J. STURM-LIOUVILLE THEORY

1. Consider the following ODE:

\[ \frac{d}{dx} \left[ r(x) \frac{dy}{dx} \right] + \left[ q(x) + \lambda w(x) \right] y = 0 \quad \text{for} \quad a \leq x \leq b \]

\[ \text{eigenvalue} \quad \text{weighting function} \]

or

\[ L(y) + \lambda w(x) y = 0 \quad \text{where} \quad L(y) = \frac{d}{dx} \left( r(x) \frac{dy}{dx} \right) + q(x) y \]

Assume the boundary conditions have the form

\[ \alpha_1 y'(a) + \beta_1 y(a) = 0 \]
\[ \alpha_2 y'(b) + \beta_2 y(b) = 0 \]

Remark: This problem represents a generalization of the eigenvalue problem discussed earlier. As we have seen, these types of problems have an infinite number of solutions \( y_n(x) \), one solution for each eigenvalue \( \lambda_n \).

We now wish to demonstrate

\[ \int_a^b w(x) y_n(x) y_m(x) \, dx = 0 \quad \text{for} \quad n \neq m \]

in many applications \( q(x) < 0 \).

(ii) Assume \( r(x), \frac{dr}{dx}, q(x) \) and \( w(x) \) are continuous on \([a, b]\) and \( r(x) > 0, w(x) > 0 \).

2. Proof: orthogonality of the eigenfunctions = same method as used previously.

From (i)

\[ y_n \frac{d}{dx} \left[ r \frac{dy}{dx} \right] + \left[ q + \lambda_n w \right] y_n = 0 \]

and

\[ y_n \frac{d}{dx} \left[ r \frac{dy}{dx} \right] + \left[ q + \lambda_m w \right] y_m = 0 \]

Subtract and integrate \( a \to b \):

\[ \left[ y_n \frac{d}{dx} \left( r \frac{dy}{dx} \right) - y_n \frac{d}{dx} \left( r \frac{dy}{dx} \right) \right] \frac{d}{dx} + (\lambda_n - \lambda_m) \int_a^b w(x) y_n(x) y_m(x) \, dx = 0 \]

Integrating by parts yields

\[ r(x) y_n \frac{dy_n}{dx} \bigg|_a^b - r(x) y_n \frac{dy_n}{dx} \bigg|_a^b + (\lambda_n - \lambda_m) \int_a^b w(x) y_m y_n \, dx = 0 \]

From the b.c.:

\[ y_n'(a) = -\frac{\beta_1}{\alpha_1} y_n(a), \quad y_n'(b) = -\frac{\beta_2}{\alpha_2} y_n(b), \quad y_m'(a) = -\frac{\beta_1}{\alpha_1} y_m(a), \quad y_m'(b) = -\frac{\beta_2}{\alpha_2} y_m(b) \]

so that one can show that \( (4) \equiv 0 \).
2. Orthogonality of eigenfunctions (continued)

We have

\[ (\lambda_n - \lambda_m) \int_a^b w(x) y_m(x) y_n(x) \, dx = 0 \]

so that if \( \lambda_m \neq \lambda_n \)

\[ \int_a^b w(x) y_m(x) y_n(x) \, dx = 0 \]

In words: the eigenfunctions that result from the Sturm-Liouville problem are orthogonal with respect to the weighting function \( w(x) \).

3. Eigenfunction expansion theorem (a generalization of Fourier series)

a. Given the assumptions stated on the previous page, it can be proven that

(i) there is an infinite set of discrete eigenvalues: \( \lambda_1 < \lambda_2 < \lambda_3 < \cdots \)

and (ii) to each eigenvalue \( \lambda_n \) there corresponds only one eigenfunction \( y_n(x) \)

b. Hence, given an arbitrary piecewise smooth function \( f(x) \) on \([a, b]\), \( f(x) \) can be represented via a series expansion in terms of the eigenfunctions \( y_n(x) \),

\[ f(x) = \sum_{n=1}^{\infty} a_n \, y_n(x) \quad \text{where} \quad a_n = \frac{\int_a^b w(x) y_n(x) f(x) \, dx}{\int_a^b w(x) y_n^2(x) \, dx} \]

and at a discontinuity, \( x_0 \),

\[ \sum_{n=1}^{\infty} a_n \, y_n(x_0) = \frac{1}{2} \left( f(x_0^-) + f(x_0^+) \right) \]

which follows directly from the orthogonality property.

4. Examples of 'Sturm-Liouville' eqns

(i) Legendre eqn

\[ \frac{d}{dx} \left[ (1-x^2) \frac{dy}{dx} \right] + n(n+1) y = 0 \]

\( w(x) = 1 \)

\( f(x) \) are finite on \(-1 \leq x \leq 1 \) \( \Rightarrow \) \( \int_{-1}^{1} P_n(x) P_n(x) \, dx = \frac{2}{2n+1} \quad n \in \mathbb{N} \)

(ii) Bessel's eqn

\[ \frac{d}{dx} \left[ x \frac{dy}{dx} \right] + \left( \frac{x^2}{x^2} \right) y = 0 \quad \Rightarrow \quad \text{we'll see that nontrivial solutions exist only for certain values of} \, \lambda_n \)

\( \lambda_n \)

Hence, for the eigenfunctions which result from Bessel's eqn,

\[ \int_0^1 x y_n(\lambda x) y_m(\lambda x) \, dx = 0 \quad n \neq m \]

eg. if \( n = 0 \) we may find \( \rightarrow \) eigenfunction may be \( J_0(\lambda x) \), so then

\[ \int_0^1 x J_0(\lambda x) J_0(\lambda x) \, dx = 0 \quad n \neq m \]

is the appropriate orthogonality condition.
5. The eigenvalue problem for a Bessel eqn:

a. Consider the differential eqn

\[ r^2 \frac{d^2\gamma}{dr^2} + r \frac{d\gamma}{dr} + \lambda^2 r^2 \gamma = 0 \]  \hspace{1cm} (1)

subject to the boundary conditions \( \gamma(0) \) is finite (i.e., bounded) and \( \gamma(\infty) = 0 \).

We will show that nontrivial \( \gamma(0) \neq 0 \) solutions exist only for certain values of \( \lambda \).

You should recognize this ode as a form of Bessel's eqn of order zero. To see this more clearly, make the change of variables \( x = \lambda r \). Then,

\[ \frac{d\gamma}{dr} = \frac{dx}{dr} \frac{d\gamma}{dx} = \lambda \frac{d\gamma}{dx} \]

by the chain-rule.

Similarly,

\[ \frac{d^2\gamma}{dr^2} = \frac{d}{dr} \left( \lambda \frac{d\gamma}{dx} \right) = \frac{dx}{dr} \frac{d}{dx} \left( \lambda \frac{d\gamma}{dx} \right) = \lambda^2 \frac{d^2\gamma}{dx^2} \]

Since \( r = x/\lambda \), eqn (1) simplifies to

\[ x^2 \frac{d^2\gamma}{dx^2} + x \frac{d\gamma}{dx} + x^2 \gamma = 0 \]

Bessel's eqn of order zero

The solutions are (assume \( \lambda^2 > 0 \))

\[ \gamma(x) = A J_0(\lambda x) + B Y_0(\lambda x) \]

or since \( x = \lambda r \),

\[ \gamma(r) = A J_0(\lambda r) + B Y_0(\lambda r) \]

However, recall that \( Y_0(\lambda x) \to -\infty \) as \( x \to 0 \) so that in order for the solution to remain bounded as \( r \to 0 \) we require \( B = 0 \).

So:

\[ \gamma(r) = A J_0(\lambda r) \]

The second b.c. is \( \gamma(\infty) = 0 = A J_0(\lambda \infty) \)

If \( A = 0 \), then \( \gamma(r) \equiv 0 \). On the other hand, \( J_0(\lambda x) = 0 \) is true for certain values of \( \lambda \).

Recall \( J_0(x) \):

The first few zeros of \( J_0(x) \) are

\[
\begin{align*}
\lambda_1 & = 2.405 \\
\lambda_2 & = 5.520 \\
\lambda_3 & = 8.654 \\
\lambda_4 & = 11.792 \\
\vdots & \\
\lambda_m & = \lambda_m x
\end{align*}
\]

\( \{ \) these are tabulated

![Graph of J0(x)](image-url)
Hence, in this case the eigenvalue condition is

\[ J_0(\lambda_n r) = 0 \]

\[ \Rightarrow \text{There are an infinite \# of real eigenvalues, } \lambda_1, \lambda_2, ..., \]

To each eigenvalue, there corresponds the eigenfunction \( J_0(\lambda_n x) \), a solution to the ode for the specific value of \( \lambda_n \).

Similarly, the problem could have been

\[ r^2 \frac{d^2 y}{dr^2} + r \frac{dy}{dr} + (\lambda^2 r^2 - 4) y = 0 \]

which we recognize as Bessel's eqn of order 2. The general solution in this case is

\[ y(r) = A J_2(\lambda r) + B Y_2(\lambda r) \]

If we again impose the b.c. that \( y(r) \) be finite as \( r \to 0 \) then we require \( B = 0 \) (since \( Y_2(\lambda r) \to -\infty \) as \( r \to 0 \)).

Then the b.c. \( y(1) = 0 = A J_2(\lambda_1) \) \( \Rightarrow \]

and then we require the zeros of \( J_2 \) which are indicated on the graph below.

\[ J_0 \]

\[ J_1 \]

\[ J_2 \]

\[ y(r) = A J_2(\lambda r) + B Y_2(\lambda r) \]

\[ \Rightarrow \]

\[ y(1) = 0 = A J_2(\lambda_1) \]

\[ \Rightarrow \]

\[ J_2(\lambda_1) = 0 \]

\[ \Rightarrow \]

\[ \text{determines an infinite set of eigenvalues} \]

\[ \downarrow \]

\[ \text{to each eigenvalue } \lambda_n \text{ there corresponds the eigenfunction } \]

\[ J_2(\lambda_n r), \]

Finally, it is straightforward to show that the eigenfunctions that arise from solutions to Bessel's eqn satisfy an orthogonality relation with weighting function \( w(x) = x \).

So for the problem on the previous page,

\[ \int_0^\infty J_0(\lambda_n x) J_0(\lambda_m x) \, dx = 0 \quad n \neq m \]

and for the problem on this page

\[ \int_0^\infty J_2(\lambda_n x) J_2(\lambda_m x) \, dx = 0 \quad n \neq m, \]
d. On the other hand, there will be instances involving Bessel functions which do not involve the origin. For example,

$$r^2 \gamma'' + r \gamma' + \lambda^2 r^2 \gamma = 0 \quad 1 \leq r \leq R, \quad \gamma(R) = 0, \quad \gamma(1) = 0$$

As before:

$$\gamma(r) = A J_0(\lambda r) + B Y_0(\lambda r)$$

Until now, we’ve only mentioned problems which have a boundary condition requiring that the function be bounded at the origin so that \(B = 0\) since \(Y_0(\lambda r) \to -\infty\) as \(r \to 0\).

In this problem, however, the origin is not in the domain of interest and \(J_0(\lambda r)\) and \(Y_0(\lambda)\) are both bounded and well-behaved for \(1 \leq r \leq 2\). So we proceed as follows.

$$\gamma(1) = 0 \Rightarrow A J_0(\lambda) + B Y_0(\lambda) = 0 \Rightarrow B = -A \frac{J_0(\lambda)}{Y_0(\lambda)}$$

$$\Rightarrow \gamma(r) = \frac{A}{Y_0(\lambda)} \left[ J_0(\lambda r) Y_0(\lambda) - J_0(\lambda) Y_0(\lambda r) \right]$$

Now the EIGENVALUE CONDITION follows from the second b.c., as

$$\gamma(1) = 0 = \frac{A}{Y_0(\lambda)} \left[ J_0(\lambda 1) Y_0(\lambda) - J_0(\lambda) Y_0(\lambda 1) \right]$$

So that for nontrivial solutions to exist, we see that \(\lambda\) must satisfy

$$J_0(\lambda_n 1) Y_0(\lambda_n) - J_0(\lambda_n) Y_0(\lambda_n 1) = 0$$

EIGENVALUE CONDITION

It turns out that this form is rather common so the roots (i.e., values of \(\lambda\)) that satisfy this eqn are tabulated.

The corresponding eigenfunction is

$$\gamma_n(r) = J_0(\lambda_n r) Y_0(\lambda_n) - J_0(\lambda_n) Y_0(\lambda_n r)$$

and satisfies the orthogonality condition typical of Bessel eqns,

$$\int_1^R \gamma_n(r) \gamma_m(r) \, dr = 0 \quad n \neq m.$$