18.355 Fall
Handout #3
Reynold's Transport Thm

We interpret dx/dt as the time rate of change of the x-coordinate position of our observer, i.e., dx/dt is the x-component of the velocity, $\mathbf{w}$, of our observer. Thus

$$w_x = \frac{dx}{dt},$$

$$w_y = \frac{dy}{dt},$$

and

$$w_z = \frac{dz}{dt},$$

and Eq. 4.1-4 becomes

$$\frac{dS}{dt} = \left(\frac{\partial S}{\partial t}\right) + w_x\left(\frac{\partial S}{\partial x}\right) + w_y\left(\frac{\partial S}{\partial y}\right) + w_z\left(\frac{\partial S}{\partial z}\right). \tag{4.1-5}$$

In vector notation this becomes,

$$\frac{dS}{dt} = \left(\frac{\partial S}{\partial t}\right) + \mathbf{w}\cdot\nabla S, \tag{4.1-6}$$

and in index notation we express this result as

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + w_i\left(\frac{\partial S}{\partial x_i}\right). \tag{4.1-7}$$

Here the repeated indices are summed from 1 to 3 in accordance with the summation convention [2]. If our observer moves *with the fluid*, i.e., $\mathbf{w} = \mathbf{v}$ the time derivative is called the *material derivative* and is denoted by

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + \mathbf{v}\cdot\nabla S. \tag{4.1-8}$$

If our observer fixes himself in space, $\mathbf{w} = 0$, and the *total time derivative* is simply equal to the partial time derivative

$$\frac{dS}{dt} = \frac{\partial S}{\partial t}, \quad \text{for } \mathbf{w} = 0 \tag{4.1-9}$$

Now we wish to consider the total time derivative of the volume integral of S over the region $\mathscr{V}_a(t)$. Here $\mathscr{V}_a(t)$ represents an *arbitrary* (hence the subscript a) volume moving through space in some specified manner. The time derivative we seek is given by

$$\frac{d}{dt}\int_{\mathscr{V}_a(t)} S\,dV = \lim_{\Delta t\to 0}\left\{\frac{\int_{\mathscr{V}_a(t+\Delta t)} S(t+\Delta t)\,dV - \int_{\mathscr{V}_a(t)} S(t)\,dV}{\Delta t}\right\}. \tag{4.1-10}$$

DERIVATION OF REYNOLDS TRANSPORT THEOREM

To visualize the process under consideration, we must think of a volume, such as a sphere, moving through space so that the velocity of each point on the surface of the volume is given by $\mathbf{w}$. The velocity $\mathbf{w}$ may be a function of the spatial coordinates (if the volume is deforming) and time (if the volume is accelerating or decelerating). At every instant of time some quantity, denoted by S, is measured throughout the region occupied by the volume $\mathscr{V}_a(t)$. The volume integral can then be evaluated at each point in time and the time derivative obtained by Eq. 4.1-10.

In Fig. 4.1.1 we have shown a volume at the times t and $t + \Delta t$ as it moves and deforms in space. During the time interval Δt the volume sweeps out a "new" region designated by $V_{II}(\Delta t)$ and leaves behind an "old" region designated by $V_I(\Delta t)$. Clearly we can express the volume $\mathscr{V}_a(t+\Delta t)$ as

$$\mathscr{V}_a(t+\Delta t) = \mathscr{V}_a(t) + V_{II}(\Delta t) - V_I(\Delta t), \tag{4.1-11}$$

reference: S. Whitaker, Elementary Heat Transfer Analysis, Pergamon Press 1976.

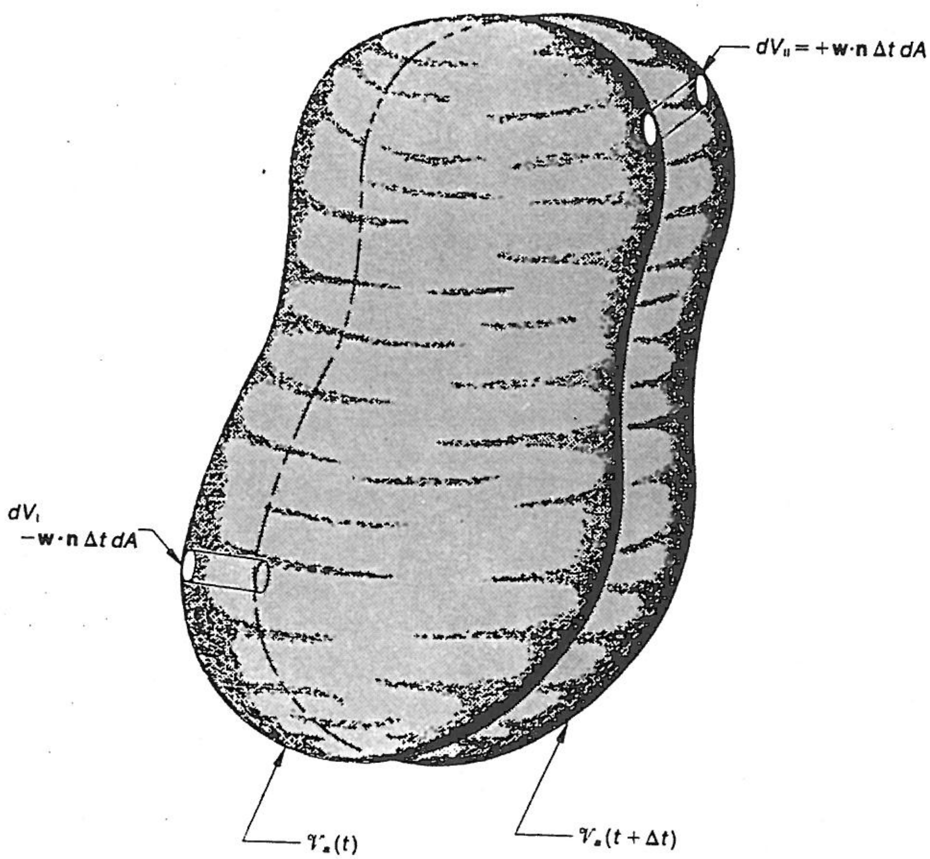

Fig. 4.1.1 A moving volume $\mathcal{V}_a(t)$.

so that the integral of $S(t+\Delta t)$ in Eq. 4.1-10 can be put in the form

$$\int_{\mathcal{V}_a(t+\Delta t)} S(t+\Delta t)\,dV = \int_{\mathcal{V}_a(t)} S(t+\Delta t)\,dV + \int_{V_{II}(\Delta t)} S(t+\Delta t)\,dV_{II} - \int_{V_I(\Delta t)} S(t+\Delta t)\,dV_I. \tag{4.1-12}$$

Substitution of Eq. 4.1-12 into Eq. 4.1-10 leads to

$$\frac{d}{dt}\int_{\mathcal{V}_a(t)} S\,dV = \lim_{\Delta t\to 0}\left\{\frac{\int_{\mathcal{V}_a(t)} S(t+\Delta t)\,dV - \int_{\mathcal{V}_a(t)} S(t)\,dV}{\Delta t}\right\} + \lim_{\Delta t\to 0}\left\{\frac{\int_{V_{II}(\Delta t)} S(t+\Delta t)\,dV_{II} - \int_{V_I(\Delta t)} S(t+\Delta t)\,dV_I}{\Delta t}\right\} \tag{4.1-13}$$

In treating the first term on the right-hand-side of Eq. 4.1-13 we note that limits of integration are the same so that the two terms can be combined to give

$$\lim_{\Delta t\to 0}\left\{\frac{\int_{\mathcal{V}_a(t)} S(t+\Delta t)\,dV - \int_{\mathcal{V}_a(t)} S(t)\,dV}{\Delta t}\right\} = \lim_{\Delta t\to 0}\left\{\frac{1}{\Delta t}\int_{\mathcal{V}_a(t)} [S(t+\Delta t) - S(t)]\,dV\right\}. \tag{4.1-14}$$

Since the limits of integration are independent of Δt the limit can be taken inside the integral sign so that Eq. 4.1-14 takes the form

$$\lim_{\Delta t\to 0}\left\{\frac{\int_{\mathcal{V}_a(t)} S(t+\Delta t)\,dV - \int_{V_a(t)} S(t)\,dV}{\Delta t}\right\} = \int_{\mathcal{V}_a(t)} \lim_{\Delta t\to 0}\left[\frac{S(t+\Delta t) - S(t)}{\Delta t}\right]. \tag{4.1-15}$$

Here we must recognize that $S(t+\Delta t)$ and $S(t)$ are evaluated at the *same point in space* so that the integrand on the right-hand-side of Eq. 4.1-15 is the partial derivative and Eq. 4.1-15 takes the form

$$\lim_{\Delta\to 0}\left\{\frac{\int_{\mathcal{V}_a(t)} S(t+\Delta t)\,dV - \int_{\mathcal{V}_a(t)} S(t)\,dV}{\Delta t}\right\} = \int_{\mathcal{V}_a(t)} \frac{\partial S}{\partial t}\,dV. \tag{4.1-16}$$

We can now return to Eq. 4.1-13 and express the time rate of change of the volume integral as

$$\frac{d}{dt}\int_{\mathcal{V}_a(t)} S\,dV = \int_{\mathcal{V}_a(t)} \left(\frac{\partial S}{\partial t}\right) dV + \lim_{\Delta t\to 0}\left\{\frac{\int_{V_{II}(\Delta t)} S(t+\Delta t)\,dV_{II} - \int_{V_I(\Delta t)} S(t+\Delta t)\,dV_I}{\Delta t}\right\}. \tag{4.1-17}$$

From Fig. 4.1.1 we note that the differential volume elements of the "new" and "old" regions can be expressed as†

$$dV_{II} = +\,\mathbf{w}\cdot\mathbf{n}\,\Delta t\,dA_{II}, \tag{4.1-18}$$

and

$$dV_I = -\,\mathbf{w}\cdot\mathbf{n}\,\Delta t\,dA_I. \tag{4.1-19}$$

Use of Eqs. 4.1-18 and 4.1-19 allows us to express the volume integrals as area integrals, thus leading to

$$\frac{d}{dt}\int_{\mathcal{V}_a(t)} S\,dV = \int_{\mathcal{V}_a(t)} \left(\frac{\partial S}{\partial t}\right) dV + \lim_{\Delta t\to 0}\left\{\frac{\int_{A_{II}} S(t+\Delta t)\,\mathbf{w}\cdot\mathbf{n}\,\Delta t\,dA_{II} + \int_{A_I} S(t+\Delta t)\,\mathbf{w}\cdot\mathbf{n}\,\Delta t\,dA_I}{\Delta t}\right\}. \tag{4.1-20}$$

On the right-hand-side of Eq. 4.1-20 we can cancel Δt in the numerator and denominator and note that

$$A_{II} + A_I \to \mathcal{A}_a(t) \quad \text{as} \quad \Delta t \to 0,$$

so that Eq. 4.1-20 takes the form

$$\frac{d}{dt}\int_{\mathcal{V}_a(t)} S\,dV = \int_{\mathcal{V}_a(t)} \left(\frac{\partial S}{\partial t}\right) dV + \int_{\mathcal{A}_a(t)} S\,\mathbf{w}\cdot\mathbf{n}\,dA. \tag{4.1-21}$$

This is known as the *general transport theorem*. A more rigorous derivation is given by Slattery [3]. If we let our arbitrary volume $\mathcal{V}_a(t)$ move *with the fluid*, the velocity $\mathbf{w}$ is equal to the fluid velocity $\mathbf{v}$, the volume $\mathcal{V}_a(t)$ becomes a *material volume* designated by $\mathcal{V}_m(t)$, and the total derivative becomes the material derivative. Under these circumstances Eq. 4.1-21 takes the form

$$\frac{D}{Dt}\int_{\mathcal{V}_m(t)} S\,dV = \int_{\mathcal{V}_m(t)} \left(\frac{\partial S}{\partial t}\right) dV + \int_{\mathcal{A}_m(t)} S\,\mathbf{v}\cdot\mathbf{n}\,dA, \tag{4.1-22}$$

and is called the *Reynolds transport theorem.*

Conservation of mass

The principle of conservation of mass can be stated as,

$$\{\text{the mass of a body}\} = \text{constant}, \tag{4.1-23}$$

or in the rate form

$$\{\text{time rate of change of the mass of a body}\} = 0. \tag{4.1-24}$$

Using the language of calculus we express Eq. 4.1-24 as

$$\frac{D}{Dt}\int_{\mathcal{V}_m(t)} \rho\,dV = 0. \tag{4.1-25}$$

†See Reference 2, Sec. 3.4 for a detailed discussion of this point.