ON THE EFFECTS OF INERTIA : THE PARADOXES OF STOKES & WHITEHEAD

REFERENCES : M. Van Dyke "Perturbation Methods in Fluid Mechanics" Chapter 8
Kaplun & Lagerstrom J. Mathematics & Mechanics, 6 p.585-593 (1957)
Proudman & Pearson J. Fluid Mechanics 2 p.237-262 (1957).

I INTRODUCTION

We will see that the problem of examining inertial effects in low Reynolds number flows is very difficult and requires ideas of singular perturbation theory. Remember, we are trying to solve the steady Navier-Stokes equations for small Reynolds numbers, $R = \frac{UL}{\nu}$,

$$\nabla^2 \underline{u} - \underline{\nabla} p = R\, \underline{u}\cdot\underline{\nabla}\underline{u}$$

and we have been assuming that for $R \ll 1$ we can instead solve the simpler eqn.

$$\nabla^2 \underline{u} - \underline{\nabla} p = 0.$$

A. Recall perturbation methods for solving (nonlinear) ode's

1. REGULAR perturbation expansion – the solution procedure is valid everywhere in the domain of interest.

example : $y' = -y + \epsilon x y^2 \qquad y(0) = 1 \qquad \epsilon \ll 1$

seek $y(x) = y_0 + \epsilon y_1 + \cdots \longrightarrow y_0' = -y_0 \quad y_0(0) = 1 \longrightarrow y_0(x) = e^{-x}$

and $y_1' = -y_1 + x y_0^2 = -y_1 + x e^{-2x} \quad y_1(0) = 0 \longrightarrow y_1(x) = e^{-x} - (x+1)e^{-2x}$

$\vdots$

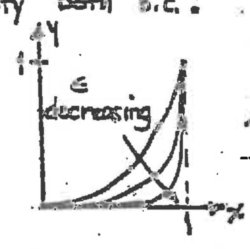
2. SINGULAR perturbation expansion – the solution obtained at leading order by letting $\epsilon \to 0$ is NOT uniformly valid in the entire domain.

standard "warning" – when set $\epsilon = 0$, the highest derivative in the governing eqn is lost, so 1 boundary condition must be dropped.

Example : $\epsilon y'' - y' = 0 \qquad y(0) = 0 \qquad y(1) = 1$

limit $\epsilon \to 0 \to y' = 0 \to y = $ constant ; can't satisfy both b.c.!

exact solution : $y(x) = \dfrac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1}$



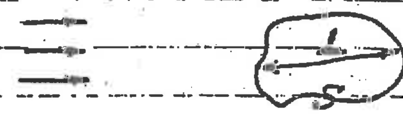
"boundary layer near x=1"

This illustrates the following "principle" : $y(x) = $ constant $(=0)$ for almost all x until near $x=1$ (a boundary) where it rapidly increases to the value of $x=1$. Hence, near $x=1$, y'' becomes very large compared to y' so that $\epsilon y''$ is the same order of magnitude as y'.

B. Well-known examples of singular perturbation in fluid dynamics (there are many)

1. Viscous flow past bodies at high Reynolds

(uniform flow) $\underline{U}$



At high Reynolds #, inertia dominates viscous forces and we choose the characteristic pressure as $p_c = \rho U^2$ ($U = |\underline{U}|$).

In this case, the appropriate form of the steady (dimensionless) Navier-Stokes eqns are $\left(R = \frac{UL}{\nu}\right)$

$$\underline{u}\cdot\underline{\nabla}\,\underline{u} = -\underline{\nabla}p + \frac{1}{R}\nabla^2\underline{u} \qquad \text{(body force neglected)}$$

$$\underline{\nabla}\cdot\underline{u} = 0.$$

b.c. $\underline{u} \to \underline{U}$ at ∞

$\underline{u} = 0$ on S

$\boxed{\text{take limit } R \to \infty}$:

$$\underline{u}\cdot\underline{\nabla}\,\underline{u} = -\underline{\nabla}p$$

$$\underline{\nabla}\cdot\underline{u} = 0$$

(Euler Eqn) — lower order eqn; because the highest derivative has been lost we ~~must now~~ → give up one boundary condition

b.c. $\underline{u} \to \underline{U}$ at ∞

$\underline{u}\cdot\underline{n} = 0$ on S.

← consistent with the neglect of viscosity in the governing eqns, we allow the tangential velocity to be nonzero at the fluid-solid interface.

It can be shown ⟹ A result of these equations and boundary conditions is that there is NO DRAG on the body. This clearly contradicts observation and everyday experience. This is known as $\boxed{\text{D'ALEMBERT'S PARADOX}}$.

"Resolution" of the paradox (Prandtl 1905): Boundary layer theory

→ near the body, viscous terms must be retained (singular region near body).

- Must satisfy no-slip condition on the body surface so very close to the body surface viscous forces will be important.

$$\Rightarrow \quad \frac{1}{R}\nabla^2\underline{u} \sim O(1) \quad \text{sufficiently close to the body.}$$

this determines the length-scale ℓ (relative to the body dimension L) where viscous effects are important → $\frac{1}{R\ell^2} = O(1)$ → $\ell \sim R^{-1/2}$ ← boundary-layer thickness ($R \gg 1$)

- a related example: Ekman boundary layer in a rotating geostrophic flow.

B. well-known examples .... (continued)

2. 2D uniform Stokes flow past a cylinder

⟹ For this boundary value problem, no solution exists satisfying the boundary conditions at ∞.

This is known as $\boxed{\text{STOKES PARADOX}}$ (1851) (see p. 5)

3. calculate the first effects of inertia for Stokes flow past a sphere

hydrodynamic force on translating sphere $\equiv$ $\underset{\sim}{F} = 6\pi\mu a \underline{U}\left(1 + g(R)\right)$

$\uparrow$ inertia – dependence on Reynolds # = ?

Try to calculate this correction using the method of successive approximations. This is equivalent to the following REGULAR perturbation expansion

$$\nabla^2 \underline{u} - \underline{\nabla} p = R\, \underline{u}\cdot\underline{\nabla}\underline{u} \qquad \left(R \ll 1 \quad p_c = \frac{\mu U}{L}\right)$$

Seek $\underline{u}(x) = \underline{u}_0 + R\, \underline{u}_1 + \cdots$

→ $O(1)$: $\nabla^2 \underline{u}_0 - \underline{\nabla} p_0 = 0$ + b.c. $\frac{\text{Stokes flow}}{\text{past a sphere}}$ $\underline{F} = 6\pi\mu a \underline{U}$

→ $O(R)$: $\nabla^2 \underline{u}_1 - \underline{\nabla} p_1 = R\, \underline{u}_0\cdot\underline{\nabla}\underline{u}_0$ ⟶ NO solution exists satisfying the b.c at ∞.

This is known as $\boxed{\text{WHITEHEAD'S PARADOX}}$ (1889)

⟹ (2) & (3) are actually SINGULAR PERTURBATION PROBLEMS. In this case, the singular region is "far" from the body (i.e., at ∞).

HISTORICAL NOTE:

(1) Stokes regarded the nonexistence of solutions for creeping flow past a cylinder to be an indication that no such steady flow existed.

(2) Whitehead concluded that the nonexistence of an approximate solution for inertial corrections to creeping flow past a sphere was an indication that discontinuities arise in the flow.

⟹ Both (1) & (2) are now known to be incorrect.

C. What's gone wrong with the solution of these Stokes flow problems?

1. The difficulty was understood by Oseen (1910) who also introduced a very useful approximate procedure for solving these problems. The "Oseen method" though is not on a sound mathematical basis. A systematic mathematical procedure, called the "method of matched asymptotic expansions" was developed in the 50s & 60s and is associated with the names of Kaplun & Lagerstrom at Caltech.

2. An order-of-magnitude estimate: examine Stokes flow past a sphere

We solved this problem and found

$$\underline{u}(\underline{x}) = \underline{U} + \underbrace{\underline{u}'_s}_{\text{(disturbance)}} = \underline{U} + \frac{3}{4}\,\underline{U}\cdot\left[\frac{\underline{\underline{I}}}{r} + \frac{\underline{x}\underline{x}}{r^3}\right] + \frac{1}{4}\,\underline{U}\cdot\left[\frac{\underline{\underline{I}}}{r^3} - \frac{3\underline{x}\underline{x}}{r^5}\right]$$

Until now, we've always assumed that the inertial terms $\underline{u}\cdot\nabla\underline{u}$ are negligible compared to the viscous terms. Examine this far from the sphere (assume all variables are dimensionless)

As $r\to\infty$, $\underline{u} \sim \underline{e}_x + O(1/r^2)$, so that $\nabla\underline{u} \sim O(1/r^2)$ and

$$\underline{u}\cdot\nabla\underline{u} \sim O(1/r^2) \quad ; \quad \text{also} \quad \nabla^2\underline{u},\ \nabla p \sim O(1/r^3)$$

$$\rightarrow \qquad R\ \underline{u}\cdot\nabla\underline{u} = -\nabla p + \nabla^2\underline{u}$$

estimating order-of-magnitude: $\quad O(R/r^2) \qquad\qquad O(1/r^3)$

$\therefore$ In order to neglect inertia: $\boxed{r\ R \ll 1}$ $\qquad R \ll 1$

$\therefore$ In an unbounded domain, no matter how small we make the Reynolds #, we can always find a distance from the sphere, $r \sim O(1/R)$ where inertial terms are no longer negligible compared to viscous terms.

$\rightarrow$ Stokes eqns do not provide a uniformly valid first approximation to the flow everywhere in an unbounded fluid.

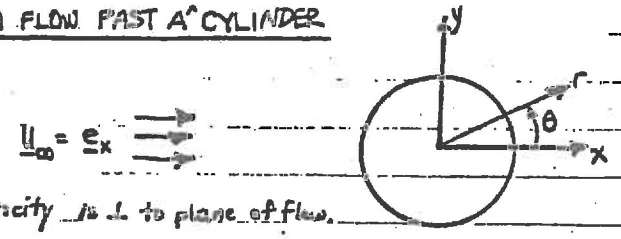
NOTE: (1) The original length-scale $l = a$ (sphere radius) is not the relevant length-scale far from the body. The appropriate characteristic length far from the body is $\frac{\nu}{U}$ (the "viscous" length-scale). The ratio of these 2 lengths is $R = \frac{aU}{\nu}$.

(2) In a bounded domain, for sufficiently small R, viscous terms will dominate inertia everywhere. (A regular perturbation expansion will be o.k. in this case)

$\rightarrow$ In a sense, it is an accident that Stokes flow past a sphere has a solution because it is based on equations that are not valid sufficiently far from the sphere.

II. TWO DIMENSIONAL STOKES FLOW PROBLEMS - UNIFORM FLOW PAST A CIRCULAR CYLINDER

(the results found below are true for arbitrary cross-sectional shape of the cylinder)



Introduce a stream function. Remember, in 2D, vorticity is $\perp$ to plane of flow.

One general way to represent this:

let $\underline{u} = \nabla \wedge (\psi \underline{e}_z)$ $\longrightarrow$ $\nabla \cdot \underline{u} = 0$ is automatically satisfied.

and $\underline{\omega} = \omega \underline{e}_z = \nabla \wedge \underline{u} = -\nabla^2 \psi \, \underline{e}_z$ $\longrightarrow$ $\boxed{\omega = -\nabla^2 \psi}$

so that

$$\nabla^2 \underline{u} = \nabla_p \xrightarrow{\text{take curl}} \nabla^2 \underline{\omega} = 0 \xrightarrow{\underline{\omega} = \omega \underline{e}_z} \boxed{\nabla^2 \omega = 0} \longrightarrow \boxed{\nabla^4 \psi = 0}$$

in cylindrical coord: $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

Hence, we wish to solve:

$$\nabla^4 \psi = 0 \qquad \left(u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \; u_\theta = -\frac{\partial \psi}{\partial r} \right)$$

with b.c.

$\psi \to r \sin\theta$ as $r \to \infty$

$\psi = 0$ at $r = 1$, all θ $\longleftarrow u_r = 0$ } on surface

$\frac{\partial \psi}{\partial r} = 0$ at $r = 1$, all θ $\longleftarrow u_\theta = 0$

+ symmetry $\psi(r, +\theta) = \psi(r, -\theta)$

let $\psi(r, \theta) = \sin\theta \, f(r)$

where $f(r)$ then satisfies

$$\left[\frac{1}{r}\frac{d}{dr}\left(r \frac{d}{dr}\right) - \frac{1}{r^2}\right]\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{f}{r^2}\right] = 0$$

Solution:

$$f(r) = c_1 r^3 + c_2 r \ln r + c_3 r + c_4 / r$$

apply b.c. $\longrightarrow$ $\psi(r, \theta) = c_2 \sin\theta \left[r \ln r - \frac{1}{2} r + \frac{1}{2r} \right]$ $\longrightarrow$ but clearly this can't satisfy b.c. at ∞ unless $c_2 = 0$ which would lead to no solution.

$\therefore$ In general, no creeping flow solution exists for streaming flow past 2D bodies in infinite domains, since b.c. at ∞ can't properly be satisfied. (Stokes realized this) $\boxed{\text{STOKES PARADOX}}$

REMARK: (1) If we were to seek a second approximation (correction to Stokes flow) for unbounded uniform flow past a 3D body we would again not be able to satisfy b.c. at ∞ $\boxed{\text{WHITEHEAD'S PARADOX}}$

(2) In 2D we are unable to even solve the Stokes flow problem. Actually, because we can fortuitously obtain a Stokes flow solution for uniform flow past 3D bodies, the singular nature of the problem is concealed until we seek an improved solution accounting for inertia.

III. MATHEMATICAL RESOLUTIONS OF THE PARADOXES

A. The Oseen Method

1. Oseen (1910) introduced an approximate cure. Although it cannot be put on a sound theoretical basis, it is useful to discuss the idea. It is also true that the Oseen equations provide a UNIFORMLY VALID first approximation to the flow field everywhere for either plane or 3D flows at low Reynolds numbers.

2. Rather than completely neglect viscous terms, approximate them by their linearized form far from the body:

$$\underline{u}\cdot\underline{\nabla}\,\underline{u} \sim \underset{\substack{\uparrow\\ \text{uniform free stream velocity}}}{\underline{U}}\cdot\underline{\nabla}\,\underline{u} \qquad \text{since} \quad \underline{u}\sim\underline{U}+O(1/r) \quad \text{as } r\to\infty$$

and this yields <u>Oseen's equations</u>

$$R\,\underline{U}\cdot\underline{\nabla}\,\underline{u} = -\underline{\nabla}p + \nabla^2\underline{u} \qquad \leftarrow \text{a linear eqn for } \underline{u}(\underline{x})$$

$$\underline{\nabla}\cdot\underline{u} = 0$$

<u>b.c.</u>

$$\underline{u}\to\underline{U} \quad \text{as } r\to\infty$$

$$\underline{u}=0 \quad \text{on } S$$

- Although this approximation may not appear to be too good near the body, it is nevertheless true that $O(R\,\underline{U}\cdot\underline{\nabla}\,\underline{u})$ is actually <u>small</u> compared to the viscous terms near the body (where the appropriate length-scale is the body dimension). Far from the body this linearization provides a proper balance between inertia & viscous terms at leading order. Hence, the solution to the Oseen eqn provides a valid solution to the velocity field everywhere.

- Oseen's eqn can be solved exactly.

B. Method of Matched Asymptotic Expansions

This formal mathematical procedure was introduced in the 50s & 60s. Rigorous mathematical proofs are difficult (if not impossible).

For lots of information regarding asymptotic methods:

Bender & Orszag *Advanced Mathematical Methods for Scientists & Engineers*

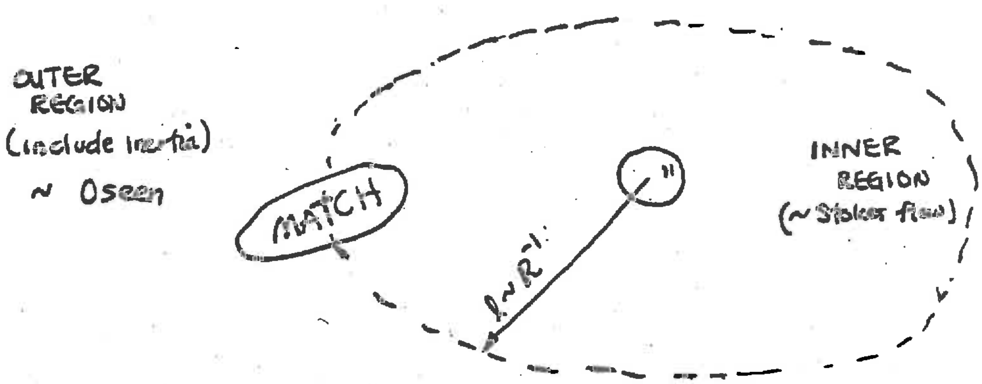
- there are 2 distinct length-scales in our problem:
  - (i) the sphere radius, representative of motion near the particle
  - (ii) ν/U_∞, the intrinsic length-scale, representative of motion far from the particle
  - → ratio: $R = \frac{aU_\infty}{\nu}$

- the idea: examine simultaneously locally valid expansions close to the particle (Stokes eqns) and far from the particle (Oseen eqns)

  the "inner" region — the "outer" region

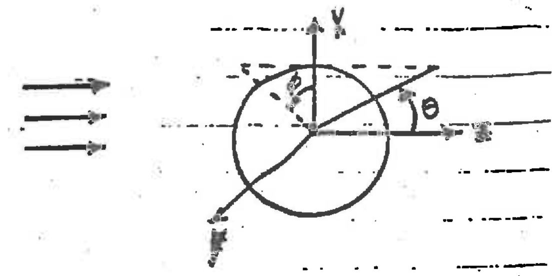
  1. develop asymptotic approximations in each region

  2. "inner" region - apply b.c. on sphere
     "outer" region - apply b.c. at ∞
     ⇒ "match" representations in intermediate regions



mathematics is difficult.

IV. LOW REYNOLDS NUMBER FLOW PAST A SPHERE



introduce a streamfunction $\psi(r,\theta)$

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \qquad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

assume flow is axisymmetric so no ϕ-dependence.

The dimensionless form of the Navier-Stokes eqns can be written

$$E^4\psi = \frac{R}{r^2\sin\theta}\left[-\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta} + \frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r} + 2\cot\theta\frac{\partial\psi}{\partial r} - \frac{2}{r}\frac{\partial\psi}{\partial\theta}\right]E^2\psi \qquad (1)$$

where E^2 is the operator

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

B.C. as $r\to\infty$ $\quad \psi \to \frac{1}{2}r^2\sin^2\theta$

$r=1$, all θ $\quad \psi = 0 \quad (u_r = 0)$

$r=1$, all θ $\quad \frac{\partial\psi}{\partial r} = 0 \quad (u_\theta = 0)$

"INNER" REGION: near the sphere we look for a solution

$$\psi(r,\theta) = \psi_0(r,\theta) + R\,\psi_1(r,\theta) + \cdots \qquad (2)$$

we know that far from the sphere this approximation will break down. Nevertheless, we solved for ψ_0 previously

$$\psi_0(r,\theta) = \sin^2\theta\left[\frac{1}{2}r^2 - \frac{3}{4}r + \frac{1}{4r}\right]$$

Eqn for ψ_1 if "naively" attempted wouldn't satisfy b.c at ∞. The appropriate boundary conditions for ψ_1 must come from "MATCHING" with an outer solution.

Substituting the perturbation expansion (2) into (1), the $O(R)$ eqn for ψ_1 is

$$E^4\psi_1 = \frac{1}{r^2\sin\theta}\left[-\frac{\partial\psi_0}{\partial r}\frac{\partial}{\partial\theta} + \frac{\partial\psi_0}{\partial\theta}\frac{\partial}{\partial r} + 2\cot\theta\frac{\partial\psi_0}{\partial r} - \frac{2}{r}\frac{\partial\psi_0}{\partial\theta}\right]E^2\psi_0$$

$$\boxed{E^4\psi_1 = -\frac{9}{4}\left(\frac{2}{r^2} - \frac{3}{r^3} + \frac{1}{r^5}\right)\sin^2\theta\cos\theta} \qquad (3)$$

"OUTER" REGION : examine the flow field far from the sphere. Must retain inertial effects independent of magnitude of R (however small).
we expect the appropriate characteristic length at large distances from the particle to be ν/U.

One way to proceed is to realize that the chosen scaling (in particular, the length-scale) in eqn (1) is not appropriate to the region far from the sphere. $\Rightarrow$ RESCALE.
Rescale eqn (1) so as to retain inertia in the limit $R \to 0$.

(*) Introduce $\rho = r R^{\alpha}$ where we expect $\alpha > 0$ since the idea of rescaling is to introduce a length-scale which is $O(1)$ for a typical variation in velocity.

$$\underset{O(1)}{\rho} = \underset{r \gg 1}{r}\ \underset{R \ll 1}{R^{\alpha}}$$

In the outer region $\psi \to \frac{1}{2} r^2 \sin^2\theta$ as $r \to \infty$

So, corresponding to the rescaling (*), we must rescale ψ so that

(**) $\Psi = \psi R^{2\alpha}$ ($= \frac{1}{2}\rho^2 \sin^2\theta$ is the uniform flow)

Substituting (*), (**) into eqn (1) : Ψ = stream-function in outer region

$$E_\rho^4 \Psi = \frac{R^{1-\alpha}}{\rho^2 \sin\theta}\left[-\frac{\partial\Psi}{\partial\rho}\frac{\partial}{\partial\theta} + \frac{\partial\Psi}{\partial\theta}\frac{\partial}{\partial\rho} + 2\cot\theta\,\frac{\partial\Psi}{\partial\rho} - \frac{2}{\rho}\frac{\partial\Psi}{\partial\theta}\right] E_\rho^2\Psi \qquad (4)$$

where $$E_\rho^2 = \frac{\partial^2}{\partial\rho^2} - \frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

$\Rightarrow$ Choose α to retain inertial terms.

$\therefore$ $\boxed{\alpha = 1}$ $\longrightarrow$ $\boxed{\begin{array}{l}\rho = r R \\ \Psi = \psi R^2\end{array}}$ (corresponds to length $L = \nu/U$ in outer region)

The solution to (4) must be "matched" to the solution found in the inner region.

Matching requires that

$$\lim_{r\to\infty} \psi(r,\theta) \xleftrightarrow[\text{(have same functional form)}]{\text{"match"}} \lim_{\rho\to 0} \frac{1}{R^2}\Psi(\rho,\theta)$$

form of INNER solution as outer region is approached

form of OUTER solution as inner region is approached

Now, if the inner solution $\psi_0(r,\theta)$ is expressed in terms of outer variables,

$r = \rho/R$

$\Psi = \psi R^2$

$$R^2 \psi_0(\rho,\theta) = \left(\frac{1}{2}\rho^2 - \frac{3}{4}\rho R + \frac{1}{4\rho} R^3\right) \sin^2\theta$$

$$= \frac{1}{2}\rho^2 \sin^2\theta \quad - \frac{3}{4}\rho \sin^2\theta \; R + \cdots$$

uniform flow at ∞

suggests that in the outer region the correction to the uniform flow is $R\,\Psi_1(\rho,\theta)$

Therefore, in the outer region we assume an expansion of the form

$$\Psi(\rho,\theta) = \frac{1}{2}\rho^2 \sin^2\theta + R\,\Psi_1(\rho,\theta) + \cdots$$

Substituting into (4) yields the following eqn for Ψ_1:

$$\left(E_\rho^2 - \cos\theta \frac{\partial}{\partial\rho} + \frac{\sin\theta}{\rho}\frac{\partial}{\partial\theta}\right) E_\rho^2 \Psi_1 = 0$$

The solution to this eqn is actually the disturbance motion to the Oseen eqn and describes the disturbance flow produced at large distances from the sphere. The solution is (Van Dyke p. 158)

$$\Psi_1(\rho,\theta) = -2c\,(1+\cos\theta)\left[1 - e^{-\frac{1}{2}\rho(1-\cos\theta)}\right]$$

The constant c is determined by matching with the inner solution.

Rewrite Ψ_1 in terms of inner variables. $\lim_{\rho\to 0} e^{-\frac{1}{2}\rho(1-\cos\theta)} \sim 1 - \frac{1}{2}\rho(1-\cos\theta) = 1 - \frac{1}{2} r R(1-\cos\theta)$

So

$$\lim_{R\to 0} \frac{1}{R^2}\Psi(\rho,\theta) \sim \frac{1}{2}\frac{\rho^2}{R^2}\sin^2\theta - \frac{1}{R}2c\,(1+\cos\theta)\left[\frac{1}{2} r R(1-\cos\theta) + \cdots\right]$$

$$\sim \frac{1}{2} r^2 \sin^2\theta - c\, r \sin^2\theta + \cdots$$

"match" with inner $\Updownarrow$

$$\psi(r,\theta) \sim \psi_0(r,\theta) \sim \left(\frac{1}{2} r^2 - \frac{3}{4} r + \frac{1}{4r}\right)\sin^2\theta \quad \Longrightarrow \quad \therefore \boxed{c = 3/4}$$

so, we've found

"INNER" solution : $\psi(r,\theta) = \left(\frac{1}{2}r^2 - \frac{3}{4}r + \frac{1}{4r}\right)\sin^2\theta + R\psi_1(r,\theta) + \cdots$

"OUTER" solution : $\Psi(\rho,\theta) = \frac{1}{2}\rho^2\sin^2\theta - R\frac{3}{2}(1+\cos\theta)\left[1 - e^{-\frac{1}{2}\rho(1-\cos\theta)}\right] + \cdots$

This analysis can be continued.

For a brief discussion of the above and solution of additional terms, see Van Dyke Perturbation Methods in Fluid Mechanics.
For a thorough discussion and a complete description of the details, see Proudman & Pearson, "Expansions at small Reynolds numbers for the flow past past a sphere and a circular cylinder" J. Fluid Mechanics 2 p. 237-262 (1957).

From the analysis one can calculate the force on the sphere moving at velocity $\underset{\sim}{U}$ in a quiescent fluid:

$$\underset{\sim}{F} = -6\pi\mu a \underset{\sim}{U}\left[\underbrace{1 + \frac{3}{8}R}_{\text{Oseen's result}} + \frac{9}{40}R^2\log R + O(R^2)\right] \qquad R \ll 1$$

unexpected logarithmic term from singular perturbation analysis

NOTE: inertia increases the drag on the particle

The analysis when carried out for a circular cylinder yields

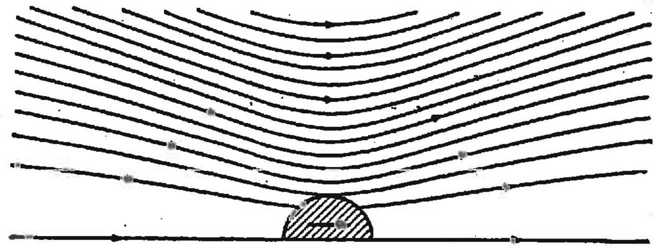
$$\begin{array}{l}\text{force/length}\\ \text{exerted by fluid}\\ \text{on cylinder}\end{array} \equiv \underset{\sim}{F} = \frac{4\pi\mu U}{\gamma - \frac{1}{2} + \log\left(\frac{R}{4}\right)}$$

$\gamma = 0.577\ldots$ (Euler's constant)

$R \ll 1$

the streamlines at zero and at small but finite Reynolds number are shown below (the coordinate system is moving with the sphere)

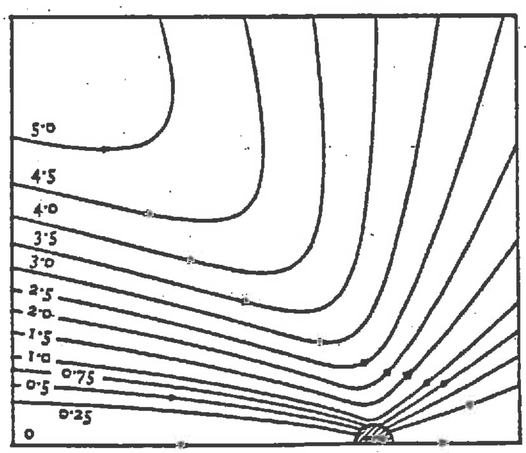
(1) Stokes flow due to a moving sphere in a quiescent fluid



streamlines are symmetric about the midplane ($\theta = \pi/2$)

Figure 4.9.1. Streamlines, in an axial plane, for flow due to a moving sphere at $R \ll 1$ (with complete neglect of inertia forces).

(2) Flow at small Reynolds numbers due to a moving sphere in a quiescent fluid, inertial effects included.



Fore-aft symmetry is lost. Far from the sphere, the streamlines tend to become radially outward, except within a narrow 'wake'-type region directly behind the sphere, $\theta \sim \pi$.

Figure 4.10.1. Streamlines in an axial plane for the outer part of the flow field due to a moving sphere, according to the Oseen equations. ψ is equal to some constant times the numbers shown on the streamlines.

reference : G.K Batchelor *An Introduction to Fluid Dynamics* p. 231, 242.