Course 18.355

Handout #7

Fall 2012

Low Reynolds Number Hydrodynamics

Vector methods for solving Stokes flow problems :

Here we wish to discuss some very powerful ideas for solving Stokes equations (and related linear equations). The techniques are limited to special geometries (for example, spherical or cylindrical coordinates) but the ideas allow many problems to be collapsed essentially to one problem.

1. Some preliminary remarks about vectors : vectors and pseudo (axial) vectors

It is important to make a distinction between proper (or true) vectors, say the position vector $\mathbf{x}$, the velocity field $\mathbf{u}(\mathbf{x})$ or a force $\mathbf{F}$, and so-called *pseudo* vectors, say the angular velocity vector $\mathbf{\Omega}$ or a torque, quantities which generally involve the notion of a cross product. Recall that in defining the cross (or vector) product of two vectors, we invoked a convention, the "right hand rule," which introduces an element of arbitrariness to the direction of the cross product.

Thus, we have the following rules relating true and pseudo vectors :

(i) the cross product is a pseudo operation

(ii) the cross product of two true vectors is a pseudo vector; *e.g.* $\mathbf{x} \wedge \mathbf{F} = \mathbf{L}$.

(iii) the cross product of a pseudo vector and a true vector is a true vector; *e.g.* $\mathbf{u} = \mathbf{\Omega} \wedge \mathbf{x}$.

(iv) pseudo scalars - for examples, the inner product of a true vector, say $\mathbf{x}$, and a pseudo vector, say $\mathbf{\Omega}$, produces a pseudo scalar $\mathbf{x} \cdot \mathbf{\Omega}$ since the arbitrariness of sign remains associated with the pseudo vector. Similarly, one may define pseudo tensors.

2. Harmonic functions

A function is called harmonic if it satisfies Laplace's equation $\nabla^2\Phi = 0$.

We will most often be concerned with problems in spherical coordinates. The variable $r = |\mathbf{x}|$ ($\mathbf{x}$ is the usual position vector) measures distance from the origin.

Exterior problems : $r > 0$

Let us first discuss *exterior* problems, for which $r > 0$. In spherical coordinates, assuming that there is no θ or ϕ dependence, $\nabla^2 = \frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr})$. Hence, by direct substitution, we see that $\Phi = 1/r$ ($r \neq 0$) is a solution to the Laplace equation. The function $1/r$ is known as the fundamental solution.

Furthermore, by taking the gradient of Laplace's equation we see that if Φ is harmonic, then so is $\nabla\Phi$ since certainly $\nabla^2(\nabla\Phi) = 0$. Hence, $\nabla(1/r) = -\mathbf{x}/r^3$ is a *harmonic vector function.* Continuing in this manner, $\nabla\nabla(1/r) = -\mathbf{I}/r^3 + 3\mathbf{x}\mathbf{x}/r^5$ is a *harmonic second rank tensor.* Notice that the harmonic functions generated in this way decay as $r \to \infty$ and are typically denoted as

$$\Phi_{-(n+1)} = \underbrace{\nabla\nabla...\nabla}_{n}\left(\frac{1}{r}\right) \tag{1}$$

The functions $\Phi_{-(n+1)}$ are called solid spherical harmonics of degree $-(n+1)$. They are clearly defined to within an arbitrary multiplicative constant and it is common to see them written as

$$\Phi_{-1} = \frac{1}{r} \qquad \Phi_{-2} = \frac{\mathbf{x}}{r^3} \qquad \Phi_{-3} = \frac{\mathbf{I}}{r^3} - \frac{3\mathbf{xx}}{r^5} \tag{2}$$

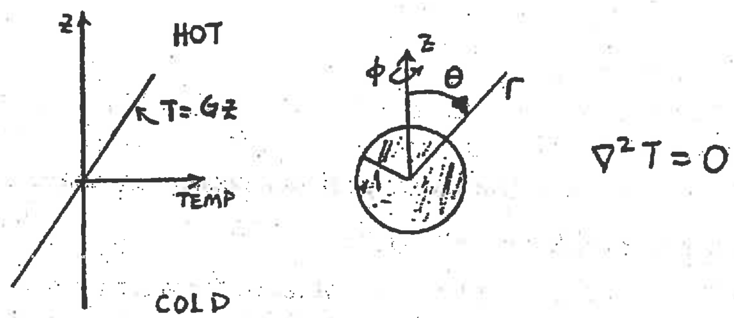
Interior problems :

For completeness, we give the form of the solid spherical harmonic functions necessary for describing *interior* problems, *i.e.* those that include the origin $r = 0$. One can verify by direct differentiation that $\Phi_n = r^{2n+1}\Phi_{-(n+1)}$ is also a solution to Laplace's equation $\nabla^2\Phi_n = 0$. Notice that because of the additional powers of r, these functions remain bounded as $r \to 0$. The first few of these functions are (again ignoring minus signs)

$$\Phi_0 = 1 \qquad \Phi_1 = \mathbf{x} \qquad \Phi_2 = r^2\mathbf{I} - 3\mathbf{xx}$$
$$\Phi_3 = 3^{(x_i\delta_{jk} + x_j\delta_{ik} + x_k\delta_{ij})} - 15^{x_ix_jx_k} \tag{3}$$

where we have used index notation for ease of writing the final expression which represents a third rank harmonic function.

3. Example : The use of linearity arguments and vector spherical harmonics to solve for heat conduction from a uniformly heated sphere (infinite conductivity) in a linear temperature gradient.



The physical problem we wish to consider is the temperature distribution outside a perfectly conducting sphere imbedded in a material of conductivity k. Assume that in the absence of the sphere the temperature varies linearly with position, $T^\infty(\mathbf{x}) = \mathbf{G}\cdot\mathbf{x}$, where $\mathbf{G}$ is the temperature gradient in the material; *i.e.* $\nabla T = \mathbf{G} = G\mathbf{e}_z$ as in the figure above. The mathematical problem statement is

$$\nabla^2 T = 0 \quad \text{for} \quad r > 1,$$

subject to boundary conditions

$$T = 0 \quad \text{on} \quad r = 1 \qquad T \to \mathbf{G}\cdot\mathbf{x} \quad \text{as} \quad r \to \infty.$$

It is often useful to rewrite the problem in terms of *disturbance variables* so that the unknown function decays as $r \to \infty$. Let $T = \mathbf{G}\cdot\mathbf{x} + T'$. Then, the disturbance temperature field T' satisfies

$$\nabla^2 T' = 0$$

$$T' = -\mathbf{G}\cdot\mathbf{x} \quad \text{on } r = 1 \qquad T' \to 0 \quad \text{as } r \to \infty.$$

To construct a solution we note that T' satisfies a linear equation and linear boundary conditions. The only "forcing" in the problem is the temperature gradient $\mathbf{G}$ so $\mathbf{G}$ (both its magnitude and direction) must determine T'. Hence, as T' is a solution to Laplace's equation, construct a decaying harmonic scalar function linear in $\mathbf{G}$ as

$$T' = \alpha\frac{\mathbf{G}\cdot\mathbf{x}}{r^3}$$

where α is a coefficient to be determined from the boundary conditions.

At $r = 1$ we have $T' = -\mathbf{G}\cdot\mathbf{x} \to \alpha = -1$. Hence, we have the solution

$$T(\mathbf{x}) \;=\; \mathbf{G}\cdot\mathbf{x}[1 - \frac{1}{r^3}]. \tag{4}$$

If $\mathbf{G} = G\mathbf{e}_z$, where $\mathbf{e}_z = \cos\theta\, \mathbf{e}_r - \sin\theta\, \mathbf{e}_\theta$ in spherical coordinates, then

$$T(r,\theta) \;=\; Gr\cos\theta[1 - \frac{1}{r^3}].$$

Exercise : Consider the same problem but allow the sphere to have a finite thermal conductivity, say $\hat{k}$. Solve for the temperature field inside and outside the sphere.

4. Lamb's general solution : a very useful method for constructing solutions to Stokes flow problems

The most common form of Lamb's general solution involves expansions in spherical harmonics; see, for example, Lamb *Hydrodynamics* §335,336 and Happel & Brenner *Low Reynolds Number Hydrodynamics* p. 62.

The alternative presentation described here involves a minimum of notation and I have found it to be the most useful form from the standpoint of applications. (I call it the "Hinch method" after Dr. E. J. Hinch of Cambridge University who taught me the method and ideas.)

Let's begin by recalling Stokes equations :

$$\mu \nabla^2 \mathbf{u} = \nabla p \qquad \nabla \cdot \mathbf{u} = 0.$$

It may be verified by direct substitution that the velocity and pressure fields can be represented by

$$\mathbf{u}(\mathbf{x}) = \nabla \phi + \mathbf{x} \wedge \nabla \psi + \nabla(\mathbf{x} \cdot \mathbf{A}) - 2\mathbf{A}, \tag{5a}$$

$$p(\mathbf{x}) = 2\mu \nabla \cdot \mathbf{A}, \tag{5b}$$

where $\phi(\mathbf{x}), \psi(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ are harmonic scalar and vector functions, respectively ($\nabla^2\phi = 0 = \nabla^2\psi; \nabla^2\mathbf{A} = 0$). Similar to the linearity arguments used to construct the solution to the heat conduction problem on the previous page, we will use the linearity of the Stokes flow problem to construct the functions ϕ, ψ and $\mathbf{A}$. (If solving a problem in dimensionless form, just drop the μ from equation 5b.)

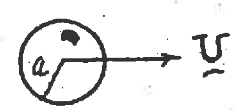
Exercise : Show by direct substitution that the representation (5) for $\mathbf{u}, p$ satisfies continuity and the Stokes equations.

It is important to recognize that the terms involving ϕ and ψ repreesent the most general velocity field which is both harmonic and divergence-free while the terms involving $\mathbf{A}$ represent the particular solution accounting for the pressure field. Also, notice that it is the divergence of $\mathbf{A}$ that completely accounts for the pressure field. Hence, any divergence-free contributions to $\mathbf{A}$ may be neglected since such contributions may be absorbed into terms involving ϕ and ψ. Finally, notice that since $\mathbf{u}$ is a true vector, it follows that ψ is a pseudo scalar function.

5. Translation of a rigid sphere - revisited

Consider the translation of a rigid spherical particle at low Reynolds numbers. The mathematical problem statement is

$$\mu \nabla^2 \mathbf{u} = \nabla p \qquad \nabla \cdot \mathbf{u} = 0$$



subject to boundary conditions

$$\mathbf{u} = \mathbf{U} \quad \text{at} \quad r = a \qquad \text{and} \qquad (\mathbf{u}, p) \to (0, p_\infty) \quad \text{as} \quad r \to \infty.$$

Nondimensionalize these equations using the particle radius a as the characteristic length scale, the particle speed $U = |\mathbf{U}|$ as the characteristic velocity and $\mu U/a$ as the characteristic pressure. This leads to the dimensionless problem statement

$$\nabla^2 \mathbf{u} = \nabla p \qquad \nabla \cdot \mathbf{u} = 0$$

with

$$\mathbf{u} = \hat{\mathbf{U}} \quad \text{at} \quad r = 1 \qquad \mathbf{u} \to 0 \quad \text{as} \quad r \to \infty,$$

where $\hat{\mathbf{U}} = \mathbf{U}/U$ is a unit vector in the direction of translation.

Let us now solve this problem using Lamb's general solution, equation 5. The only 'forcing' in the problem is the true vector $\hat{\mathbf{U}}$. Hence, we choose the harmonic functions ϕ, ψ and $\mathbf{A}$ as :

$\psi = 0$ since there is no way to form a pseudo scalar function using $\hat{\mathbf{U}}$

and

$$\phi = \alpha \frac{\hat{\mathbf{U}} \cdot \mathbf{x}}{r^3} \qquad \mathbf{A} = \beta \frac{\hat{\mathbf{U}}}{r}. \tag{6}$$

Then, using (5) to determine the form of $\mathbf{u}(\mathbf{x})$, $p(\mathbf{x})$ leads to

$$\mathbf{u}(\mathbf{x}) = \alpha[\frac{\hat{\mathbf{U}}}{r^3} - \frac{3\hat{\mathbf{U}} \cdot \mathbf{xx}}{r^5}] - \beta[\frac{\hat{\mathbf{U}}}{r} + \frac{\mathbf{U} \cdot \mathbf{xx}}{r^3}] \tag{7a}$$

$$p(\mathbf{x}) = p_\infty + \beta \frac{\hat{\mathbf{U}} \cdot \mathbf{x}}{r^3}. \tag{7b}$$

It only remains to evaluate the two unknown coefficients α and β using the boundary conditions. Since $\mathbf{u} = \hat{\mathbf{U}}$ at $r = 1$ for all positions $\mathbf{x}$ on the sphere surface we have

$$\hat{\mathbf{U}} = (\alpha - \beta)\hat{\mathbf{U}} - (3\alpha + \beta)\hat{\mathbf{U}} \cdot \mathbf{xx} \quad \text{for } |\mathbf{x}| = 1$$

from which we conclude

$$1 = \alpha - \beta \qquad \text{and } 3\alpha + \beta = 0.$$

Hence, $\alpha = 1/4$ and $\beta = -3/4$. Therefore, the dimensionless velocity and pressure fields in the fluid are given by

$$\mathbf{u}(\mathbf{x}) = \frac{3}{4}\frac{\hat{\mathbf{U}}}{r} \cdot [\mathbf{I} + \frac{\mathbf{xx}}{r^2}] - \frac{1}{4}\frac{\hat{\mathbf{U}}}{r^3} \cdot [\mathbf{I} - \frac{3\mathbf{xx}}{r^2}] \tag{8a}$$

$$p(\mathbf{x}) = p_\infty + \beta \frac{\hat{\mathbf{U}} \cdot \mathbf{x}}{r^3}. \tag{8b}$$

It remains to calculate the hydrodynamic force the fluid exerts on the particle. As always the hydrodynamic force exerted by the fluid on the particle is given by integrating the stress over the particle surface :

$$\mathbf{F} = \int_S \mathbf{n} \cdot \mathbf{T}\, dS.$$

Calculating the stress requires some tedious algebra, most of which amounts to taking the gradient of $\mathbf{x}$ and $1/r$ and is greatly simplified using index notation. For example, the rate of strain tensor $\mathbf{E}$ is given by

$$\mathbf{E} = \frac{1}{4}\left[-6\frac{\hat{\mathbf{U}}\mathbf{x} + \mathbf{x}\hat{\mathbf{U}} + \hat{\mathbf{U}}\cdot\mathbf{x}\mathbf{I}}{r^5} + 30\frac{\hat{\mathbf{U}}\cdot\mathbf{xxx}}{r^7}\right] + \frac{3}{4}\left[\frac{2\hat{\mathbf{U}}\cdot\mathbf{x}\mathbf{I}}{r^3} - \frac{6\hat{\mathbf{U}}\cdot\mathbf{xxx}}{r^5}\right].$$

Evaluated on the particle surface $r = 1, \mathbf{x} = \mathbf{n}$, we find the dimensionless force per unit area to be

$$\begin{aligned}\mathbf{n}\cdot\mathbf{T}|_{r=1} &= -p\mathbf{n} + 2\mathbf{n}\cdot\mathbf{E}\,|_{r=1} \\ &= \underbrace{-p_\infty\mathbf{n} - \frac{3}{2}\hat{\mathbf{U}}\cdot\mathbf{nn}}_{\text{pressure contribution}} + \underbrace{\frac{3}{2}\hat{\mathbf{U}}\cdot\mathbf{nn} - \frac{3}{2}\hat{\mathbf{U}}}_{\text{viscous contribution}}\end{aligned}$$

It remains to do the surface integration. Here we make use of the following identity valid for integration over spherical surfaces

NOTE: $\int_S \underline{n}\, dS = 0$ by the Divergence Theorem

$\int_S dS = 4\pi$ for a sphere

$$\int_S \mathbf{nn}\, dS = \frac{4\pi}{3}\mathbf{I}.$$

Hence we have

$$\mathbf{F} = \int_S \mathbf{n}\cdot\mathbf{T}\, dS = \underbrace{-2\pi\hat{\mathbf{U}}}_{\text{pressure contribution}} - \underbrace{4\pi\hat{\mathbf{U}}}_{\text{viscous contribution}} = -6\pi\hat{\mathbf{U}} \qquad (9a)$$

or in dimensional terms

$$\boxed{\mathbf{F} = -6\pi\mu a\mathbf{U}. \quad \text{Stokes drag}} \qquad (9b)$$

From 9*a* we see that the pressure gradient across the particle contributes one-third the value of the drag, while the viscous stress contributes two-thirds of the drag. These ratios change for different particle shapes and different Reynolds numbers. The drag arising from the presure distribution over the particle is called *form* drag and the drag arising from viscous stresses is called *friction* drag.

We conclude with a simple force balance on a spherical particle sedimenting at steady state. A force balance requires that the hydrodynamic drag be balanced by the gravitational force and the buoyancy force due to the displaced fluid. Hence

$$\begin{aligned}0 = \sum \text{Forces} &= \mathbf{F}_{hydro} + \mathbf{F}_{gravity} + \mathbf{F}_{buoyant} \\ &= -6\pi\mu a\mathbf{U} + \frac{4\pi a^3}{3}\rho_p\mathbf{g} - \frac{4\pi a^3}{3}\rho\mathbf{g}.\end{aligned}$$

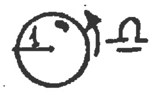
$\rho_p \equiv$ particle density

Therefore, the steady translation of the particle is

$$\mathbf{U} = \frac{2}{9}\frac{a^2}{\mu}(\rho_p - \rho)\mathbf{g}. \qquad \text{Stokes setting velocity} \tag{10}$$

6. Example : A rotating spherical particle in a quiescent fluid

As an additional example of this vector method of solving Stokes flow problems, determine the velocity field due to a rotating sphere in a fluid that is otherwise at rest.



The angular velocity of the sphere is denoted by $\mathbf{\Omega}$. Since the definition of the angular velocity involves the right-hand rule, $\mathbf{\Omega}$ is a pseudo vector.

Now construct the velocity using Lamb's general solution, equation (5). Choose the harmonic functions ϕ, ψ and $\mathbf{A}$ as

$$\psi = \alpha\frac{\mathbf{\Omega}\cdot\mathbf{x}}{r^3} \qquad \phi = 0 \qquad \mathbf{A} = 0.$$

→ $\beta\left(\Omega \wedge \frac{x}{r^3}\right)$

Hence,

$$\mathbf{u}(\mathbf{x}) = \mathbf{x}\wedge\nabla\psi = -\alpha\frac{\mathbf{\Omega}\wedge\mathbf{x}}{r^3}$$

← gives same term in u

$$p(\mathbf{x}) = 0.$$

Apply the boundary conditions $\mathbf{u} = \mathbf{\Omega}\wedge\mathbf{x}$ on $r = 1$ and $\mathbf{u} \to 0$ as $r \to \infty$. Clearly the latter boundary condition is automatically satisfied and the boundary condition on the sphere surface is satisfied by choosing $\alpha = -1$. Therefore,

$$\boxed{\mathbf{u}(\mathbf{x}) = \frac{\mathbf{\Omega}\wedge\mathbf{x}}{r^3}.} \tag{11}$$

Now consider the torque acting on the particle. At steady state, the external torque $\mathbf{L}^{ext}$ required to rotate the particle is equal and opposite to the torque exerted by the fluid on the particle. Hence, $\mathbf{L}^{ext}$ is calculated as

$$\mathbf{L}^{ext} = -\int_S \mathbf{x}\wedge(\mathbf{n}\cdot\mathbf{T})\, dS.$$

The stress tensor is straightforward to calculate from $\mathbf{u}(\mathbf{x})$. Using index notation

$$T_{il} = -3\Omega_j x_k\left[\epsilon_{jki}x_l + \epsilon_{jkl}x_i\right].$$

On the sphere surface $\mathbf{x} = \mathbf{n}, r = 1$, so that

$$\mathbf{x}\wedge(\mathbf{n}\cdot\mathbf{T})\,|_{r=1} = -3\mathbf{n}\wedge(\mathbf{\Omega}\wedge\mathbf{n}) = 3\left[\mathbf{n}(\mathbf{n}\cdot\mathbf{\Omega}) - \mathbf{\Omega}\right]$$

or

$$\int_S \mathbf{n}\wedge(\mathbf{n}\cdot\mathbf{T})\, dS = 3\int_S \mathbf{nn}\, dS\cdot\mathbf{\Omega} - 3\int_S dS.$$

Using the identity for integrating $\mathbf{nn}$ over a spherical surface we have

$$\mathbf{L}^{ext} = -\left[4\pi\mathbf{\Omega} - 12\pi\mathbf{\Omega}\right] = 8\pi\mathbf{\Omega}. \tag{12}$$

Therefore, the dimensional torque required to spin the particle at angular velocity $\mathbf{\Omega}$ is $\mathbf{L}^{ext} = 8\pi\mu a^3\mathbf{\Omega}$.

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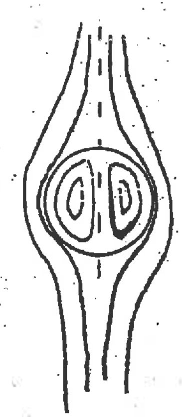
Low Reynolds Number Hydrodynamics

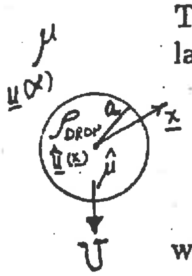
Example : A sedimenting liquid drop (also, see Batchelor, p. 235-238)

Consider the translation of a spherical liquid drop in an otherwise quiescent fluid. To be specific, consider the situation where the drop fluid is a higher density ρ_{drop} than the surrounding fluid, density ρ, so that the drop sediments. We are interested in determining the detailed velocity field inside and outside the drop, as well as the translational velocity of the drop.

We will now construct an exact solution to the low Reynolds number flow problem using the vector methods introduced earlier. Both the fluid motion inside and outside the droplet must be studied and appropriate boundary conditions applied along the fluid-fluid interface. For simplicity, we will assume at the outset that the drop remains spherical.



Choose a coordinate system fixed to the instantaneous center of the sphere. The sphere translates with steady velocity $\mathbf{U}$. Relative to an observer fixed in the laboratory, the governing equations are



$$\mu\nabla^2\mathbf{u} = \nabla p \qquad \hat{\mu}\nabla^2\hat{\mathbf{u}} = \nabla\hat{p}$$
$$\nabla\cdot\mathbf{u} = 0 \qquad \nabla\cdot\hat{\mathbf{u}} = 0. \tag{1}$$

where $\hat{}$ is used to denote the droplet phase. One boundary condition on the velocity field is the continuity of velocity

$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{at} \quad r = a. \tag{2a}$$

Also, since the drop remains spherical, the normal component of the surface velocity is given by

$$\mathbf{u}\cdot\mathbf{n} = \hat{\mathbf{u}}\cdot\mathbf{n} = \mathbf{U}\cdot\mathbf{n} \qquad \text{at} \quad r = a. \tag{2b}$$

The dynamic condition, which states that the tangential stresses across the interface are continuous is written

$$\mathbf{s}\cdot\mathbf{n}\cdot\mathbf{T} = \mathbf{s}\cdot\mathbf{n}\cdot\hat{\mathbf{T}} \qquad \text{at} \quad r = a \tag{2c}$$

where $\mathbf{T}$ is the stress tensor. In these equations $\mathbf{n}$ is the unit normal directed from the drop fluid into the external fluid and $\mathbf{s}$ is the unit tangent vector along the drop interface.

Finally, far from the translating drop the velocity vanishes, $\mathbf{u} \to 0$ as $r \to \infty$. It is worth noting that if contaminants or a temperature gradient are present at the fluid-fluid interface, the tangential stress balance must be modified to account for the resulting tangential interfacial stresses. Also, no statement is made about normal stresses since the drop shape has been assumed to be spherical. The normal stress balance is checked *a posteriori* and is used to calculate corrections to the drop shape. We will not discuss this here.

At this point the translational velocity of the drop $\mathbf{U}$ is an unknown constant vector and must be determined, after the velocity field is known, from an overall force balance on the drop.

Nondimensionalization :

Choose the drop radius a as the characteristic length, $\mathbf{U}$ as the characteristic velocity and choose characteristic pressures for the two fluid phases as $p_c = \mu U/a$ and $\hat{p}_c = \hat{\mu} U/a$, with $U = |\mathbf{U}|$.

The dimensionless problem statement is

$$\nabla^2 \mathbf{u} = \nabla p \qquad \nabla^2 \hat{\mathbf{u}} = \nabla \hat{p}$$
$$\nabla \cdot \mathbf{u} = 0 \qquad \nabla \cdot \hat{\mathbf{u}} = 0, \tag{3}$$

with

$$\mathbf{u} = \hat{\mathbf{u}} \qquad \mathbf{u} \cdot \mathbf{n} = \hat{\mathbf{u}} \cdot \mathbf{n} = \hat{\mathbf{U}} \cdot \mathbf{n} \qquad \mathbf{s} \cdot \mathbf{n} \cdot \mathbf{T} = \lambda \mathbf{s} \cdot \mathbf{n} \cdot \hat{\mathbf{T}} \qquad \text{at } r = 1, \quad (4a-c)$$

and

$$\mathbf{u} \to \infty \qquad \text{as } r \to \infty.$$

Here $\lambda = \hat{\mu}/\mu$ is the viscosity ratio of the two fluids and $\hat{\mathbf{U}} = \mathbf{U}/U$ is a unit vector in the direction of motion.

Now apply Lamb's general solution. Clearly the solution to this linear problem depends completely on $\hat{\mathbf{U}}$. For the flow outside the spherical drop we use exterior harmonics, so similar to the rigid sphere case, choose

$$\phi = \alpha \frac{\hat{\mathbf{U}} \cdot \mathbf{x}}{r^3} \qquad \psi = 0 \qquad \mathbf{A} = \beta \frac{\hat{\mathbf{U}}}{r}.$$

The external velocity and pressure fields are then given by

$$\mathbf{u}(\mathbf{x}) = \alpha \left[\frac{\hat{\mathbf{U}}}{r^3} - \frac{3\hat{\mathbf{U}} \cdot \mathbf{x}\mathbf{x}}{r^5} \right] - \beta \left[\frac{\hat{\mathbf{U}}}{r} + \frac{\hat{\mathbf{U}} \cdot \mathbf{x}\mathbf{x}}{r^3} \right] \tag{5a}$$

$$p(\mathbf{x}) = p_\infty + \beta \frac{\hat{\mathbf{U}} \cdot \mathbf{x}}{r^3}, \tag{5b}$$

where p_∞ is the constant reference pressure far from the drop (for an incompressible flow, the pressure is only determined to within an additive constant).

For the flow internal to the drop, use internal harmonics which remain bounded as $r \to 0$. Hence, choose

$$\hat{\phi} = \hat{\alpha}\, \hat{\mathbf{U}} \cdot \mathbf{x} \qquad \hat{\psi} = 0 \qquad \hat{\mathbf{A}} = \hat{\beta}\, [r^2 \hat{\mathbf{U}} - 3\hat{\mathbf{U}} \cdot \mathbf{x}\mathbf{x}\,].$$

The internal velocity field is then given by

$$\hat{\mathbf{u}}(\mathbf{x}) = \hat{\alpha}\,\hat{\mathbf{U}} - 2\,\hat{\beta}\,[2r^2\hat{\mathbf{U}} - \hat{\mathbf{U}}\cdot\mathbf{x}\mathbf{x}\,] \tag{6a}$$

$$\hat{p}(\mathbf{x}) = p_o - 20\,\hat{\beta}\,\hat{\mathbf{U}}\cdot\mathbf{x}, \tag{6b}$$

where p_o is a constant reference pressure inside the drop (p_o and p_∞ are not independent; their difference is proportional to the interfacial tension between the two fluids).

It remains to determine the 4 unknown coefficients $\alpha, \beta, \hat{\alpha}$ and $\hat{\beta}$ from the boundary conditions.

Continuity of velocity, $\mathbf{u} = \hat{\mathbf{u}}$ at $r = 1, \mathbf{x} = \mathbf{n}$, for all $\mathbf{x}$ on the surface, yields the two relationships

$$\alpha - \beta = \hat{\alpha} - 4\hat{\beta} \qquad -3\alpha - \beta = 4\hat{\beta}. \tag{7a, b}$$

From the normal velocity relationship $\hat{\mathbf{u}}\cdot\mathbf{n} = \hat{\mathbf{U}}\cdot\mathbf{n}$ we have

$$\hat{\alpha} - 2\hat{\beta} = 1. \tag{7c}$$

Finally, it is necessary to compute the stress tensors $\mathbf{T}, \hat{\mathbf{T}}$, where, for example, in dimensionless terms, $\mathbf{T} = -p\mathbf{I} + 2\mathbf{E}$. This requires some tedious alebra (index notation is highly recommended), which one can do to find

$$\mathbf{T}(\mathbf{x}) = -p_\infty\mathbf{I} + \alpha\Big[-6\frac{(\hat{\mathbf{U}}\mathbf{x} + \mathbf{x}\hat{\mathbf{U}} + \mathbf{x}\cdot\hat{\mathbf{U}}\mathbf{I})}{r^5} + 30\frac{\hat{\mathbf{U}}\cdot\mathbf{x}\mathbf{x}\mathbf{x}}{r^7}\Big] + 6\beta\frac{\hat{\mathbf{U}}\cdot\mathbf{x}\mathbf{x}\mathbf{x}}{r^5}, \tag{8a}$$

so that

$$\mathbf{n}\cdot\mathbf{T}|_{r=1,\mathbf{x}=\mathbf{n}} = -p_\infty\mathbf{n} + \alpha\,[\,18\hat{\mathbf{U}}\cdot\mathbf{n}\mathbf{n} - 6\hat{\mathbf{U}}\,] + 6\beta\,\hat{\mathbf{U}}\cdot\mathbf{n}\mathbf{n} \tag{8b}$$

Note : for points on the spherical interface, $\mathbf{x} = \mathbf{n}$.
Also,

$$\hat{\mathbf{T}}(\mathbf{x}) = -p_o\mathbf{I} + \hat{\beta}[24\hat{\mathbf{U}}\cdot\mathbf{x}\mathbf{I} - 6(\hat{\mathbf{U}}\mathbf{x} + \mathbf{x}\hat{\mathbf{U}})], \tag{9a}$$

so that

$$\mathbf{n}\cdot\hat{\mathbf{T}}|_{r=1,\mathbf{x}=\mathbf{n}} = -p_o\mathbf{n} + \hat{\beta}[18\hat{\mathbf{U}}\cdot\mathbf{n}\mathbf{n} - 6\hat{\mathbf{U}}]. \tag{9b}$$

Hence, the tangential stress balance yields ($\mathbf{s}\cdot\mathbf{n} = 0$)

$$-6\alpha = -6\lambda\hat{\beta} \quad \rightarrow \quad \boxed{\alpha = \hat{\beta}} \tag{10}$$

Using equations $7a - c$ and 10 to solve for the four coefficients, we find

$$\alpha = \frac{\lambda}{4(1+\lambda)} \qquad \beta = \frac{-(2+3\lambda)}{4(1+\lambda)} \qquad \hat{\alpha} = \frac{(3+2\lambda)}{2(1+\lambda)} \qquad \hat{\beta} = \frac{1}{4(1+\lambda)}.$$

The velocity fields are now known. We can now use this solution to determine the translational velocity of the drop. To calculate the hydrodynamic force exerted by the outer fluid on the drop, integrate the force per unit area $\mathbf{n} \cdot \mathbf{T}$ over the drop surface (use equation 8b) :

$$\mathbf{F} = \int_S \mathbf{n} \cdot \mathbf{T}|_{r=1} \, dS = (18\alpha + 6\beta)\hat{\mathbf{U}} \cdot \int_S \mathbf{nn} \, dS - 6\alpha\hat{\mathbf{U}} \int_S dS = 8 \, \pi \, \beta \, \hat{\mathbf{U}}.$$

Therefore, returning to dimensional variables we have

$$\boxed{\mathbf{F} = -2\pi \frac{2+3\lambda}{1+\lambda} \mu a \mathbf{U}.} \tag{11}$$

We see that in the limit $\lambda \to \infty$ $\mathbf{F} = -6\pi\mu a\mathbf{U}$ which is the Stokes drag result. Also, in the limit $\lambda \to 0$ $\mathbf{F} = -4\pi\mu a\mathbf{U}$ which is the appropriate result for a rising bubble. As one might expect, a bubble of the same radius rises faster than a sphere with a rigid surface. Balancing the hydrodynamic force with the gravitational and buoyancy forces yields an explicit relationship for the rise velocity as a function of the density difference between the two phases :

$$\boxed{\mathbf{U} = \frac{2}{3} \frac{1+\lambda}{2+3\lambda} \frac{(\rho_{drop} - \rho)a^2}{\mu} \mathbf{g}} \quad \text{Hadamard-Rybczynski formula (1912).} \tag{12}$$