Lecture 5 : Classical Mechanics

Particle motion in classical mechanics : displacement of particle $\underset{\sim}{x}_p(t)$ evolves according to Newton's 2nd Law:

Trajectory Eqn : $m\ddot{\underset{\sim}{x}}_p(t) = \underset{\sim}{F}(t, \underset{\sim}{x}_p(t))$

Initial conditions : $\underset{\sim}{x}_p(0) = \underset{\sim}{x}_0$, $\dot{\underset{\sim}{x}}_p(0) = \underset{\sim}{V}_0$

Laplace's Demon : given ICs, can predict $\underset{\sim}{x}_p(t)$ for all time.

- this perspective upended by both the probabilism of QM, and the limits on predictability imposed by chaos.

Chaotic Dynamics : for sufficiently complex systems, uncertainty in ICs ($\underset{\sim}{x}_p(0) = \underset{\sim}{x}_0 + \underset{\sim}{\varepsilon}_p$, $\dot{\underset{\sim}{x}}_p(0) = \underset{\sim}{V}_0 + \underset{\sim}{\varepsilon}_v$) are amplified exponentially with time

$\Rightarrow$ loss of predictability

Eg. 1 the Butterfly affect

Eg. 2 in orbital pilot-wave dynamics, chaos characterized by intermittent switching between unstable periodic states

Hereditary Systems : classical systems whose evolution depends explicitly on their past

$\Rightarrow$ a.k.a. time-delay, non-Markovian, or temporally nonlocal systems

- to predict the future of a hereditary system, one needs to know its history
- originally considered by Brillouin, Boltzmann, Volterra (~1910)

Eg. 1 For elastic solids, the state of stress depends on the system's history of strain

Eg. 2 For an electric charge influenced by its own field, the force on the charge may depend on its history

Eg. 3 For particle motion in a fluid, the stress field will generally depend on the particle history

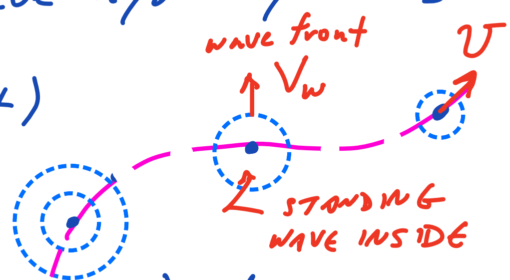
e.g. a fish swimming in a circle [SLIDE]

Trajectory equation : $m\ddot{\underline{x}}_p(\tau) = \underline{F}(\tau, \underline{x}_p(t<\tau))$

- described by delay differential equations : the particle acceleration depends on particle history

Eg. the Stroboscopic Model of pilot-wave hydrodynamics

$$\underbrace{m\ddot{\underline{x}}_p}_{\text{INERTIA}} = \underbrace{-D\dot{\underline{x}}_p}_{\text{DRAG}} \underbrace{- mg\,\underline{\nabla} h(\underline{x}_p, t)}_{\text{WAVE FORCE}}$$



where $\underline{\nabla} h(\underline{x}_p, t) \sim \int_{-\infty}^{t} \underbrace{\underline{k}\left(\underline{x}_p(t) - \underline{x}_p(s)\right)}_{\text{WAVE KERNEL}} ds$

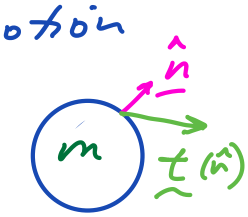
NB : $V_w \gg U$

Note : the system appears to be hereditary only when the trajectory equation is considered in isolation

- a complete formulation requires description of both the particle motion and the associated fluid motion

FLUID MECHANICS

Trajectory equation for a mass moving in a fluid : $m\ddot{\underline{x}}_p = \int_S \underbrace{\hat{\underline{n}} \cdot \underline{\underline{T}}}_{\underline{t}(\hat{\underline{n}})\ \text{stress vector}} dS$



where $\underline{\underline{T}}(p, \underline{u})$ is the stress tensor, and the fluid pressure p and velocity $\underline{u}$ evolve according to Navier-Stokes equations.

Throughout the course, we restrict our attention to incompressible, Newtonian fluids with constant density ρ and dynamic viscosity $\mu = \rho\nu$.

Note: the kinematic viscosity $\nu = \frac{\mu}{\rho}$ has units $\left[\frac{L^2}{T}\right]$ so prescribes the rate of diffusion of momentum and vorticity within the flow

For such fluids, the Navier-Stokes equations are a set of 4 equations in 4 unknown field variables, specifically the pressure $p(\underline{x},t)$ and the 3 components of the velocity $\underline{u}(x,t)$. They follow from the application of the classical concepts of conservation of mass, momentum and angular momentum to an infinitesimal fluid element.

Continuum Hypothesis (ASSUMPTION): the fluid is still smooth at this scale $\delta \gtrsim 10$ fluid molecules, so the graininess associated with molecular structure need not be considered.

Conservation of Mass: $\underline{\nabla} \cdot \underline{u} = 0$

Conservation of Momentum:

$$\underbrace{\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \underline{\nabla}\,\underline{u}\right)}_{\text{INERTIA}} = \underbrace{\underline{f}}_{\text{APPLIED FORCE}} + \underbrace{\underline{\nabla} \cdot \underline{\underline{T}}}_{\text{FORCE ASSOCIATED WITH STATE OF STRESS}}$$

is Cauchy's Momentum Eqn

Conservation of Angular Momentum

Stress tensor symmetric : $\underline{\underline{T}} = \underline{\underline{T}}^T$

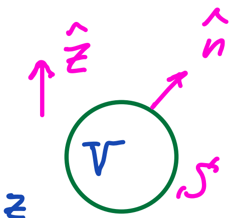
Express stress tensor : $\underline{\underline{T}} = -p \underline{\underline{I}} + \underline{\underline{\tau}}(\underline{u})$ DEVIATORIC STRESS TENSOR

STATIC DYNAMIC

Static component consistent with our notion of pressure.

Force on a submerged body in a static fluid

$$\underline{F}_h = \int_S \hat{n} \cdot \underline{\underline{T}} \, dS = -\int_S p \hat{n} \, dS$$



e.g. with hydrostatic pressure $p(z) = p_0 - \rho g z$

$$\underline{F}_h = -\int_S (p_0 - \rho g z) \hat{n} \, dS = \rho g \int_V \underline{\nabla} z \, dV = \rho g V \hat{z}$$

GEN DIV THM

BUOYANCY FORCE = weight of displaced fluid

$\Rightarrow$ Archimedes Principle

We now need only deduce the form of the deviatoric stress tensor $\underline{\underline{\tau}}(\underline{u})$.

Since $\underline{\underline{T}} = -p \underline{\underline{I}} + \underline{\underline{\tau}}$ is symmetric, so too must be $\underline{\underline{\tau}}$.

Moreover, $\underline{\underline{\tau}}$ cannot depend on translational or rotational velocity, or it would be frame dependent. We thus expect $\underline{\underline{\tau}}$ to depend on the symmetric rate of strain tensor

$$\underline{\underline{E}} = \frac{1}{2}\left[\underline{\nabla}\underline{u} + (\underline{\nabla}\underline{u})^T \right]$$

Newtonian Fluids

(1.) Relation between $\underline{\underline{\tau}}$ and $\underline{\underline{E}}$ is local in space + time.

Counterexample : fluids with 'memory', for which $\underline{\underline{\tau}}$ depends on the history of strain eg. elastic fluids, polymers

(2.) Relation is linear. The most general / such relationship may be expressed as $\underline{\underline{\tau}} = \underline{\underline{\underline{\underline{A}}}} : \underline{\underline{E}}$ where $\underline{\underline{A}}$ is a 4th rank constant tensor.

(3.) Fluid is isotropic : there is no preferred orientation in space

$\Rightarrow$ $\underline{\underline{\underline{A}}}$ must be an isotropic 4th rank tensor

For the incompressible Newtonian fluids of interest, this relation assumes the simple form

$\underline{\underline{\tau}} = 2\mu \underline{\underline{E}}$ where μ is the fluid viscosity

We may thus express the stress tensor :

$$\underline{\underline{T}} = -p\underline{\underline{I}} + 2\mu \underline{\underline{E}}$$

Substituting into Cauchy's Momentum Eqn and noting

$$\underline{\nabla} \cdot \underline{\underline{T}} = -\underline{\nabla} p + 2\mu \underline{\nabla} \cdot \underline{\underline{E}} = -\underline{\nabla} p + \mu \nabla^2 \underline{u}$$

then yields

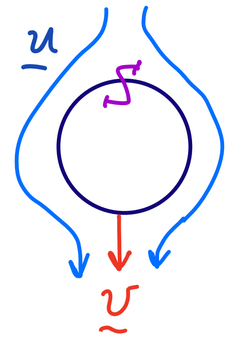
Navier-Stokes Eqns for an incompressible Newtonian fluid with constant ρ, μ :

$$\boxed{\begin{aligned} &\rho \left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \underline{\nabla} \underline{u} \right) = -\underline{\nabla} p + \mu \nabla^2 \underline{u} + \underline{f} \\ &\underline{\nabla} \cdot \underline{u} = 0 \end{aligned}}$$

4 eqns in 4 unknowns ($\underline{u}, p$) must be solved subject to appropriate initial and boundary conditions.

Fluid - Solid BCs : no-slip

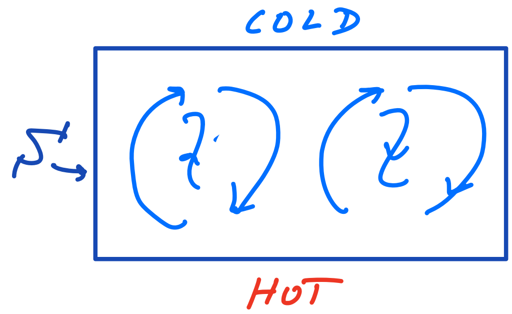
$$\Rightarrow \quad \underset{\sim}{u} = \underline{U}_{SOLID}$$



Eg. 1 Falling sphere : $\underline{u} = \underline{U}$ on S

Eg. 2 Convection in a box :

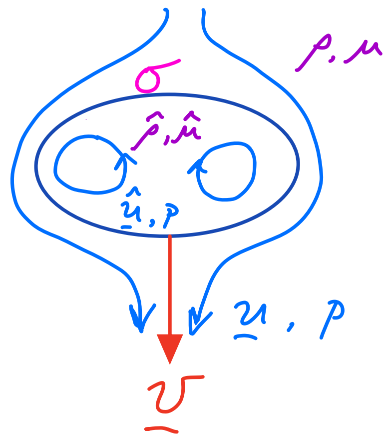
$$\underline{u} = 0 \text{ on } S$$

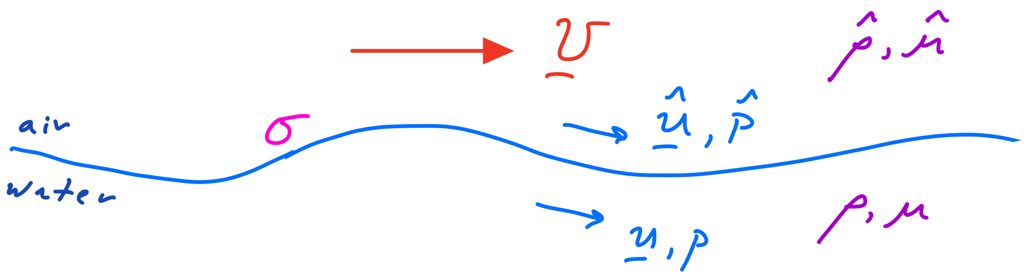


In this course, we are concerned with flows dominated by interfacial effects,

e.g. drop motion

e.g. water waves





Note : these interfaces are free to move ; thus, this of problems are known as FREE BOUNDARY PROBLEMS

Continuity of Velocity at an interface requires that

$$\underline{u} = \hat{\underline{u}}.$$

We shall proceed by developing the interfacial stress boundary conditions.

But first, what is an interface, and what is surface tension?

$\Longrightarrow$ SLIDES