18.357: Problem Set 2

Do any 5 of the 7 following problems. Each is worth 20 points. Due November 23, 2022.

1. Self-propulsion

a) **Drops on a needle:** A drop of silicone oil is deposited on a conical copper wire (mean radius of 250μm), whose radius increases from the right to the left with a gradient dr/dz of the order of 1 %. Note that silicone oil totally wets copper. Four successive photographs are taken. The interval between two snapshots is about 1 second.

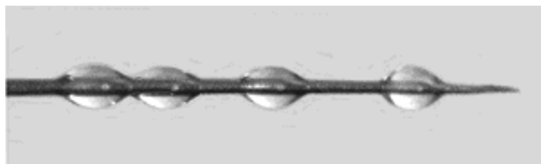


Figure 1: Silicon oil on a conical wire

In which direction does the drop move? Why? Why is the interval between two drops not constant? Might this be useful for cacti?

b) **A fluid slug in a tube:** A fluid drop is placed in a tube and forced along the tube by air. The liquid totally wets the tube, so that a film (of thickness h) is deposited on the wall in the wake of the drop. When the forcing is stopped, the pressure is atmospheric on both sides of the drop.

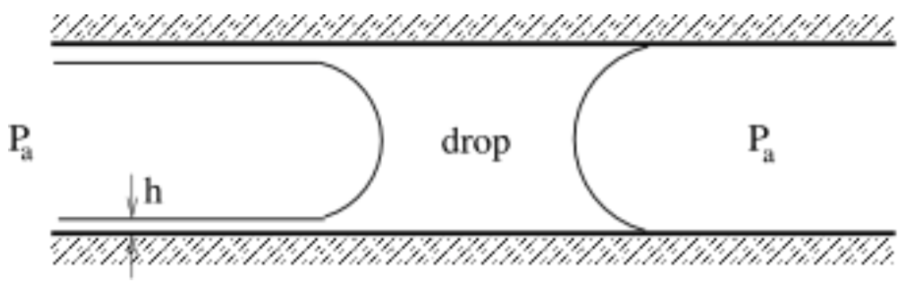


Figure 2: Fluid slug in a tube

Does the slug move ? Calculate the force acting on it.

c) **Self-propelling floater:** A cube of volume $V = L^3$ floats at the air-water interface. Its density is the mean of air and water; thus, provided L is large relative to the capillary length, it will float with half its volume submerged. Assume that it floats with top and bottom faces horizontal. Explain how a lateral force can be generated by changing the chemistry of the body, that is, by having variable contact angles on the front and rear faces. Give rough estimates for the propulsion speed in the high and low Re limits. How do you expect the physical picture to change if the body is small relative to the capillary length?

2. Retraction of a fluid thread

Consider a fluid that initially assumes the form of a cylinder of radius a with hemispherical caps. Neglect the influence of gravity.

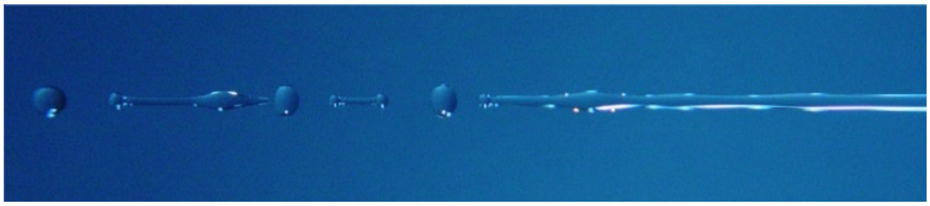


Figure 3: Breakup of a fluid thread

a) How do you expect its shape to evolve? Why?

b) In the limit where viscosity is negligible, deduce the rate of retraction of the thread.

c) By comparing the retraction rate with the growth rate of the Rayleigh-Plateau instability, estimate under what circumstances the thread will retract into a single droplet.

d) How will this physical picture change when the retraction and pinch-off are dominated by the influence of viscosity?

3. The *Michela Geri* Problem: Thermal delay of drop coalescence

In 2016, a student in 18.357 solved the following problem as her course project. It was subsequently published as an article in JFM, which would serve as a valuable reference. This paper, along with my lecture notes on the related 'squeeze film' problem, are also posted on line.

A small drop of undeformed radius R is emplaced on the surface of a fluid bath.

a) Provided the Bond number based on R is small, show that the drop will deform in such a way that it will have an effective contact radius with the bath given by $R_d \sim R^2/\ell_c$, where ℓ_c denotes the capillary length. One can then treat the air gap between the drop and bath as a cylindrical disk of radius R_d and height $h(t)$. The evolution of $h(t)$ defines the rate of approach of the drop to the bath. Typically, coalescence arises when $h(t)$ reaches a critical value (typically $\sim 100\mu$m) at which Van der Waals forces initiate coalescence.

b) If there is a temperature difference ΔT between the drop and bath, describe the Marangoni flow within the gap. Show that one expects opposing radially divergent or convergent flow along the drop and bath surfaces (according to the sign of ΔT), and estimate the magnitude of these Marangoni flows. You may assume that the surface tension is a linearly decreasing function of the temperature, $\sigma = \sigma_0 - \beta(T - T_0)$, where β is a constant

c) Solve the lubrication flow within the gap in order to deduce the pressure distribution within the gap $p(r)$ in terms of $\frac{dh}{dt}$ and the velocity difference Δv between the upper and lower surfaces.

d) Write a quasi-steady force balance for the droplet, wherein its weight is supported by the pressure in the gap. Manipulate this equation to find an expression for $\frac{dh}{dt}$. Solve this equation to deduce $h(t)$ in the isothermal limit.

e) Show that there is a critical temperature difference above which the drop never coalesces, and find an expression for this critical ΔT in terms of the system parameters.

4. Viscous Rayleigh-Taylor instability with Marangoni effects

An enthusiastic amateur chef notices that there is a slick of bacon fat on the ceiling above his stove that looks roughly as shown below. He terms the resulting undulations 'stalardtites', and wonders if they will ever drip. A graduate of 18.357, he is lead to consider the following problem.

Figure 4: Gravity-induced instability of a thin film of liquid.

A liquid of density ρ and high viscosity μ is placed on a flat plate, resulting in a thin layer relative to the capillary length. The plate is then inverted, so that the fluid layer is hanging upside down, and so subject to a gravitational instability. Suppose further that the plate is heated, inducing a temperature profile, $T = T(z)$, across the thin film. We will use lubrication theory to study the evolution of the interface $\eta = \eta(x, y, t)$, between the liquid and the air. We will assume that the surface tension is a linearly decreasing function of the temperature, $\sigma = \sigma_0 - \beta(T - T_0)$, where β is a constant.

a) To simplify the calculations, we assume that the Peclet number is small, $Pe \ll 1$. Assume further that the temperature on the plate is fixed at $T = T_0$ and that the heat flux at the interface between the fluid and the air is constant, $\frac{\partial T}{\partial z} = -\kappa_0$. Calculate the linear temperature profile in the thin film.

b) Write down the boundary conditions satisfied by the velocity field $\mathbf{u} = (u, v, 0)$ on the plate, $z = 0$, and at the interface, $z = h$. Calculate the pressure jump across the interface arising from gravity and curvature. Then solve the lubrication theory equations for the flow $\mathbf{u}$ in terms of the pressure gradient ∇p.

c) Mass conservation requires that $\frac{\partial \eta}{\partial t} + \nabla \cdot \mathbf{q} = 0$, where $\mathbf{q}$ is the depth averaged volume flux $\mathbf{q} = \int_0^\eta \mathbf{u}\, dz$. Use this law to derive the following evolution equation for the interface η:

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \left[\frac{\rho g}{3\mu} \eta^3 \nabla \eta + \frac{1}{3\mu} \eta^3 \nabla \left(\sigma \nabla^2 \eta \right) + \frac{\kappa_0 \beta}{2\mu} \eta^2 \nabla \eta \right] = 0$$

d) Following the approach taken in lecture, linearize about a film of constant thickness η_0 to find the fastest growing wavelength of instability. [*Hint*: write $\eta = \eta_0 + \eta_1 e^{i\mathbf{k}\cdot\mathbf{x} + \omega t}$, and linearize to find an equation for the growth rate $\omega = \omega(k)$; then, maximise ω with respect to k.] Discuss how and why the most unstable wavelength is influenced by both surface tension and Marangoni stresses.

5. Bouncing drops

Consider a drop of diameter 1mm bouncing on a fluid bath at 50Hz, rising a few diameters above the interface and distorting the interface by approximately 100 microns on impact.

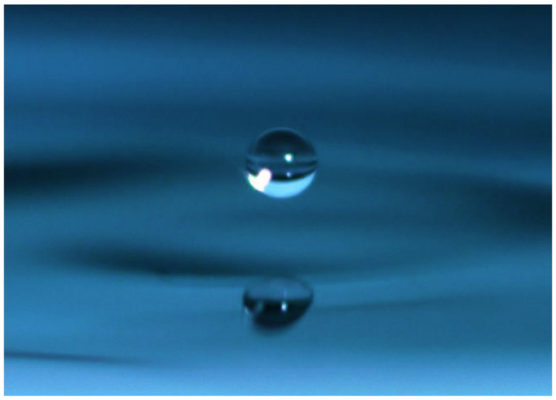


Figure 5: Droplet bouncing on a vibrating bath of silicon oil.

a) By evaluating familiar dimensionless groups, estimate the relative magnitudes of the interfacial, inertial and hydrostatic forces that resist the intrusion of the drop into the bath.

b) Deduce a model for the force imparted by the bath to the drop as a function of its intrusion depth by making the following simplifying assumptions: i) the drop does not deform during impact, ii) the interface conforms to precisely match the shape of the drop as the drop intrudes into the bath and is otherwise undeformed, iii) hydrostatic and inertial effects are negligible. If the interface behaves like a linear spring, calculate its spring constant. *Hint*: calculate the rate of increase of surface energy with intrusion depth. Discuss the shortcomings of this crude model on the basis of your estimates in part a).

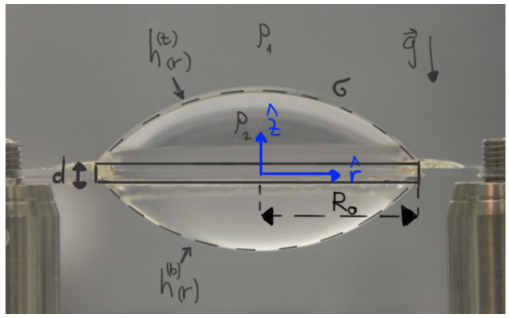
c) If inertia and gravity are indeed negligible, show that the interface shape beyond the drop?s contact area must correspond to a catenoid to leading order.

6. Photo essay

Take a photograph or video of a striking surface-tension related phenomenon. (Google downloads unacceptable). Describe and rationalize the phenomenon captured.

7. Shaping liquids into lenses

Consider a liquid chamber filled with liquid of density, ρ_1, containing a submerged ring-shaped frame with radius R_0 and vertical thickness d. An immiscible liquid of density ρ_2, which is injected into the bounding surface, is pinned at the boundaries, forming two interfaces, $h^{(t)}(r)$ and $h^{(b)}(r)$, with the immersion liquid. The capillary length for this system is given by $\ell_c = \sqrt{\sigma/\Delta\rho g}$, where $\Delta\rho = |\rho_2 - \rho_1|$.



a) Find the equations describing the steady-state shapes of the top and bottom interfaces by minimizing the free energy functional of the system:

$$F = 2\pi\sigma \int_0^{R_0} \left(\sqrt{1 + (h_r^{(t)})^2} + \sqrt{1 + (h_r^{(b)})^2} - \frac{\Delta\rho g}{2\sigma} \left[(h^{(t)})^2 - (h^{(b)})^2 \right] \;+\; \frac{\lambda}{\sigma}(h^{(t)} - h^{(b)}) \right) r \; dr \;,$$

with the boundary conditions:

$$h^{(b)}(R_0) = h_0 \;\;,\;\; h^{(t)}(R_0) = h_0 + d \;\;,\;\; V_{lens} = 2\pi \textstyle\int_0^{R_0} (h^{(t)}(r) - h^{(b)}(r)) \; r \; dr \;,$$

where the first two terms under the integral represent surface energy, the third term represents gravitational energy, and the last term represent the volume constraint, with λ being the Lagrange multiplier. Show that at neutral buoyancy condition (*i.e.* $\Delta\rho = 0$), these equations reduce to the Young-Laplace form, in which case $h^{(t)}(r)$ and $h^{(b)}(r)$ take the shape of spherical lenses. Note that if the lens liquid can be solidified (*e.g.* UV curable polymer), then the liquid lens can be turned into solid form without any mechanical processing.

b) Nondimensionalize the equations you found, using the following scaling:

$$R = \frac{r}{R_0} \;\;,\;\; H^{(t)}(R) = \frac{h^{(t)}(r)}{h_c} \;\;,\;\; H^{(b)}(R) = \frac{h^{(b)}(r)}{h_c} \;\;,\;\; p = \frac{\lambda h_c}{\sigma \epsilon^2} \;,$$

where h_c is some characteristic vertical deformation length scale, and $\epsilon = (h_c/R_0)^2$. What is the dimensionless number governing this problem?

(c) Linearize the nondimensional equations by keeping only the leading order terms in ϵ, and solve them using the boundary conditions (note that these are given in a dimensional form):

$$h^{(b)}(R_0) = h_0 \;\;,\;\; h^{(t)}(R_0) = h_0 + d \;\;,\;\; V_{lens} = 2\pi \textstyle\int_0^{R_0} (h^{(t)}(r) - h^{(b)}(r)) \; r \; dr.$$

Plot the solutions for different values of the dimensionless number, and compare them to the solution you found for neutral buoyancy conditions (*i.e.* $\Delta\rho = 0$).