18.355 Handout #5

Thx to Howard Stone! (for the notes)

Greenberg p. 363-366
Hildebrand p. 203-208

J. STURM-LIOUVILLE THEORY

$\Rightarrow$ This discussion will generalize the concept of Fourier series to include eigenfunctions that arise from other ode's.

1. Consider the following ode:

$$(1) \quad \frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + \left[q(x) + \underset{\text{eigenvalue}}{\lambda}\, \underset{\text{weighting function}}{w(x)}\right] y = 0 \qquad a \le x \le b$$

or $\quad L(y) + \lambda w(x) y = 0 \quad$ where $\quad L(y) = \frac{d}{dx}\left(r(x)\frac{dy}{dx}\right) + q(x)y$

Assume the boundary conditions have the form

$$\left.\begin{aligned} \alpha_1 y'(a) + \beta_1 y(a) &= 0 \\ \alpha_2 y'(b) + \beta_2 y(b) &= 0 \end{aligned}\right\} \text{"homogeneous b.c."}$$

Remark: (i) This problem represents a generalization of the eigenvalue problem discussed earlier. As we have seen, these types of problems have an infinite number of solutions $y_n(x)$, one solution for each eigenvalue λ_n.

We now wish to demonstrate $\int_a^b w(x)\, y_n(x)\, y_m(x)\, dx = 0$ for $n \neq m$

(ii) Assume $r(x)$, dr/dx, $q(x)$ and $w(x)$ are continuous on $[a,b]$ and $r(x) > 0$, $w(x) > 0$.

in many applications, $q(x) < 0$.

2. Proof: orthogonality of the eigenfunctions – same method as used previously

From (1) $\quad y_m \frac{d}{dx}\left[r\frac{dy_n}{dx}\right] + \left[q + \lambda_n w\right] y_m y_n = 0$

and

$$y_n \frac{d}{dx}\left[r\frac{dy_m}{dx}\right] + \left[q + \lambda_m w\right] y_m y_n = 0$$

Subtract and integrate $a \to b$:

$$\int_a^b \left[y_m \frac{d}{dx}\left(r\frac{dy_n}{dx}\right) - y_n \frac{d}{dx}\left(r\frac{dy_m}{dx}\right)\right] dx + (\lambda_n - \lambda_m)\int_a^b w(x)\, y_m(x)\, y_n(x)\, dx = 0$$

Integrating by parts yields

$$\underbrace{r(x)\, y_m \frac{dy_n}{dx}\Big|_a^b - r(x)\, y_n \frac{dy_m}{dx}\Big|_a^b}_{*} + (\lambda_n - \lambda_m)\int_a^b w(x)\, y_m(x)\, y_n(x)\, dx = 0$$

From the b.c.,

$$y_n'(a) = -\frac{\beta_1}{\alpha_1} y_n(a), \quad y_n'(b) = -\frac{\beta_2}{\alpha_2} y_n(b), \quad y_m'(a) = -\frac{\beta_1}{\alpha_1} y_m(a), \quad y_m'(b) = -\frac{\beta_2}{\alpha_2} y_m(b)$$

so that one can show that $(*) \equiv 0$.

2. Orthogonality of eigenfunctions (continued)

We have $$(\lambda_n - \lambda_m)\int_a^b w(x)\, y_m(x) y_n(x)\, dx = 0$$

so that if $\lambda_m \neq \lambda_n$,

$$\therefore \int_a^b w(x) y_m(x)\, y_n(x)\, dx = 0$$

In words: the eigenfunctions that result from the Sturm-Liouville problem are orthogonal with respect to the weighting function $w(x)$.

3. Eigenfunction expansion theorem (a generalization of Fourier series)

a. Given the assumptions stated on the previous page, it can be proven that
(i) there is an infinite set of discrete eigenvalues : $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$
and (ii) to each eigenvalue λ_n there corresponds only one eigenfunction $y_n(x)$

b. Hence, given an arbitrary <u>piecewise</u> smooth function $f(x)$ on $[a,b]$, $f(x)$ can be represented via a series expansion in terms of the eigenfunctions $y_n(x)$,

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x) \qquad \text{where} \qquad a_n = \frac{\int_a^b w(x)\, y_n(x)\, f(x)\, dx}{\int_a^b w(x)\, y_n^2(x)\, dx}$$

(which follows directly from the orthogonality property)

and at a discontinuity, x_0,

$$\sum_{n=1}^{\infty} a_n y_n(x) = \tfrac{1}{2}\left[f(x_0^+) + f(x_0^-)\right]$$

4. Examples of 'Sturm-Liouville' eqns

(i) Legendre eqn $$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0$$

$w(x) = 1$

$P_n(x)$ are finite on $-1 \le x \le 1$ $\rightarrow$ $$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \begin{cases} 0 & n \neq m \\ \frac{2}{2n+1} & n = m \end{cases}$$

(ii) Bessel's eqn $$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \Big(\underset{\uparrow\, w(x)}{\lambda^2 x^2} - \nu^2/x\Big)y = 0$$ $\Rightarrow$ we'll see that nontrivial solutions exist only for certain values of λ, $\rightarrow \lambda_n$

Hence, for the eigenfunctions which result from Bessel's eqn,

$$\int_0^1 x\, y_\nu(\lambda_n x)\, y_\nu(\lambda_m x)\, dx = 0 \qquad n \neq m$$

e.g., if $\nu = 0$ we may find $\rightarrow$ eigenfunctions may be $J_0(\lambda_n x)$, so then

$$\int_0^1 x\, J_0(\lambda_n x)\, J_0(\lambda_m x)\, dx = 0 \quad n \neq m$$ is the appropriate orthogonality condition

5. The eigenvalue problem for a Bessel eqn :

a. Consider the differential eqn

$$r^2 \frac{d^2y}{dr^2} + r\frac{dy}{dr} + \lambda^2 r^2 y = 0 \qquad (1)$$

subject to the boundary conditions $y(0)$ is finite (i.e., bounded) and $y(l) = 0$.

$\Rightarrow$ We will show that nontrivial ($y(r) \neq 0$) solutions exist only for certain values of λ.

You should recognize this ode as a form of Bessel's eqn of order zero. To see this more clearly, make the change of variables $x = \lambda r$. Then,

$$\frac{dy}{dr} = \frac{dx}{dr}\frac{dy}{dx} = \lambda\frac{dy}{dx} \quad \text{by the chain-rule.}$$

Similarly,

$$\frac{d^2y}{dr^2} = \frac{d}{dr}\left(\lambda\frac{dy}{dx}\right) = \overbrace{\frac{dx}{dr}}^{\lambda}\frac{d}{dx}\left(\lambda\frac{dy}{dx}\right) = \lambda^2\frac{d^2y}{dx^2}$$

Since $r = x/\lambda$, eqn (1) simplifies to

Bessel's eqn of order zero $\Rightarrow$ $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2 y = 0$$

recall Bessel's eqn of order ν: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0.$

The solutions are (assume $\lambda^2 > 0$)

$$y(x) = A\,J_0(x) + B\,Y_0(x)$$

or since $x = \lambda r$, the solution to eqn (1) is

$$\boxed{y(r) = A\,J_0(\lambda r) + B\,Y_0(\lambda r)}$$

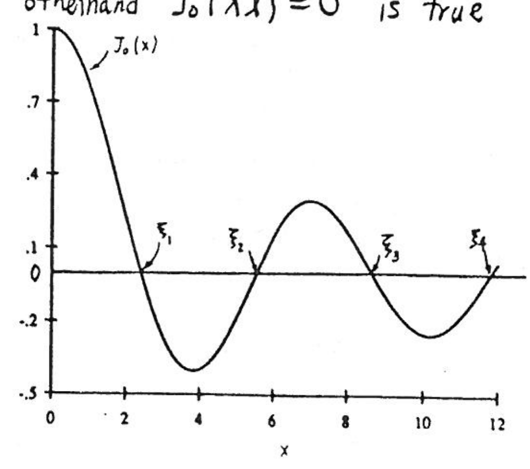
However, recall that $Y_0(x) \to -\infty$ as $x \to 0$ so that in order for the solution to remain bounded as $r \to 0$ we require $B = 0$.

$$\therefore \quad \boxed{y(r) = A\,J_0(\lambda r)}$$

The second b.c. is $\underline{\underline{y(l) = 0 = A\,J_0(\lambda l)}}$

If $A = 0$, then $y(r) \equiv 0$. On the other hand $J_0(\lambda l) = 0$ is true for certain values of λ.

Recall $J_0(x)$:



The first few zeros of $J_0(x)$ are

m	$\xi_m = \lambda_m l$	
1	2.405	these are tabulated
2	5.520	
3	8.654	
4	11.792	
$\vdots$	$\vdots$	

Hence, in this case the eigenvalue condition is

$$J_0(\lambda_n l) = 0$$

$\Rightarrow$ There are an infinite # of real eigenvalues, $\lambda_1, \lambda_2, \dots$

To each eigenvalue, there corresponds the eigenfunction $J_0(\lambda_n x)$ ← a solution to the ode for the specific value of λ_n.

b. Similarly, the problem could have been

$$r^2 \frac{d^2y}{dr^2} + r\frac{dy}{dr} + (\lambda^2 r^2 - \underset{\nu^2}{4})\, y = 0$$

which we recognize as Bessel's eqn of order 2.
The general solution in this case is

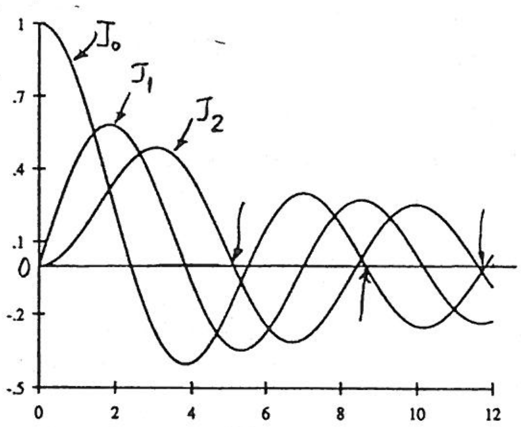
$$y(r) = A J_2(\lambda r) + B Y_2(\lambda r)$$

If we again impose the b.c. that $y(r)$ be finite as $r \to 0$ then we require $B = 0$ (since $Y_2(r) \to -\infty$ as $r \to 0$).

Then the b.c. $y(l) = 0 = A J_2(\lambda l) \Rightarrow \boxed{J_2(\lambda_n l) = 0 \quad \text{EIGENVALUE CONDITION}}$

determines an infinite set of eigenvalues

and then we require the zeros of J_2 which are indicated on the graph below.



to each eigenvalue λ_n there corresponds the eigenfunction $J_2(\lambda_n r)$.

c. Finally, it is straightforward to show that the eigenfunction that arise from solutions to Bessel's eqn satisfy an orthogonality relation with weighting function $w(x) = x$.
So for the problem on the previous page, $\int_0^l x J_0(\lambda_n x) J_0(\lambda_m x)\, dx = 0 \quad n \neq m$

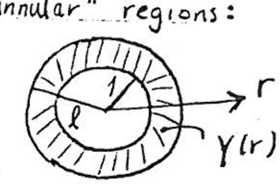
and for the problem on this page $\int_0^l x J_2(\lambda_n x) J_2(\lambda_m x)\, dx = 0 \quad n \neq m.$

d. On the otherhand, there will be instances involving Bessel functions which do not involve the origin. For example,

$$r^2 y'' + r y' + \lambda^2 r^2 y = 0 \qquad 1 \le r \le l \quad , \; y(1) = 0 \; , \; y(l) = 0$$

As before : $\quad y(r) = A\, J_0(\lambda r) + B\, Y_0(\lambda r)$

NOTE: Bessel's eqn often arises in problems involving cylindrical coords, so this problem is typical of "annular" regions:



Until now, we've only mentioned problems which have a boundary condition requiring that the function be bounded at the origin so that $B \equiv 0$ since $Y_0(\lambda r) \to -\infty$ as $r \to 0$.

In this problem, however, the origin is not in the domain of interest and $J_0(\lambda r)$, $Y_0(\lambda r)$ are both bounded and well-behaved for $1 \le r \le 2$. So we proceed as follows.

$$y(1) = 0 \to \quad A\, J_0(\lambda) + B\, Y_0(\lambda) = 0 \quad \to \quad B = -A \frac{J_0(\lambda)}{Y_0(\lambda)}$$

$$\Rightarrow \quad \boxed{y(r) = \frac{A}{Y_0(\lambda)} \left[J_0(\lambda r)\, Y_0(\lambda) - J_0(\lambda)\, Y_0(\lambda r) \right]}$$

Now the EIGENVALUE CONDITION follows from the second b.c. as

$$y(l) = 0 = \frac{A}{Y_0(\lambda)} \left[J_0(\lambda l)\, Y_0(\lambda) - J_0(\lambda)\, Y_0(\lambda l) \right]$$

So that for nontrivial solutions to exist, we see that λ must satisfy

$$J_0(\lambda_n l)\, Y_0(\lambda_n) - J_0(\lambda_n)\, Y_0(\lambda_n l) = 0 \qquad \text{EIGENVALUE CONDITION}$$

It turns out that this form is rather common so the roots (i.e., values of λ) that satisfy this eqn are <u>tabulated</u>.

The corresponding eigenfunction $y_n(r)$ is

$$y_n(r) = J_0(\lambda_n r)\, Y_0(\lambda_n) - J_0(\lambda_n)\, Y_0(\lambda_n r)$$

and satisfies the orthogonality condition typical of Bessel eqns,

$$\int_1^l r\, y_n(r)\, y_m(r)\, dr = 0 \qquad n \ne m.$$