18.355
Handout #2
Derivation of Governing Equations

1.1 Lagrangian vs. Eulerian points of view

In fluid mechanics we describe the motion of liquids and gases (such as water and air) using the approach of continuum mechanics, wherein the fluid is characterized by properties that are aggregates over a large number of individual molecules. When we talk about a 'fluid particle', we mean an infinitesimally small region of fluid when discussing mathematical formulations (when taking limits for derivatives, for example) but we understand that the region is nevertheless large in comparison with the mean spacing between molecules. Each fluid particle has associated with it various physical properties, such as temperature and density, and is assumed to have a well defined position and velocity.

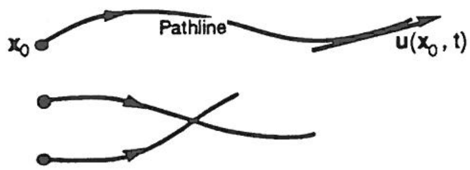
There are two different mathematical representations of fluid flow: the **Lagrangian** picture in which we keep track of the locations of individual fluid particles; and the **Eulerian** picture in which coordinates are fixed in space (the laboratory frame).

The Lagrangian picture is not often used for theoretical developments but can provide a useful picture of fluid flow in experiments. For example, in oceanography, buoys and patches of dye are deposited on the sea surface and their positions are noted as they vary in time. The density ρ and velocity $\mathbf{u}$ are described mathematically by

Velocity $$\mathbf{u} = \mathbf{u}(\mathbf{x}_0, t),$$

Density $$\rho = \rho(\mathbf{x}_0, t),$$

i.e., the field values are those of a fluid particle at some time t after the particle was 'released' at the initial position $\mathbf{x}_0$.



The loci of fluid particles are called 'pathlines' and it is clear that these lines may cross, since two different fluid particles may occupy the same position in space at different times.

Since these coordinates describe the motion of individual particles, the acceleration of a particle is given simply by

Acceleration $$\mathbf{a} = \frac{\partial \mathbf{u}}{\partial t}.$$

If the fluid is incompressible then the density of each fluid particle remains constant in time, which is expressed mathematically as

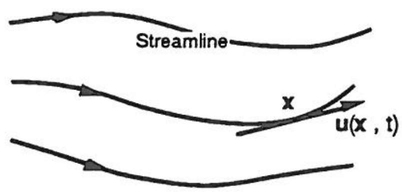
Incompressibility $$\frac{\partial \rho}{\partial t} = 0.$$

In the Eulerian picture, the velocity and density are given by

Velocity $$\mathbf{u} = \mathbf{u}(\mathbf{x}, t),$$

Density $$\rho = \rho(\mathbf{x}, t),$$

where $\mathbf{x}$ is a fixed location in the laboratory frame, and thus $\mathbf{u}$ and ρ are the velocity and density of the fluid particle that is instantaneously at position $\mathbf{x}$ at time t.



The velocity vectors form a vector field that is assumed to be differentiable and hence there are 'streamlines' that are everywhere parallel to the local velocity vector. Streamlines can never cross except at point sources or sinks of fluid.

In order to compute the acceleration of a fluid particle with these coordinates, we must realise that after a small time δt the particle is at the new position $\mathbf{x} + \delta\mathbf{x}$ with velocity

$$\mathbf{u}(\mathbf{x} + \delta\mathbf{x}, t + \delta t) = \mathbf{u}(\mathbf{x}, t) + (\delta\mathbf{x} \cdot \nabla)\mathbf{u} + \delta t \frac{\partial \mathbf{u}}{\partial t} + O(\delta\mathbf{x}^2, \delta t^2).$$

Thus the acceleration of the fluid particle is

Acceleration
$$\lim_{\delta t \to 0} \frac{\mathbf{u}(\mathbf{x} + \delta\mathbf{x}, t + \delta t) - \mathbf{u}(\mathbf{x}, t)}{\delta t} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \equiv \frac{D\mathbf{u}}{Dt}.$$

The operator $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ is called the 'material derivative' or 'substantial derivati- vative'. It is the rate of change with time following a fluid particle.

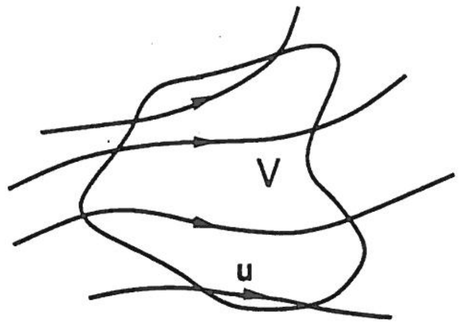
In the Eulerian picture, incompressibility is expressed by

Incompressibility
$$\frac{D\rho}{Dt} = 0,$$

since it is the density of a fluid particle that remains constant, not the density of the fluid at a fixed position in space.

1.2 Conservation of mass

Consider an arbitrary fixed control volume V in the laboratory frame



The rate of change of the mass of fluid contained within V is equal to the mass inflow through the boundary ∂V of V. Thus

$$\frac{d}{dt}\int_V \rho\, dV = -\int_{\partial V} \rho\mathbf{u}\cdot\mathbf{n}\, dS,$$

where $\mathbf{n}$ is the outward normal to ∂V. Applying the divergence theorem to this equation, we obtain

$$\int_V \frac{\partial \rho}{\partial t}\, dV = -\int_V \nabla\cdot(\rho\mathbf{u})\, dV.$$

Since these integrals are equal for arbitrary control volumes, it can be deduced that the integrands must also be equal. Thus the differential equation expressing conservation of mass is

Mass conservaton $$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0.$$

This equation is readily rearranged into the form

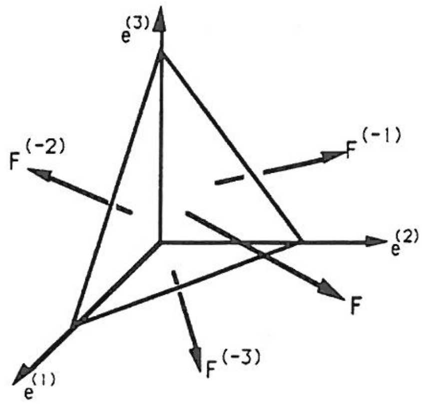
$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0,$$

from which we see that if the fluid is incompressible

Incompressibility $$\nabla\cdot\mathbf{u} = 0.$$

1.3 The Stress Tensor

Consider a small tetrahedron of fluid aligned with local, rectangular coordinate axes $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, $\mathbf{e}^{(3)}$. The forces exerted by the fluid exterior to the tetrahedron on the surfaces of the tetrahedron are $\mathbf{F}^{(-i)}$ on the three surfaces having outward normals in the three negative coordinate directions $-\mathbf{e}_i$ and $\mathbf{F}$ on the sloping face of the tetrahedron, which has outward normal $\mathbf{n}$.



The magnitude of the surface forces, which are due to molecular jostling and to short-range van der Waals forces, are proportional to the surface area of the tetrahedron, which is of order $V^{2/3}$, where V is the volume of the tetrahedron, whereas the inertial forces (mass $\times$ acceleration) and long-range body forces, such as gravity, are proportional to V. Thus the surface forces must balance by themselves in the limit as $V \to 0$ and we obtain

$$\begin{aligned} \mathbf{F} &= -\sum_k \mathbf{F}^{(-k)} \\ &= \sum_k \mathbf{F}^{(k)} \qquad \text{(by Newton's 3rd law)} \\ \Rightarrow A\tau &= \sum_k A^{(k)}\tau^{(k)} \end{aligned}$$

where τ is the **stress**, which is the force per unit area acting on a surface, and $A^{(k)}$ is the area of the k^{th} surface of the tetrahedron. From projective geometry, we have that

$A^{(k)} = A\mathbf{n} \cdot \mathbf{e}^{(k)}$. Thus the stress can be written as

$$\tau = \left(\sum_k \tau^{(k)} \mathbf{e}^{(k)} \right) \cdot \mathbf{n}$$
$$= \sigma \cdot \mathbf{n}$$

where $\sigma = \sum_k \tau^{(k)} \mathbf{e}^{(k)}$ is the **stress tensor**, which is independent of the direction $\mathbf{n}$. The components of the stress tensor are given by

$$\sigma_{ij} = \sum_k \tau_i^{(k)} \mathbf{e}_j^{(k)}.$$

But $\mathbf{e}_j^{(k)} = \delta_{jk}$, so

$$\sigma_{ij} = \tau_i^{(j)},$$

which is the i th component of the force per unit area exerted by the fluid on a surface with normal in the j th coordinate direction.

The most important statement relating to the stress tensor is that the force per unit area (stress) exerted by the fluid on a surface with unit normal $\mathbf{n}$ pointing into the fluid is given by

Stress $$\tau = \sigma \cdot \mathbf{n}$$

1.4 The momentum equation

Consider the arbitrary fixed control volume of section 1.2. The rate of change of the total momentum within the control volume is effected by the inflow of momentum through the boundary, and the forces acting on the fluid, which comprise both body forces (total per unit volume) and surface forces. Thus

$$
\begin{aligned}
\frac{d}{dt}\int_V \rho\mathbf{u}\,dV = &-\int_{\partial V}(\rho\mathbf{u})\mathbf{u}\cdot\mathbf{n}\,dS && \text{momentum flux} \\
&+\int_V \mathbf{f}\,dV && \text{body forces} \\
&+\int_{\partial V}\sigma\cdot\mathbf{n}\,dS && \text{surface forces}
\end{aligned}
$$

Use of the divergence theorem gives

$$\int_V \frac{\partial}{\partial t}(\rho u_i)\,dV = -\int_V \frac{\partial}{\partial x_j}(\rho u_i u_j)\,dV + \int_V f_i\,dV + \int_V \frac{\partial}{\partial x_j}(\sigma_{ij})\,dV$$

Again, since this expression holds for arbitrary control volumes, the integrands must equate to give

$$\rho\frac{D\mathbf{u}}{Dt} + \mathbf{u}\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right) = \mathbf{f} + \nabla\cdot\sigma.$$

The second term is zero by conservation of mass, so

The momentum equation $$\rho\frac{D\mathbf{u}}{Dt} = \mathbf{f} + \nabla\cdot\sigma.$$

1.5 Stress tensor for a Newtonian fluid

In this course, we shall be concerned solely with **Newtonian** fluids, which are those that are assumed to have two fundamental properties: the fluid should be isotropic; and there should be a linear relationship between stress and the rate of strain of the fluid. In addition, we require that the long-range forces exert no couple on individual molecules (a counter example to this last requirement is provided by magnetic fluids – see homework). With this latter condition, we can show that the stress tensor is symmetric as follows.

The rate of change of the angular momentum of a fluid particle is equal to the moment of the forces acting on the particle. Thus

$$\frac{d}{dt}\int_V \mathbf{x}\wedge(\rho\mathbf{u})\,dV = \int_V \mathbf{x}\wedge\mathbf{f}\,dV + \int_{\partial V}\mathbf{x}\wedge(\sigma\cdot\mathbf{n})\,dS$$

The term on the left-hand side and the first term on the right-hand side are each of order $V^{4/3}$ as $V\to 0$, while the last term, representing the couple exerted by the surface forces is of order V. Thus the surface moments dominate the equation and must tend to zero as $V\to 0$. We can apply the divergence theorem to this equation to give

$$\begin{aligned}
0 &= \int_{\partial V}\mathbf{x}\wedge(\sigma\cdot\mathbf{n})\,dS \\
&= \int_V \frac{\partial}{\partial x_m}(\epsilon_{ijk}x_j\sigma_{km})\,dV \\
&= \int_V \epsilon_{ijk}\delta_{jm}\sigma_{km}\,dV + \int_V \epsilon_{ijk}x_j\frac{\partial}{\partial x_m}\sigma_{km}\,dV \\
&= \int_V \epsilon_{ijk}\sigma_{kj}\,dV + \int_V \epsilon_{ijk}x_j\frac{\partial}{\partial x_m}\sigma_{km}\,dV
\end{aligned}$$

Now, provided that the stress tensor is differentiable so that $\nabla\cdot\sigma$ is finite, the second term in this last equation is of order $V^{4/3}$ while the first term is of order V as $V\to 0$. So the first term dominates the equation and shows that

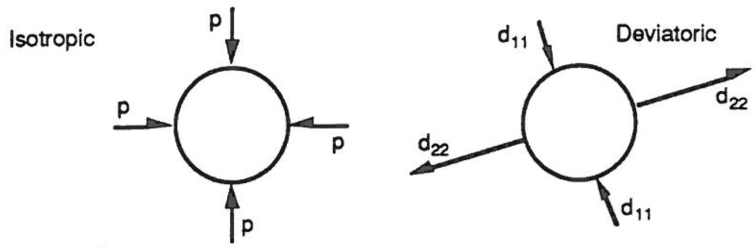
$$\epsilon_{ijk}\sigma_{kj} = 0$$

i.e., that the stress tensor is symmetric ($\sigma_{ij}=\sigma_{ji}$).

Next, we note that we can always write

$$\sigma_{ij} = -p\delta_{ij} + d_{ij} \qquad \text{with } d_{ii}=0$$

thus splitting σ into an isotropic part and a non-isotropic part called the deviatoric stress tensor.



The isotropic part of the stress tensor gives a force that pushes equally in all directions and so we interpret the constant p as a pressure. The deviatoric stress arises from deviations of the flow local to a fluid particle and we assume therefore that $\mathbf{d}$ is a function of the velocity gradient $\nabla \mathbf{u}$ with $\mathbf{d} = 0$ when $\nabla \mathbf{u} = 0$.

Here is where we assume that Newtonian fluids are **linear**, by which we mean that $\mathbf{d}$ is a linear function of $\nabla \mathbf{u}$, so that

$$d_{ij} = A_{ijkl} \frac{\partial u_k}{\partial x_l}.$$

Finally, we assume that the fluid is isotropic so that $\mathbf{A}$ is isotropic and hence is given by

$$A_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \nu \delta_{il} \delta_{kj},$$

where λ, μ, ν are constants, this being the most general isotropic fourth-rank tensor.

From symmetry, we deduce that

$$d_{ij} = d_{ji} \Rightarrow A_{ijkl} = A_{jikl} \Rightarrow \mu = \nu,$$

while the fact that $\mathbf{d}$ is traceless gives

$$d_{ii} = 0 \Rightarrow A_{iikl} = 0 \Rightarrow 3\lambda + \mu + \nu = 0,$$

whence $\lambda = -\frac{2}{3}\mu$. Hence

$$d_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \tfrac{2}{3}\mu \frac{\partial u_k}{\partial x_k} \delta_{ij}$$

or

$$\mathbf{d} = 2\mu\mathbf{e} - \tfrac{2}{3}\mu(\nabla\cdot\mathbf{u})\mathbf{I},$$

where **e** is the symmetric part of the velocity gradient $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. If the fluid is incompressible, so that $\nabla\cdot\mathbf{u} = 0$ then

$$\sigma = -p\mathbf{I} + 2\mu\mathbf{e}.$$

The constant μ is called the **dynamic viscosity** of the fluid.

Putting this stress tensor in the general momentum equation yield the Navier-Stokes equations

Navier Stokes

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla\mathbf{p} + \mu\nabla^2\mathbf{u}$$
$$\nabla\cdot\mathbf{u} = 0.$$