Problem Set 7

18.355, Fall 2021
Due Wednesday Dec.1

1. Verify that the Stokes and continuity equations are satisfied by the representation

$$\mathbf{u}(\mathbf{x}) \;=\; \nabla\phi \;+\; \mathbf{x}\wedge\nabla\Psi \;+\; \nabla(\mathbf{x}\cdot\mathbf{A}) \;-\; 2\mathbf{A} \quad , \quad p(\mathbf{x}) \;=\; 2\mu\,\nabla\cdot\mathbf{A} \quad ,$$

where ϕ, Ψ and $\mathbf{A}$ are harmonic functions.

2. A rigid sphere of radius a is held stationary in a pure straining flow, so that

$\mathbf{u}(\mathbf{x}) = 0$ on the particle surface, and $\quad \mathbf{u} \to \mathbf{E}\cdot\mathbf{x} \quad$ as $\quad |\mathbf{x}| \to \infty$.

Determine the velocity and pressure fields in the fluid. Note: $\mathbf{E}$ is the constant traceless and symmetric rate-of-strain tensor of the undisturbed flow in the absence of the particle.

3. A rigid sphere of radius a_1 rotates steadily inside a larger concentric sphere of radius a_2. The rotational velocity of the inner sphere is $\mathbf{\Omega}$.

a) Assuming that the Reynolds number is small, determine the velocity and pressure fields in the fluid occupying the annular region between the spheres.

b) Deduce an expression for the torque required to rotate the inner sphere. Evaluate the torque for the case of extremely large a_2/a_1.

BONUS (1 point and $1 million from the Clay Institute)

The Clay Prize question: Prove, or give a counter-example to, the following statement...

In three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier-Stokes equations.