Problem Set 2

18.355

The purpose of this homework assignment is to review some basic manipulations of vector calculus that will be used throughout much of this course. It is (strongly) suggested that index notation be used to prove the vector identities shown below.

1. Evaluate the following expresions :

   (i) $\delta_{ij}\delta_{ij}$ (ii) $\epsilon_{ijk}\epsilon_{kji}$ (iii) $\epsilon_{ijk}a_i a_k$ (iv) $\epsilon_{ijk}\frac{\partial^2\phi}{\partial x_i \partial x_j}$

2. The meaning of $\partial A/\partial x$ is simply the rate-of-change of the scalar quantity A with respect to the direction x at a point in space. Briefly discuss the physical significance of ∇A? If a denotes a vector, what about $\nabla \mathbf{a}$?

3. Prove : $(\mathbf{a}\wedge\mathbf{b})\cdot(\mathbf{a}\wedge\mathbf{b}) = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2$

4. Prove the following vector identities: $\mathbf{a},\mathbf{b}$ = vectors, ϕ = scalar function

   (i) $\nabla\cdot(\phi\mathbf{a}) = \phi\nabla\cdot\mathbf{a} + \mathbf{a}\cdot\nabla\phi$
   (ii) $\nabla\wedge(\phi\mathbf{a}) = \phi\nabla\wedge\mathbf{a} + (\nabla\phi)\wedge\mathbf{a}$
   (iii) $\nabla\cdot(\nabla\wedge\mathbf{a}) = 0$ $\Leftarrow$ true for any vector
   (iv) $\nabla\wedge(\nabla\phi) = 0$ $\Leftarrow$ true for any scalar
   (v) $\nabla\wedge(\mathbf{a}\wedge\mathbf{b}) = (\mathbf{b}\cdot\nabla)\,\mathbf{a} - \mathbf{b}\,(\nabla\cdot\mathbf{a}) - (\mathbf{a}\cdot\nabla)\,\mathbf{b} + \mathbf{a}\,(\nabla\cdot\mathbf{b})$
   (vi) $\nabla\cdot(\mathbf{a}\wedge\mathbf{b}) = (\nabla\wedge\mathbf{a})\cdot\mathbf{b} - \mathbf{a}\cdot(\nabla\wedge\mathbf{b})$
   (vii) $\nabla\cdot(\mathbf{a}\mathbf{b}) = (\nabla\cdot\mathbf{a})\,\mathbf{b} + \mathbf{a}\cdot(\nabla\mathbf{b})$
   (viii) $\nabla\wedge(\nabla\wedge\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$

   Furthermore, if C is a second order tensor, prove

   (ix) $\mathbf{a}\cdot\mathbf{C} = \mathbf{C}\cdot\mathbf{a}$ if and only if $\mathbf{C} = \mathbf{C}^T$, i.e., C is symmetric.
   (x) If $\mathbf{C} = -\mathbf{C}^T$ (C is anti-symmetric), then $\mathbf{a}\cdot\mathbf{C}\cdot\mathbf{a} = 0$.

5. Let x represent the usual position vector. Evaluate

   (i) $\nabla\cdot\mathbf{x}$ (ii) $\nabla\wedge\mathbf{x}$ (iii) $\nabla^2\mathbf{x}$
   (iv) Let $r^2 = x_j x_j$; differentiate both sides with respect to x_i and show that $\partial r/\partial x_i = x_i/r$ (this is a very useful formula).
   (v) $\nabla^2 r$

6. Prove :

   (i) $\int_S \mathbf{n}\, dS = 0$ where n denotes the normal to the surface S.
   (ii) $\int_S (\mathbf{x}\wedge\mathbf{a})\wedge\mathbf{n}\, dS = 2V\mathbf{a}$ where x denotes the position vector to a point on the surface S, a = constant vector and V = volume bounded by S.
   (iii) $\int_C \phi\nabla\phi\cdot d\mathbf{l} = 0$ where C denotes a closed curve.

7. The Reynolds Transport Theorem (RTT) states that

$$\frac{d}{dt}\int_{V(t)} f(\mathbf{x},t)\ dV = \int_{V(t)} [\frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{u}f)]\ dV$$

where $V(t)$ indicates a material control volume, *i.e.*, a volume element that moves with the local fluid velocity $\mathbf{u}(\mathbf{x},t)$. If the density ρ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

prove the special form of the RTT:

$$\frac{d}{dt}\int_{V(t)} \rho\ g(\mathbf{x},t)\ dV = \int_{V(t)} \rho \frac{Dg}{Dt}\ dV,$$

where D/Dt represents the material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla.$$