{sub}}{2\pi R \sigma} \sim \frac{\rho g R^3}{R \sigma} \sim \frac{R^2}{\ell_c^2} = B_o$$

where Bond number $B_o = \dfrac{\rho g R^2}{\sigma}$

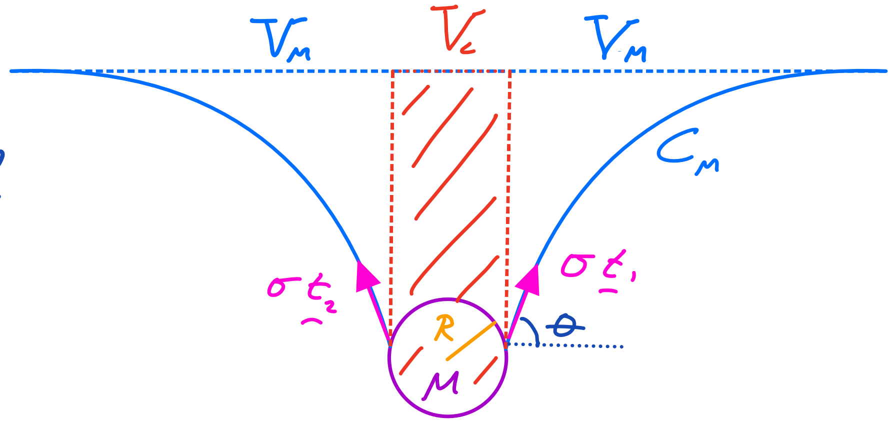
$\Rightarrow \sigma$ negligible provided $R \gg \ell_c$

Case 2: Small bodies ($R \leq \ell_c$)

Vertical force balance: $Mg = F_b + F_c$

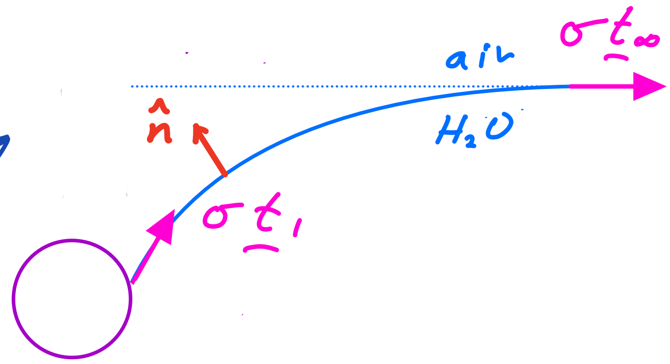
Buoyancy :

$$F_b = \int_c \rho g z \, \underline{n} \cdot \hat{z} \, dl$$
$$= \rho g V_c$$



where V_c = volume above body, but inside the contact line

Capillary : $F_c = 2\sigma \sin\theta$



Now, pressure balance along the meniscus C_M :

$$\rho g z \, \underline{n} = \sigma \, \underline{\nabla} \cdot \underline{n} \, \underline{n} \Rightarrow \text{integrate along } C_M$$

$$\Rightarrow \int_{C_M} \rho g z \, \underline{n} \, dl = \int_{C_M} \sigma \underbrace{\underline{\nabla} \cdot \underline{n} \, \underline{n}}_{\frac{d\underline{t}}{dl}} \, dl \quad \text{via Frenet-Serret}$$

$$\Rightarrow \rho g V_M = \sigma (\underline{t}_\infty - \underline{t}_1) \cdot \hat{z} = \sigma \sin\theta$$

$$\Rightarrow 2\sigma \sin\theta = 2\rho g V_M$$

CAP FORCE   WT DISPACED OUTSIDE CONTACT LINE by MENISCUS

Archimedes Principle is still valid !