18.355
Mathematical Preliminaries
Handout #1 (Thx to H.A. Stone!)
64

II. CALCULUS OF VECTORS, DYADICS AND TENSORS

A. *Introduction & Review*

1. scalars & vectors

scalar : magnitude only e.g., mass, temperature

vector : characterized by magnitude & direction ; represented geometrically as an arrow ↗

→ 2 vectors are equal if they have the same magnitude & direction ; "parallel transport of vectors"

$\Rightarrow A = B$

(Nevertheless, it is important to keep in mind that the effect of a given vector may depend upon its location)

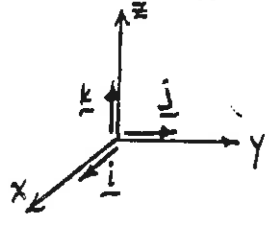
NOTATION : I will typically indicate a vector quantity by an underline, e.g. $\underset{\sim}{a}$ or $\underset{\sim}{b}$.

Another common method is to use arrows, $\vec{a}$, $\vec{b}$.

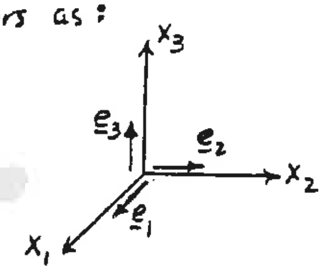
2. *Cartesian Coordinate system*

unit vector has length = 1

a. We will indicate the unit base vectors as :



or



$\underset{\sim}{e}_3 = (0,0,1)$
$\underset{\sim}{e}_2 = (0,1,0)$
$\underset{\sim}{e}_1 = (1,0,0)$

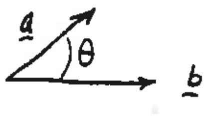
b. In order to describe a vector you must give both the components and the base vectors

e.g., $\underset{\sim}{a} = a_x \, \underline{i} + a_y \, \underline{j} + a_z \, \underset{\sim}{k}$

c. Recall the definition of the SCALAR PRODUCT of 2 vectors:

also called DOT or INNER PRODUCT

(i) $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underline{a}| \, |\underline{b}| \cos\theta$



where $|\underline{a}|$, $|\underline{b}|$ are the magnitudes of $\underline{a}$ and $\underset{\sim}{b}$

NOTE : If $\underline{a} \cdot \underline{b} = 0$ and $|\underline{a}| \neq 0$, $|\underline{b}| \neq 0$, then $\underline{a} \perp \underline{b}$.

Also, since $\underline{i} \cdot \underline{i} = 1$, $\underline{i} \cdot \underline{j} = 0$ $\underline{i} \cdot \underline{k} = 0$, etc., then

$\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$

c. scalar product (continued)

(ii) Clearly, we also have $\underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a}$ and $\underline{a}\cdot(\underline{b}+\underline{c}) = \underline{a}\cdot\underline{b} + \underline{a}\cdot\underline{c}$ and

$$|\underline{a}|^2 = \underline{a}\cdot\underline{a} = a^2$$

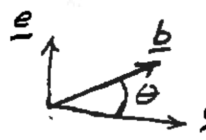
d. VECTOR PRODUCT (also called CROSS PRODUCT)

(i) The vector product of 2 vectors $\underline{a}, \underline{b}$ is defined as

NOTICE: My notation for this operation is ^; many others write X.

$$\underline{a} \wedge \underline{b} = |\underline{a}|\,|\underline{b}| \sin\theta \ \underline{e}$$

NOTE $\underline{a}\wedge\underline{a} = 0$.



where $\underline{e}$ is a unit vector in the direction perpendicular to the plane formed by $\underline{a}$ & $\underline{b}$, as given by the RIGHT-HAND RULE.

(ii) From the definition: $\underline{a}\wedge\underline{b} = -\underline{b}\wedge\underline{a}$ and $\underline{a}\wedge(\underline{b}+\underline{c}) = \underline{a}\wedge\underline{b} + \underline{a}\wedge\underline{c}$

It also follows that $\underline{i}\wedge\underline{j} = \underline{k}$, $\underline{i}\wedge\underline{k} = -\underline{j}$ $\underline{j}\wedge\underline{k} = \underline{i}$ $\underline{i}\wedge\underline{i} = 0$ etc.

(iii) You may also remember writing something like

$$\underline{a}\wedge\underline{b} = \det\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \underline{i}\,(a_y b_z - a_z b_y) + \underline{j}\,(a_z b_x - a_x b_z) + \underline{k}\,(a_x b_y - a_y b_x)$$

⇒ Much of the above is cumbersome & frightfully lengthy to write.

We now introduce a special notation which will simplify many manipulations.

B. EINSTEIN INDEX NOTATION AND THE SUMMATION CONVENTION

1. Let us reconsider some of the above. From now on keep in mind that we are representing vectors in a three-dimensional world.

So, we will now label (x, y, z) coordinates by $(1, 2, 3)$

Let the vector $\underline{a}$ have components a_i, base vectors $\underline{e}_i$.

Then,

$$\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_2 \underline{e}_3 = \sum_{i=1}^{3} a_i \underline{e}_i \equiv a_i \underline{e}_i \quad (= a_j \underline{e}_j)$$

↑ dummy index

This idea must be clear in your mind before you move on.

⇒ From now on, we will **not** write the summation symbol. Instead we will invoke the SUMMATION CONVENTION — if an index appears twice, we will know that we should do a summation $i = 1, 2, 3$.

IMPORTANT: Use a different index for each vector.

2. scalar product revisited

Consider two vectors $\underline{a} = a_i \underline{e}_i \qquad \underline{b} = b_j \underline{e}_j$

Then, $\rightarrow \underline{a} \cdot \underline{b} = \underbrace{\sum_{i=1}^{3} a_i \underline{e}_i \cdot \sum_{j=1}^{3} b_j \underline{e}_j} = \sum_{i=1}^{3} a_i b_i = a_i b_i \; (= a_1 b_1 + a_2 b_2 + a_3 b_3)$

(do you understand the notation?)

base vectors are orthogonal $\left\{ \begin{array}{ll} \underline{e}_i \cdot \underline{e}_j = 0 & \text{if } i \neq j \\ \phantom{\underline{e}_i \cdot \underline{e}_j} = 1 & \text{if } i = j \end{array} \right.$

where we again invoke the summation convention and drop the summation symbol

3. Kronecker delta δ_{ij} $\quad (i = 1,2,3 \quad j = 1,2,3)$

a. Definition :

$$\boxed{\delta_{ij} = \left\{ \begin{array}{ll} 0 & i \neq j \\ 1 & i = j \end{array} \right.}$$

NOTE: If you like, you may think about δ_{ij} as the components of the identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

clearly $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$

b. With this shorthand we write

δ_{ij} is a 'replacement' operator

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_i \underline{e}_i \underset{\uparrow}{\cdot} b_j \underline{e}_j = a_i b_j \overbrace{\underline{e}_i \cdot \underline{e}_j}^{\delta_{ij}} = \underbrace{a_i b_j \overset{\downarrow}{\delta_{ij}}} = a_i b_i$$

only non-zero if $i = j$

vector operation; only acts on the base vectors not the components

implies the double sum $\sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \delta_{ij}$

In this eqn, i would be called the summation index. $\rightarrow$

$$\therefore \boxed{\underset{\sim}{a} \cdot \underset{\sim}{b} = a_i b_i = a_j b_j}$$

and we again remark that a different dummy index was used for each vector $(a_i \underline{e}_i, \; b_j \underline{e}_j)$. NEVER write $a_i \underline{e}_i \cdot b_i \underline{e}_i$ — this is very confusing.

c. Remarks: (i) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$ using the summation convention; $\boxed{\delta_{ii} = 3}$

an important idea $\longrightarrow$ (ii) $\delta_{ij} \rightarrow$ the REPLACEMENT OPERATOR: $\boxed{\delta_{ij} c_j = c_i}$

(iii) Very often, one will not write the unit vectors $\underline{e}_i$, and will write A_i where it is understood that i may be either 1, 2 or 3. In this case i would be called a free index since it is free to take on the values 1, 2 or 3.

Similarly, the vector eqn $\underline{a} = \underline{b}$ may be written

$$a_i \underline{e}_i = b_i \underline{e}_i \quad \text{or} \quad a_i = b_i \quad \text{and}$$

since i only appears once on each side of the eqn, it is free to take on the value 1, 2, or 3 so this stands for 3 seperate equalities: $a_1 = b_1 \quad a_2 = b_2 \quad a_3 = b_3$.

Another example:

$$(\underline{a} \cdot \underline{b}) \underline{c} = a_i b_i \underset{\sim}{c} = a_i b_i c_j \underline{e}_j \quad \text{or} \quad \underbrace{a_i b_i}_{} c_j$$

i appears twice so we sum $i = 1 \rightarrow 3$

j is free to take on the values 1, 2 or 3 this symbol will be useful whenever vector products arise.

4. Permutation Symbol ε_{ijk} $i=1,2,3$ $j=1,2,3$ $k=1,2,3$

a. Definition :
$$\varepsilon_{ijk} = \begin{cases} +1 \text{ or } -1 & \text{if } i,j,k \text{ are all different} \\ 0 & \text{if any two indices are the same} \end{cases}$$

In particular,

$\varepsilon_{ijk} = +1$ if if i,j,k are an EVEN permutation of 1,2,3

$\rightarrow$ $\varepsilon_{123} = 1$ $\varepsilon_{312} = 1$ $\varepsilon_{231} = 1$

$\varepsilon_{ijk} = -1$ if i,j,k are an ODD permutation of 1,2,3

$\varepsilon_{213} = -1$ $\varepsilon_{132} = -1$ $\varepsilon_{321} = -1$

NOTE: By even permutation we mean that an even # of interchanges of the indices must occur to get back to the order 123 ; analogous for meaning of odd permutation.

b. This definition has the following cyclic and interchange property :

$$\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$$

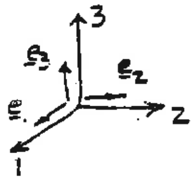
and if two indices are simply interchanged, the sign changes,

$$\varepsilon_{ijk} = -\varepsilon_{ikj} \quad \text{or} \quad \varepsilon_{ijk} = -\varepsilon_{jik}$$

Also, since i,j,k can each independently take on the values 1,2,3, then ε_{ijk} represents 27 quantities.

∴ Cross-product of base vectors (or any 2 vectors) will always involve the permutation symbol ε_{ijk}.

c. We also have $\boxed{\underline{e}_i \wedge \underline{e}_j = \varepsilon_{ijk}\,\underline{e}_k}$



and by referring to the figure at right, everything is o.k. : $\underline{e}_1 \wedge \underline{e}_2 = +\underline{e}_3 = \underbrace{\varepsilon_{123}}_{+1}\underline{e}_3$ etc.

d. We now have an effective shorthand notation for representing the vector product.

let $\underline{c} = \underline{a} \wedge \underline{b}$; write $\underline{a} = a_i\,\underline{e}_i$, $\underline{b} = b_j\,\underline{e}_j$

$\rightarrow$ $= a_i\,\underline{e}_i \wedge b_j\,\underline{e}_j = a_i b_j (\underline{e}_i \wedge \underline{e}_j)$

$\boxed{\underline{a} \wedge \underline{b} = a_i b_j\, \varepsilon_{ijk}\, \underline{e}_k}$ ← NOTE CAREFULLY THE ORDER OF THE INDICES

or with $\underline{c} = c_k\,\underline{e}_k$, we have $c_k = \underbrace{a_i b_j\, \varepsilon_{ijk}}_{\text{use summation convention on } i,j}$ which represents 3 eqns for k=1,2 or 3.

EXERCISE: Verify that this is in agreement with the 'matrix' definition on pg. 57.

e. triple scalar product: $\underline{a}\cdot(\underline{b}\wedge\underline{c})$

Again we are careful to use different dummy indices for each vector so

$$\underline{a}\cdot(\underline{b}\wedge\underline{c}) = a_i\,\underline{e}_i\cdot(b_j\,\underline{e}_j\wedge c_k\,\underline{e}_k) = a_i\,\underline{e}_i\cdot(b_j c_k\,\varepsilon_{jkl}\,\underline{e}_l)$$

$$= a_i b_j c_k\,\varepsilon_{jkl}\,\underbrace{\underline{e}_i\cdot\underline{e}_l}_{\delta_{il}}$$

$$= \varepsilon_{jki}\,a_i b_j c_k = \varepsilon_{ijk}\,a_i b_j c_k = (\underline{a}\wedge\underline{b})\cdot c = (\underline{c}\wedge\underline{a})\cdot b$$

by using cyclic property of ε_{ijk}. Exercise - convince yourself that these last 2 identities follow from index expression.

Recall also that

$$\underline{a}\cdot(\underline{b}\wedge\underline{c}) = \det\begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} = \underbrace{\varepsilon_{ijk}\,a_i b_j c_k}$$

index representation of the 3x3 determinant

5. Useful identities involving ε and δ

$$\boxed{\varepsilon_{ijk}\,\varepsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}}$$

i, j, l, m can each independently take on values 1, 2, 3. Hence, this eqn corresponds to 81 quantities.

Proof: verify by brute force for each of the 81 eqns! However, it is best to make your life easier by noticing that both sides change sign if either $i \leftrightarrow j$ or $l \leftrightarrow m$ are interchanged. Also, both sides vanish if $i = j$ or $l = m$. Then, consider remaining terms like:

$$\varepsilon_{12k}\,\varepsilon_{k12} = \overset{0}{\cancel{\varepsilon_{121}\,\varepsilon_{112}}} + \overset{0}{\varepsilon_{122}}\,\varepsilon_{212} + \varepsilon_{123}\,\varepsilon_{312} = 1$$

and

$$\delta_{11}\,\delta_{22} - \overset{0}{\cancel{\delta_{12}}}\,\delta_{12} = 1 \qquad \text{so o.k.}$$

Likewise

$$\varepsilon_{12k}\,\varepsilon_{k13} = \varepsilon_{121}\,\varepsilon_{113} + \varepsilon_{122}\,\varepsilon_{213} + \varepsilon_{123}\,\varepsilon_{313} = 0\text{; also }\ \delta_{11}\delta_{23} - \delta_{13}\delta_{23} = 0 \text{ so o.k.}$$

etc. //

Example 1: show that $\underline{e}_i = \frac{1}{2}\,\varepsilon_{mni}\,\underline{e}_m\wedge\underline{e}_n$

Well, $\varepsilon_{mni}\,\underline{e}_m\wedge\underline{e}_n = \varepsilon_{mni}\,\varepsilon_{mnj}\,\underline{e}_j = \overbrace{\varepsilon_{nim}\,\varepsilon_{mnj}}^{\delta_{nn}\delta_{ij} - \delta_{nj}\delta_{ni}}\,\underline{e}_j$

$$= (3\delta_{ij} - \delta_{ij})\,\underline{e}_j = 2\,\underline{e}_i \qquad \times$$

Example 2: show that $\underline{a}\wedge(\underline{b}\wedge\underline{c}) = \underline{b}(\underline{a}\cdot\underline{c}) - \underline{c}(\underline{a}\cdot\underline{b})$

$$\underline{a}\wedge(\underline{b}\wedge\underline{c}) = a_i\,\underline{e}_i\wedge(b_j\,\underline{e}_j\wedge c_k\,\underline{e}_k) = a_i\,\underline{e}_i\wedge(b_j c_k\,\varepsilon_{jkl}\,\underline{e}_l) = a_i b_j c_k\,\varepsilon_{jkl}(\underline{e}_i\wedge\underline{e}_l)$$

$$= a_i b_j c_k\,\varepsilon_{jkl}\,\varepsilon_{ilm}\,\underline{e}_m = a_i b_j c_k\,\varepsilon_{jkl}\,\varepsilon_{lmi}\,\underline{e}_m = a_i b_j c_k(\delta_{jm}\delta_{ki} - \delta_{ji}\delta_{km})\underline{e}_m$$

$$= a_i b_m c_i\,\underline{e}_m - a_i b_i c_m\,\underline{e}_m = (a_i c_i)\,b_m\,\underline{e}_m - (a_i b_i)\,c_m\,\underline{e}_m = (\underline{a}\cdot\underline{c})\,\underline{b} - (\underline{a}\cdot\underline{b})\,\underline{c} \qquad \times$$

Some additional Examples of the use of index notation

First, a brief summary of the important ideas

(i) $\underline{e}_i \cdot \underline{e}_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

(ii) $\underline{e}_i \wedge \underline{e}_j = \varepsilon_{ijk}\, \underline{e}_k$ $\qquad \varepsilon_{ijk} = \begin{cases} +1 & i,j,k \text{ an even permutation of } 1,2,3 \\ -1 & i,j,k \text{ an odd permutation of } 1,2,3 \\ 0 & \text{any two indices the same} \end{cases}$

(iii) summation convention : whenever a subscript appears twice, a summation from $1 \rightarrow 3$ is implied.

Examples :

(i) $\delta_{ik}\,\delta_{jk} = \delta_{ij}$ since δ_{jk} is only nonzero when $j=k$ so the k in δ_{ik} may be replaced by j.

(ii) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$ note : Since i was a dummy index, $\delta_{ii} = \delta_{kk} = \delta_{mm}$ etc,

(iii) $\delta_{ij}\,\varepsilon_{ijk} = \varepsilon_{iik} = 0$ since two of the indices are the same.

(iv) $\varepsilon_{ijk}\,\varepsilon_{njk} = \varepsilon_{ijk}\,\varepsilon_{knj}$ by first rotating the indices on the second ε.

Next, use the identity : $\varepsilon_{ijk}\,\varepsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$

$\Rightarrow$ $\varepsilon_{ijk}\,\varepsilon_{knj} = \delta_{in}\underbrace{\delta_{jj}}_{3} - \delta_{ij}\,\delta_{nj} = 3\,\delta_{in} - \delta_{in} = \underline{\underline{2\,\delta_{in}}}$

(v) $a_m b_n \varepsilon_{mnq} - a_n b_m \varepsilon_{mnq} = ?$

$\Rightarrow$ m & n appear twice in each term so summation is implied.

But, m & n are simply dummy variables, i.e., we could just as well use another letter.

So, examine the second term $a_n b_m \varepsilon_{mnq}$.

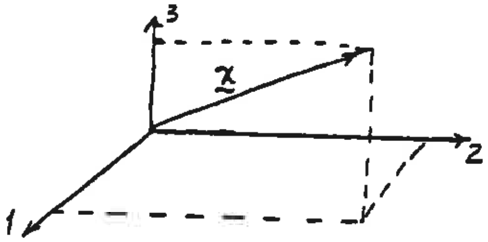
$$\begin{aligned} a_n b_m \varepsilon_{mnq} &= -a_n b_m \varepsilon_{nmq} \quad ; \text{ now let } j=n,\ m=k \\ &= -a_j b_k \varepsilon_{jkq} \quad \text{which is the same as the above since Summation on } j,k \text{ is implied.} \\ &= -a_m b_n \varepsilon_{mnq} \quad \text{by letting } j=m \ \ k=n \end{aligned}$$

So, we see that

$$a_m b_n \varepsilon_{mnq} - a_n b_m \varepsilon_{mnq} = 2\underbrace{a_m b_n \varepsilon_{mnq}}_{(\underline{a}\wedge\underline{b})_q} \longleftarrow q^{\text{th}} \text{ component of } \underline{a}\wedge\underline{b}$$

C. Some vector calculus (taking derivatives of vector functions)

1. Notation : we will use the vector $\underset{\sim}{x}$ to denote the vector location of a point in space.



$\underset{\sim}{x} = (x_1, x_2, x_3) \longleftarrow (x, y, z)$

the value of ϕ depends on location in space

one can discuss <u>scalar</u> fields $\phi(\underset{\sim}{x}) = \phi(x_1, x_2, x_3)$ or just $\phi(x_i)$

and one can discuss <u>vector</u> fields $\underset{\sim}{a}(\underset{\sim}{x}) = a_1(x_1,x_2,x_3)\,\underline{e}_1 + a_2(x_1,x_2,x_3)\,\underline{e}_2 + a_3(x_1,x_2,x_3)\,\underline{e}_3$

each of the <u>components</u> of the vector $\underline{a}$ depends on location in space $\Rightarrow$ could simply write $a_j(x_i)$

2. Differentiation of vectors

a. Suppose $\underline{a} = \underset{\sim}{a}(t) = a_i(t)\,\underline{e}_i$

Then $\dfrac{d\underline{a}}{dt} = \dfrac{da_i(t)}{dt}\,\underline{e}_i$ since the cartesian base vectors <u>i.e.</u> constant vectors.

clearly $\dfrac{d}{dt}(\underline{a}\cdot\underline{b}) = \dfrac{d\underline{a}}{dt}\cdot\underline{b} + \underline{a}\cdot\dfrac{d\underline{b}}{dt}$

We will now consider spatial derivatives of vectors, e.g., $\dfrac{\partial}{\partial x}\underline{b}(\underline{x})$ or $\dfrac{\partial}{\partial y}\underline{a}(\underline{x})$

3. Gradient operator – Section 9.3 Greenberg; Sections 6.7, 6.6 Hildebrand

Let $\phi(\underline{x})$ be a scalar function which varies with position x, y, z in space.

The rate of variation of ϕ in the x-direction is $\dfrac{\partial\phi}{\partial x} = \dfrac{\partial\phi}{\partial x_1}$, in the y-direction is $\dfrac{\partial\phi}{\partial y} = \dfrac{\partial\phi}{\partial x_2}$, in the z-direction is $\dfrac{\partial\phi}{\partial z} = \dfrac{\partial\phi}{\partial x_3}$

We introduce the <u>vector</u>,

grad ϕ (or gradient of ϕ) $\equiv \underset{\sim}{\nabla}\phi = \underline{e}_1\dfrac{\partial\phi}{\partial x_1} + \underline{e}_2\dfrac{\partial\phi}{\partial x_2} + \underline{e}_3\dfrac{\partial\phi}{\partial x_3} = \underline{e}_i\dfrac{\partial\phi}{\partial x_i} = \underbrace{\underline{e}_i\,\phi_{,i}}$

"comma" notation to indicate differentiation with respect to x_i

$$\text{gradient operator} \equiv \underset{\sim}{\nabla}(\,) = \underline{e}_1\frac{\partial}{\partial x_1}(\,) + \underline{e}_2\frac{\partial}{\partial x_2}(\,) + \underline{e}_3\frac{\partial(\,)}{\partial x_3}$$

$$= \underline{e}_i\frac{\partial}{\partial x_i}(\,)$$

3. more about the gradient operator

⇒ Relation between the gradient and the <u>directional</u> <u>derivative</u>

consider a small displacement $d\underset{\sim}{r}$, where $|d\underset{\sim}{r}| = ds$.

The unit tangent vector $\underset{\sim}{t}$ in the direction of $d\underset{\sim}{r}$ is $\underset{\sim}{t} = \frac{d\underset{\sim}{r}}{ds}$

Then, the rate-of-change of ϕ in the direction of $\underset{\sim}{t}$ is

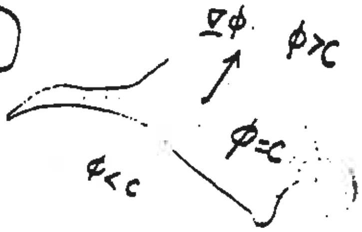
$\boxed{d\underset{\sim}{r} = dx_1 \underset{\sim}{e}_1 + dx_2 \underset{\sim}{e}_2 + dx_3 \underset{\sim}{e}_3}$

$$\underset{\sim}{t} \cdot \underset{\sim}{\nabla} \phi = t_i \underbrace{\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j}_{\delta_{ij}} \frac{\partial \phi}{\partial x_j} = t_i \frac{\partial \phi}{\partial x_i} = t_1 \frac{\partial \phi}{\partial x_1} + t_2 \frac{\partial \phi}{\partial x_2} + t_3 \frac{\partial \phi}{\partial x_3}$$

$$\underset{\sim}{t} \cdot \underset{\sim}{\nabla} \phi = \frac{dx_1}{ds} \frac{\partial \phi}{\partial x_1} + \frac{dx_2}{ds} \frac{\partial \phi}{\partial x_2} + \frac{dx_3}{ds} \frac{\partial \phi}{\partial x_3} = \frac{d\phi}{ds}$$

$$\therefore \boxed{\frac{d\phi}{ds} = \underset{\sim}{t} \cdot \underset{\sim}{\nabla} \phi}$$ Directional derivative of ϕ in the $\underset{\sim}{t}$ direction

Now, consider a surface $\phi(\underset{\sim}{x}) = \text{constant}$



Clearly $d\phi = 0$ for any displacement <u>along</u> the surface, then since $\underset{\sim}{t}$ is a tangent vector to the surface, it follows that that $\underset{\sim}{t} \perp \underset{\sim}{\nabla}\phi$, i.e., $\underset{\sim}{\nabla}\phi$ is a vector perpendicular to the surface $\phi = \text{constant}$.

→ $\underset{\sim}{\nabla}\phi$ is a vector <u>normal</u> to the surface $\phi = \text{constant}$

4. Divergence of a vector field : $\underset{\sim}{\nabla} \cdot \underset{\sim}{f}$ or $\text{div}\, \underset{\sim}{f}$

a. Simply compute using standard ideas.

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{f} = \left(\underset{\sim}{e}_i \frac{\partial}{\partial x_i}\right) \cdot \left(f_j \underset{\sim}{e}_j\right) = \underset{\sim}{e}_i \cdot \frac{\partial f_j}{\partial x_i} \underset{\sim}{e}_j + f_j \underset{\sim}{e}_i \cdot \underbrace{\frac{\partial \underset{\sim}{e}_j}{\partial x_i}}_{0}$$ using the product rule

0 since the $\underset{\sim}{e}_j$'s are unit vectors which do not vary with position in space

$$= \delta_{ij} \frac{\partial f_j}{\partial x_i}$$

comma notation sometimes used

$$= \frac{\partial f_j}{\partial x_j} \left(= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) = f_{j,j}$$

NOTE: Now that you have gone through this, make your life easier. The $\underset{\sim}{e}_j$ are <u>constant</u> vectors with respect to differentiation so we know it is o.k. to simply write

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{f} = \underset{\sim}{e}_i \frac{\partial}{\partial x_i} \cdot \left(f_j \underset{\sim}{e}_j\right) = \underset{\sim}{e}_i \cdot \underset{\sim}{e}_j \frac{\partial f_j}{\partial x_i} = \frac{\partial f_j}{\partial x_j}$$

same index.

Also, whenever you see a term like $\frac{\partial f_k}{\partial x_k}$, you now know $\frac{\partial f_k}{\partial x_k} = \underset{\sim}{\nabla} \cdot \underset{\sim}{f}$.

b. An identity using index notation: let $\phi(\underline{x})$ be a scalar field

$$\underline{\nabla}\cdot(\phi\,\underline{f}) = \underline{e}_i\frac{\partial}{\partial x_i}\cdot\left(\phi f_j\,\underline{e}_j\right)$$

← (i) use product rule
(ii) $\underline{e}_j$ are constant vectors
(iii) the inner product $(\cdot)$ only operates on vectors, <u>not</u> the scalar components ϕf_j

$$= (\underline{e}_i\cdot\underline{e}_j)\frac{\partial}{\partial x_i}(\phi f_j)$$

$$= \delta_{ij}\left[\frac{\partial\phi}{\partial x_i}f_j + \phi\frac{\partial f_j}{\partial x_i}\right]$$

$$= \frac{\partial\phi}{\partial x_j}f_j + \phi\frac{\partial f_j}{\partial x_j} = (\underline{\nabla}\phi)\cdot\underline{f} + \phi\,\underline{\nabla}\cdot\underline{f}$$

$$\therefore\ \underline{\nabla}\cdot(\phi\,\underline{f}) = (\underline{\nabla}\phi)\cdot\underline{f} + \phi\,\underline{\nabla}\cdot\underline{f}$$

← notice how similar this is to the normal product rule of differentiation

c. Interpretation of the Divergence of a vector field (similar to electric field lines in a medium; also magnetic field lines)

Recall the Divergence Theorem which relates certain volume integrals to integrals over a bounding surface:

$$\int_V \underline{\nabla}\cdot\underline{f}\ dV = \int_S \underline{f}\cdot\underline{n}\ dS$$

In the field of fluid dynamics we find a very nice physical interpretation of the divergence of a vector field.

Consider the flow of a fluid of <u>constant density</u> (e.g., water – such a flow is called INCOMPRESSIBLE)

Let $\underline{v}(\underline{x})$ be the velocity of the fluid at a pt $\underline{x}$.

Let S be some fixed boundary drawn in the fluid surrounding the fixed volume V. The net flow rate through a surface with differential area dS is $(\underline{v}\cdot\underline{n})dS$ ←

The total flow through the surface is found by integrating over S

$$\int_S \underline{v}\cdot\underline{n}\ dS = 0$$ since for a fluid of constant density: $\left\{\begin{matrix}\text{in}\\\text{flow}\end{matrix}\right\} - \left\{\begin{matrix}\text{out}\\\text{flow}\end{matrix}\right\} = 0.$

$$\parallel$$

$$\int_V \underline{\nabla}\cdot\underline{v}\ dV = 0$$ by the Divergence Theorem – you may remember this from Math 21. If not, we will discuss shortly.

And since this must be true for any choice of the volume element V we conclude $\underline{\nabla}\cdot\underline{v} = 0$ for all $\underline{x}$

For an incompressible fluid, the vanishing of the divergence of the velocity field is associated with conservation of mass. Alternatively, if there is a source (or sink) of mass. $\nabla\cdot v$ measures net flux away from a point.

4. Curl of a vector field $\underline{\nabla}\wedge\underset{\sim}{f}$ or curl $\underset{\sim}{f}$

a. Again, simply compute using standard ideas

$$\underline{\nabla}\wedge\underline{f} = \underline{e}_i\frac{\partial}{\partial x_i}\wedge\left(f_j\,\underline{e}_j\right)$$

⇐ As before, the $\underline{e}_j$ are constant vectors and the curl ($\wedge$) operation only affects vectors

$$= (\underline{e}_i\wedge\underline{e}_j)\frac{\partial f_j}{\partial x_i}$$

$$\boxed{\underline{\nabla}\wedge\underset{\sim}{f} = \varepsilon_{ijk}\frac{\partial f_j}{\partial x_i}\,\underline{e}_k}$$

Sometimes people will write this as

$$\left(\underline{\nabla}\wedge\underline{f}\right)_k = \varepsilon_{ijk}\frac{\partial f_j}{\partial x_i}$$

↑ indicates the k^{th} component of the vector $\underline{\nabla}\wedge\underline{f}$

b. Alternatively, lets just go through and show that the above agrees with what you have seen in earlier vector calculus courses.

First, $\underline{\nabla}\wedge\underline{f} = (\underline{e}_i\wedge\underline{e}_j)\dfrac{\partial f_j}{\partial x_i}$ and since the summation convention has been assumed and the variables i, j appear twice, we must sum $i = 1\to 3$, $j = 1\to 3$.

or

$$\underline{\nabla}\wedge\underline{f} = \cancelto{0}{(\underline{e}_1\wedge\underline{e}_1)}\frac{\partial f_1}{\partial x_1} + \underset{\underset{\underline{e}_3}{\parallel}}{(\underline{e}_1\wedge\underline{e}_2)}\frac{\partial f_2}{\partial x_1} - \underset{\underset{-\underline{e}_2}{\parallel}}{(\underline{e}_1\wedge\underline{e}_3)}\frac{\partial f_3}{\partial x_1} - \underset{\underset{-\underline{e}_3}{\parallel}}{(\underline{e}_2\wedge\underline{e}_1)}\frac{\partial f_1}{\partial x_2} + \cancelto{0}{(\underline{e}_2\wedge\underline{e}_2)}\frac{\partial f_2}{\partial x_2} + \cdots$$

$$= \underline{e}_1\left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) + \underline{e}_2\left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) + \underline{e}_3\left(\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}\right)$$

$$= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{vmatrix} \quad //$$

which is probably how you previously saw it represented.

c. Another identity:

$$\underline{\nabla}\wedge\underline{\nabla}\phi = \underline{e}_i\frac{\partial}{\partial x_i}\wedge\left(\underline{e}_j\frac{\partial\phi}{\partial x_j}\right) = \underline{e}\wedge\underline{e}_j\frac{\partial^2\phi}{\partial x_i\partial x_j} = \varepsilon_{ijk}\frac{\partial^2\phi}{\partial x_i\partial x_j}\underline{e}_k$$

But notice that by using the properties of ε_{ijk},

$$\varepsilon_{ijk}\frac{\partial^2\phi}{\partial x_i\partial x_j} = -\varepsilon_{jik}\frac{\partial^2\phi}{\partial x_i\partial x_j} = -\varepsilon_{jik}\frac{\partial^2\phi}{\partial x_j\partial x_i} = 0$$

by comparing with the first term.

relabel: $i\to j$, $j\to i$

twice continuously differentiable

$$\therefore \boxed{\underline{\nabla}\wedge\underline{\nabla}\phi = 0 \text{ for any scalar function } \phi}$$

Prove: $\underline{a}\wedge\underline{a} = 0$

i.e., $a_i a_j \varepsilon_{ijk}\underline{e}_k = 0$.

each component = 0.

4. Curl of a vector field (continued)

d. Evaluate $\underline{\nabla} \cdot (\underline{a} \wedge \underline{b})$

constant, so may be taken outside of parenthesis

$$\Rightarrow \underline{e}_i \frac{\partial}{\partial x_i} \cdot \left(a_j \underline{e}_j \wedge b_k \underline{e}_k \right) = \underline{e}_i \frac{\partial}{\partial x_i} \cdot \left(a_j b_k \right) \epsilon_{jkl} \underline{e}_l$$

$$= \underset{\overset{\shortparallel}{\delta_{il}}}{\left(\underline{e}_i \cdot \underline{e}_l \right)} \frac{\partial}{\partial x_i} (a_j b_k) \epsilon_{jkl} = \frac{\partial}{\partial x_i} (a_j b_k) \epsilon_{jki}$$

$\epsilon_{jki} = \epsilon_{ijk} = \epsilon_{kij}$

$$= \left(\frac{\partial a_j}{\partial x_i} \right) \epsilon_{ijk} b_k + \frac{\partial b_k}{\partial x_i} \epsilon_{kij} a_j$$

$$= \frac{\partial a_j}{\partial x_i} \epsilon_{ijk} b_k - \frac{\partial b_k}{\partial x_i} \epsilon_{ikj} a_j = \left(\underline{\nabla} \wedge \underline{a} \right)_k b_k - \left(\underline{\nabla} \wedge \underline{b} \right)_j a_j$$

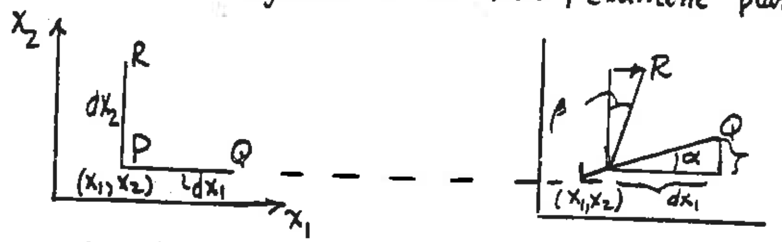
or in vector notation,

$$\underline{\nabla} \cdot \left(\underline{a} \wedge \underline{b} \right) = (\underline{\nabla} \wedge \underline{a}) \cdot \underline{b} - (\underline{\nabla} \wedge \underline{b}) \cdot \underline{a}$$

5. Interpretation of the curl of a vector field – Again, use the velocity field of a fluid flow as an example

Let $\underline{v}(\underline{x})$ be the velocity of a fluid flow. We will now see that $\underline{\nabla} \wedge \underline{v}$ provides a measure of the average angular velocity.

Consider 2 line segments in the flow; examine planar motions for simplicity



at time t

at time t + dt

For small rotations, $\tan\alpha \sim \alpha$; $\tan\beta \sim \beta$ so that in a short time dt

$$\alpha \sim \tan\alpha = \frac{\left[v_2(x_1 + dx_1) - v_2(x_1) \right] dt}{dx_1} \sim \frac{\partial v_2}{\partial x_1} dt \quad \text{via a Taylor series}$$

and

$$\beta \sim \tan\beta = \frac{\left[v_1(x_2 + dx_2) - v_1(x_2) \right] dt}{dx_2} \sim \frac{\partial v_1}{\partial x_2} dt$$

$\Rightarrow$ average rate of counterclockwise rotation of fluid particle about the x_3-axis $= \frac{1}{2}\left[\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right] = \frac{1}{2}(\underline{\nabla} \wedge \underline{v})_3$

and, in general, the average rate of rotation of a fluid particle about the x_1-axis is $= \frac{1}{2}(\underline{\nabla} \wedge \underline{v})_i$; $\underline{\omega} = \underline{\nabla} \wedge \underline{v} \equiv$ Vorticity vector

$d\underset{\sim}{S} = \underline{n}\, dS$

D. Integral Theorems

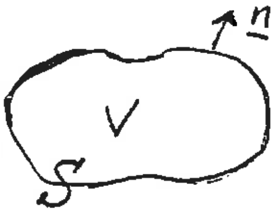
1. Divergence Theorem (or Gauss' Theorem)

⇒ This theorem relates integrals over volumes to integrals over the bounding surface(s).

The theorem states that given a continuous vector function $\underset{\sim}{f}$ with continuous first partial derivatives, then

when writing this thrm in vector form, it is useful to get in the habit of writing $\underline{n}$ on the left

DIVERGENCE THEOREM
$$\int_V \underline{\nabla}\cdot\underset{\sim}{f}\, dV = \int_S \underline{n}\cdot\underline{f}\, dS$$



where n is the unit outward normal from V.

Using index notation, we write $\int_V \frac{\partial f_i}{\partial x_i}\, dV = \int_S n_i f_i\, dS$

(Recall the proof of the theorem from Math 21)

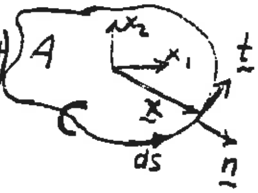
Written out in 3D:

$$\int_V \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) dV = \int_S (n_1 f_1 + n_2 f_2 + n_3 f_3)\, dS$$

2. Planar versions of the Divergence Theorem

Consider some area A in the plane bounded by the curve C

let $\underset{\sim}{n}$ and $\underset{\sim}{t}$ be the unit outward normal and unit tangent vectors along the boundary. $(ds = |d\underline{x}|$, $d\underline{x} \equiv$ small displacement along boundary$)$



Since $\underset{\sim}{t} = \frac{d\underline{x}}{ds} \rightarrow \underset{\sim}{t}\, ds = dx_1\, \underline{e}_1 + dx_2\, \underline{e}_2$

and

Since $\underline{n}\cdot\underline{t} = 0 \qquad \underset{\sim}{n}\, ds = dx_2\, \underline{e}_1 - dx_1\, \underline{e}_2$

So,

integral over area

integral around the boundary

$$\int_A \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) dA = \int_C \underline{f}\cdot\underline{n}\, ds = \int_C (f_1\, dx_2 - f_2\, dx_1)$$

let $f_1 = N(x_1, x_2)$
$f_2 = -M(x_1, x_2)$

$$\Rightarrow \int_A \left(\frac{\partial N}{\partial x_1} - \frac{\partial M}{\partial x_2}\right) \overset{dx\,dy}{\widetilde{dA}} = \int_C (M\, dx_1 + N\, dx_2)$$

which you may recall seeing in a previous course.

2. Planar versions (continued)

On the other hand, if we begin with the last eqn but identify N, M with the components of a vector as

$\underset{\sim}{a}(\underline{x}) = a_1(x_1, x_2)\,\underset{\sim}{e}_1 + a_2(x_1, x_2)\,\underset{\sim}{e}_2$ and let $a_1 = M$, $a_2 = N$,

then

$$\int_C (a_1\, dx_1 + a_2\, dx_2) = \int_A \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right) dA$$

or

$$\int_C \underset{\sim}{a} \cdot \underset{\sim}{t}\, ds = \int_A (\underline{\nabla} \wedge \underset{\sim}{a}) \cdot \underset{\sim}{e}_3\, dA$$

which is the planar version of <u>Stokes Theorem</u>
↓
we will come back to Stokes Thrm shortly

3. Some Theorems which follow directly from the Divergence Theorem

Begin with

$$\int_V \frac{\partial f_i}{\partial x_i}\, dV = \int_S n_i f_i\, dS$$

i.e, $\int_V \underline{\nabla} \cdot \underset{\sim}{f}\, dV = \int_S \underline{n} \cdot \underset{\sim}{f}\, dS$

(i) let $\underset{\sim}{f} = \phi\, \underset{\sim}{b}$ where $\phi = \phi(\underline{x})$, but $\underset{\sim}{b}$ is an arbitrary <u>constant</u> vector

So, we have

$$\left(\int_V \underline{\nabla}\phi\, dV\right) \cdot \underset{\sim}{b} = \left(\int_S \underline{n}\phi\, dS\right) \cdot \underline{b} \Leftarrow \left(\int_V \frac{\partial \phi}{\partial x_i}\, dV\right) b_i = \left(\int_S n_i \phi\, dS\right) b_i$$

where b_i can be taken outside of the integrals because $\underset{\sim}{b}$ is a <u>constant</u> vector.

"standard argument"

$\left[\int \underline{\nabla}\phi\, dV - \int \underline{n}\phi\, dS\right] \cdot \underline{b} = 0$

so either
(i) $[\;] = 0$
(ii) $\underline{b} = 0$ or
(iii) $[\;] \perp b$
but $\underline{b}$ is an <u>arbitrary</u> <u>constant</u> vector so (i) & (iii) cannot hold

Since this eqn must be true for arbitrary $\underset{\sim}{b}$, we conclude

$$\boxed{\int_V \frac{\partial \phi}{\partial x_i}\, dV = \int_S n_i\, \phi\, dS} \qquad i = 1, 2, \text{or } 3$$

or in vector notation

and this is a VECTOR EQUALITY so it holds for each component. →

$$\boxed{\int_V \underline{\nabla}\, \phi\, dV = \int_S \underline{n}\, \phi\, dS}$$

which is Gauss' Theorem for a scalar function.

(ii) Another theorem which follows from the Divergence Theorem is obtained by letting

$$\underset{\sim}{f} = \underset{\sim}{\nabla}\phi$$

Then since $\underset{\sim}{\nabla}\cdot\underset{\sim}{f} = \underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla}\phi) = \overbrace{\underset{\sim}{e}_i \frac{\partial}{\partial x_i}\cdot\left(\underset{\sim}{e}_j}^{\delta_{ij}} \frac{\partial\phi}{\partial x_j}\right)$ $\qquad \frac{\partial^2\phi}{\partial x_i^2}$

$$= \frac{\partial^2\phi}{\partial x_j \partial x_i} = \frac{\partial^2\phi}{\partial x_i^2} = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} \equiv \underbrace{\nabla^2\phi}_{\text{the LAPLACIAN}}$$

we have

$$\boxed{\int_V \nabla^2\phi \, dV = \int_S \underbrace{\underset{\sim}{n}\cdot\underset{\sim}{\nabla}\phi}_{\frac{\partial\phi}{\partial n}} \, dS}$$

← gradient of ϕ in the direction of the unit normal $\underset{\sim}{n}$ (i.e., directional derivative along $\underset{\sim}{n}$)

4. Green's Theorem (Hildebrand, p. 301-2)

(i) Begin with the Divergence Theorem and let $\underset{\sim}{f} = \psi\underset{\sim}{\nabla}\phi$

Then,

$$\int_S \psi(\underset{\sim}{n}\cdot\underset{\sim}{\nabla}\phi) \, dS = \int_V \underset{\sim}{\nabla}\cdot(\psi\underset{\sim}{\nabla}\phi) \, dV = \int_V \overbrace{\underset{\sim}{e}_i \frac{\partial}{\partial x_i}\cdot\left(\psi \frac{\partial\phi}{\partial x_j}\underset{\sim}{e}_j\right)}^{\delta_{ij}}$$

$$= \int_V \frac{\partial}{\partial x_i}\left(\psi\frac{\partial\phi}{\partial x_i}\right) dV = \int_V \left(\frac{\partial\psi}{\partial x_i}\frac{\partial\phi}{\partial x_i} + \psi\frac{\partial^2\phi}{\partial x_i^2}\right) dV$$

$$\therefore \boxed{\int_S \psi \, \underbrace{\underset{\sim}{n}\cdot\underset{\sim}{\nabla}\phi}_{\partial\phi/\partial n} \, dS = \int_V \left[\underset{\sim}{\nabla}\psi\cdot\underset{\sim}{\nabla}\phi + \psi\nabla^2\phi\right] dV}$$

Green's 1st form or Green's 1st Identity

(ii) Interchange ψ, ϕ in the above eqn, then subtract from the eqn just derived.

$$\Rightarrow \boxed{\int_S \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right) dS = \int_V \left(\psi\nabla^2\phi - \phi\nabla^2\psi\right) dV}$$

Green's 2nd form or Green's Second Identity

Exercise : derive this eqn

An interesting aside:

(iii) Green's Theorems are often useful for proving some very general results.

For example, begin with Green's first form; let $\psi = \phi$

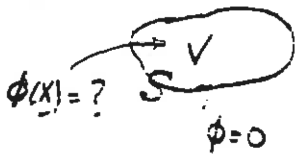
Then,

$$\int_S \phi \frac{\partial \phi}{\partial n} \, dS = \int_V \left[\underset{\sim}{\nabla}\phi \cdot \underset{\sim}{\nabla}\phi + \phi \nabla^2 \phi \right] dV$$

(a p.d.e.)

⇒ Now suppose you wish to solve $\nabla^2\phi = 0$ (Laplace's eqn) in V subject to the b.c $\phi = 0$ on S

What is $\phi(\underset{\sim}{x})$ for $\underset{\sim}{x} \in V$?



Well, we are given $\phi = 0$ on the boundary S so,

$$\int_S \phi \frac{\partial \phi}{\partial n} \, dS = 0,$$

Furthermore, since $\nabla^2\phi = 0$ in V, the eqn above reduces to

$$\int_V \left[\underset{\sim}{\nabla}\phi \cdot \underset{\sim}{\nabla}\phi \right] dV = 0$$

But the integrand $\underset{\sim}{\nabla}\phi \cdot \underset{\sim}{\nabla}\phi$ is the sum of squares, so is always positive. For the integral to vanish we must consequently require $\underset{\sim}{\nabla}\phi \cdot \underset{\sim}{\nabla}\phi = 0$ everywhere in V

or $\underset{\sim}{\nabla}\phi = 0$, i.e., ϕ doesn't vary with spatial position in V.

$\Rightarrow \phi =$ constant in V

But $\phi = 0$ on S and so therefore $\phi = 0$ throughout V.

//

5. A further generalization of the Divergence Theorem:

a. We began by stating the Divergence Theorem $\int_V \underset{\sim}{\nabla}\cdot \underset{\sim}{f}\, dV = \int_S \underset{\sim}{n}\cdot\underset{\sim}{f}\, dS$

Where S represents the closed surface enclosing the volume V and $\underset{\sim}{n}$ is the unit outward normal from the volume.

We then proved a form suitable for scalar functions,

$$\int_V \underset{\sim}{\nabla}\phi \, dV = \int_S \underset{\sim}{n}\,\phi\, dS$$

or using index notation

$$\int_V \frac{\partial\phi}{\partial x_i}\, dV\, \underset{\sim}{e}_i = \int_S n_i\,\phi\, dS \quad \underset{\sim}{e}_i$$

If two vectors are equal their corresponding components are equal, so we see that

$$(*) \qquad \int_V \frac{\partial \phi}{\partial x_i}\, dV = \int_S n_i\, \phi\, dS \qquad \text{for } i = 1, 2 \text{ or } 3.$$

b. We now construct a form useful for when the <u>cross-product</u> appears.

For example, consider integrals of the form $\int_V \underset{\sim}{\nabla}\wedge \underset{\sim}{f}\, dV$

It is simplest to work in index notation. Notice that $\underset{\sim}{\nabla}\wedge\underset{\sim}{f}$ represents a vector so the result of the integration is also a vector. Basically, we then proceed by considering each component of the vector separately.

So using index notation

$$\int_V \underset{\sim}{\nabla}\wedge\underset{\sim}{f}\, dV = \int_V \frac{\partial f_k}{\partial x_j}\,\epsilon_{jki}\,\underset{\sim}{e}_i\, dV = \int_V \frac{\partial f_k}{\partial x_j}\, dV\; \underbrace{\epsilon_{jki}\,\underset{\sim}{e}_i}$$

these are just constants (remember - the summation convention is in effect so this actually represents the sum of lots of terms)

But for each k and i we know that

$$\int_V \frac{\partial f_k}{\partial x_j}\, dV = \int_S n_j\, f_k\, dS$$

In other words, for each k, i use eqn (*) and let $f_k = \phi$ so that

$$\int_V \frac{\partial\phi}{\partial x_j}\, dV\, \epsilon_{jki}\,\underset{\sim}{e}_i = \int_S n_j\,\phi\, dS\, \epsilon_{jki}\,\underset{\sim}{e}_i$$

Hence,

$$\int_V \underset{\sim}{\nabla}\wedge\underset{\sim}{f}\, dV = \int_V \frac{\partial f_k}{\partial x_j}\, dV\, \epsilon_{jki}\,\underset{\sim}{e}_i = \int_S n_j\cdot f_k\, dS\, \epsilon_{jki}\,\underset{\sim}{e}_i = \int_S \underset{\sim}{n}\wedge\underset{\sim}{f}\, dS$$

c. Notice, that we can conveniently summarize all of the above results concerning the Divergence Theorem as follows:

$$\int_V \underline{\nabla} * \underline{\Phi} \, dV = \int_S \underline{n} * \underline{\Phi} \, dS$$

where Φ is any quantity, scalar or vector, and $*$ is any operation (scalar product, vector product or a simple gradient operation) that <u>makes</u> <u>sense</u>.

d. EXAMPLES:

(i) let $\underline{a} \equiv$ constant vector $\int_S \underline{n} \cdot \underline{a} \, dS = \int_V \underbrace{\underline{\nabla} \cdot \underline{a}}_{0 \text{ if } \underline{a} = \text{constant vector}} dV = 0$

(ii) Evaluate $\int_S \underline{n} \cdot (\underline{\nabla} \wedge \underline{f}) \, dS$

By the Divergence Theorem,

$$\int_S \underline{n} \cdot (\underline{\nabla} \wedge \underline{f}) \, dS = \int_V \underbrace{\underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{f})}_{0} dV = 0.$$

0 for any twice continuously differentiable vector function $\underline{f}$.

(iii) Evaluate $\int_S \underline{n} \cdot \underline{\nabla} r^2 \, dS$

Using index notation, $\underline{n} \cdot \underline{\nabla} r^2 = n_i \frac{\partial}{\partial x_i}(r^2) \rightarrow \frac{\partial}{\partial x_i}(r^2) = 2r \frac{\partial r}{\partial x_i} = 2x_i$

So, $\int_S \underline{n} \cdot \underline{\nabla} r^2 \, dS = \int_S n_i \frac{\partial}{\partial x_i}(r^2) \, dS$ since $\frac{\partial r}{\partial x_i} = x_i/r$ (Homework 5, problem 5c)

$$= \int_S 2 n_i x_i \, dS = 2 \int_V \underbrace{\frac{\partial x_i}{\partial x_i}}_{\delta_{ii}} dV \text{ by the Divergence Theorem}$$

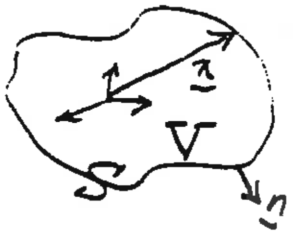
$$= 2 \cdot \delta_{ii} \underbrace{\int_V dV}_{} = V \equiv \text{volume of domain}; \ \delta_{ii} = 3$$

∴ $\boxed{\int_S \underline{n} \cdot \underline{\nabla} r^2 \, dS = 6V}$ where $V \equiv$ volume of domain bounded by S.

e. For those of you who would like more exercises, show

(i) $$\int_S \underline{n} \wedge \underline{\nabla}\, r^2 \, dS = 0$$

(ii) $$\int_S \underline{n} \cdot \underline{\nabla} (\underline{x} \wedge \underline{a}) \, dS = 0$$ where $\underline{a}$ is a constant vector and $\underline{x}$ is a position vector to a pt on the surface; $r = |\underline{x}|$

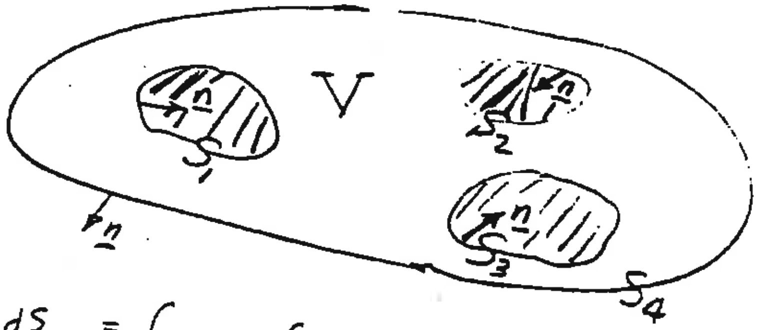


You will <u>not</u> be held responsible for this.

6. There is one final important point concerning the Divergence Theorem and that is if there are <u>multiple</u> bounding surfaces, you must include all of them.

In other words $$\int_V \underline{\nabla} \cdot \underline{f} \, dV = \int_{S \nwarrow} \underline{n} \cdot \underline{f} \, dS$$ all bounding surfaces

So, if V is as shown below:

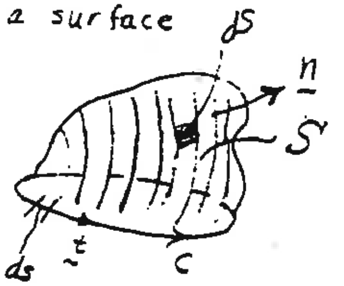


Then $$\int_S \underline{n} \cdot f \, dS = \int_{S_1 + S_2 + S_3 + S_4} \underline{n} \cdot \underline{f} \, dS$$

and notice how on each surface, $\underline{n}$ points <u>outward</u> from V

7. Stokes' Theorem – This allows one to express an integral around a <u>closed curve</u> as an integral over that <u>area</u> having the curve as a boundary.

a. Let C be a closed curve and let S be a surface which has C as a bounding edge.
(imagine a 'hat-shaped' surface)



let $\underline{n}$ ≡ unit normal to S with direction given by the right-hand rule – as you curl your hand in the direction indicated about C, your thumb points in the direction of $\underline{n}$.

$\underline{t}$ ≡ unit tangent vector to C.

sometimes people write $\underset{\sim}{ds} = \underline{t}\, ds$

Stokes Theorem :

$$\boxed{\oint_C \underset{\sim}{f} \cdot \underset{\sim}{t}\; \underset{\uparrow}{ds} = \int_S \underline{n} \cdot (\underline{\nabla} \wedge \underline{f})\, dS}$$

differential element along C

Recall proof given in Math 21.

← distinguish 'little s' and 'big S'

Using index notation :

$$\oint_C f_i t_i \, ds = \int_S n_i \frac{\partial f_k}{\partial x_j} \varepsilon_{ijk} \, dS$$

Writing this all out

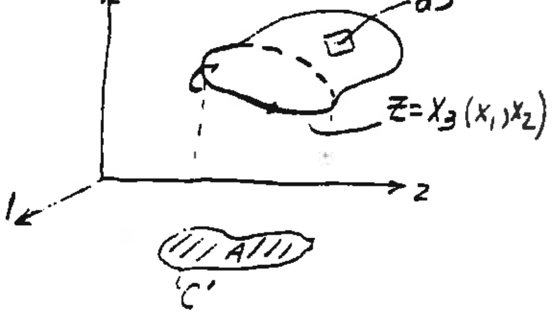
$$\oint_C (f_1 t_1 + f_2 t_2 + f_3 t_3)\, ds = \iint_S \left[n_1 \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) - n_2 \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \right) + n_3 \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) \right] dS$$

NOTE: This is true for an arbitrary surface S with C as a bounding edge.

b. We can actually construct a simple proof using results developed so far

Using s to denote arclength along the bounding curve C,

$$\frac{d\underline{x}}{ds} = \underline{t} \quad \text{then} \quad dx_i = t_i\, ds$$



So, considering the f_1 term :

$$\oint_C f_1 t_1 \, ds = \oint_{\substack{C \\ \uparrow}} f_1 \, dx_1 = \oint_{C'} \mathcal{F}(x_1(x_3), x_2(x_3))\, dx_1$$

function of x, y, z

$\nearrow$ $z = z(x, y)$ represents curve **C** given C' in xy-plane

<u>and</u> we have written $f_1(x_1, x_2, x_3(x_1, x_2)) = \mathcal{F}(x_1, x_2)$

where we have essentially <u>projected information down to the xy-plane</u>.

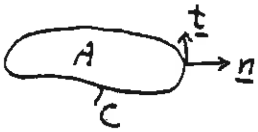
b. "simple proof" (continued)

With information "in the plane" we can now use the identity given on pg. 68 :

one form of the 'Divergence Theorem' was $\int_S n_i \phi \, dS = \int_V \frac{\partial \phi}{\partial x_i} dV$

which has the 'planar version'

$$\int_A \frac{\partial \phi}{\partial x_i} \, dA = \int_C \phi \, n_i \, ds$$



or

$$\int_A \frac{\partial \phi}{\partial x_1} dA = \int_C \phi \, dx_2 \quad \text{and} \quad \int_A \frac{\partial \phi}{\partial x_2} dA = -\int_C \phi \, dx_1$$

p. 67 $\begin{cases} \underline{t} \, ds = dx_1 \, \underline{e}_1 + dx_2 \, \underline{e}_2 \\ \underline{n} \, ds = dx_2 \, \underline{e}_1 - dx_1 \, \underline{e}_1 \end{cases}$

Hence, beginning with the eqn on the bottom of the last page

$$\oint_{C'} \mathcal{F} \, dx_1 = -\int_A \frac{\partial \mathcal{F}}{\partial x_2} dA \qquad \text{but} \quad \underbrace{\frac{\partial}{\partial x_2} \mathcal{F}(x_1, x_2)}_{} = \frac{\partial}{\partial x_2} f_1(x_1, x_2, x_3(x_1 x_2))$$

$$= -\int_A \left[\frac{\partial f_1}{\partial x_2} + \frac{\partial x_3}{\partial x_2} \frac{\partial f_1}{\partial x_3} \right] dA \quad \text{by the chain-rule}$$

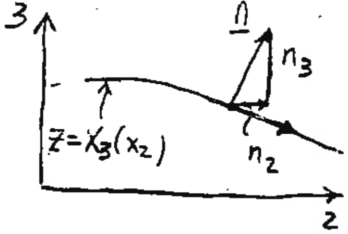
Now, a differential element dS in space is related to its projection in the xy-plane by

$$dA = n_3 \, dS$$

and furthermore one can show

i.e,

$$\frac{\partial x_3}{\partial x_2} = -\frac{n_2}{n_3}$$



so that we have

$$\oint_{C'} \mathcal{F} \, dx_1 = -\int_S \left[\frac{\partial f_1}{\partial x_2} n_3 - \frac{\partial f_1}{\partial x_3} n_2 \right] dS$$

or

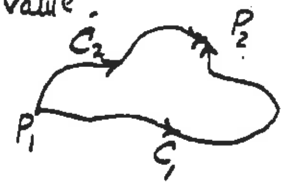
$$\oint_C f_1 \, dx_1 = \oint_C f_1 \, t_1 \, ds = \int_S \left(\frac{\partial f_1}{\partial x_3} n_2 - \frac{\partial f_1}{\partial x_2} n_3 \right) dS \quad \Leftarrow$$

which accounts for two of the terms in Stokes theorem. In a similar manner one can account for the other 2 terms. As in standard versions of these proofs, it is necessary to imagine that S is subdivided into sections which can be projected onto the xy-plane, even when the whole of S does not have a one-to-one projection.

8. Potentials for vector fields

a. Suppose that in some region $\underset{\sim}{\nabla} \wedge \underset{\sim}{f} = 0$

Then, from Stokes theorem, it immediately follows that all line integrals between any two points P_1 and P_2 have the same value.
In other words, $\underset{\sim}{\nabla} \wedge \underset{\sim}{f} = 0$, so by Stokes' Theorem,



$$\oint_C \underset{\sim}{f} \cdot \underset{\sim}{t}\, ds = 0$$

and since

$$\oint_C (\;)\, ds = \int_{C_1} (\;)\, ds - \int_{C_2} (\;)\, ds$$

minus sign since we've sketched the path C_2 as clockwise

such a vector field $\underset{\sim}{f}$ is called a CONSERVATIVE field

Clearly

$$\int_{C_1} \underset{\sim}{f} \cdot \underset{\sim}{t}\, ds = \int_{C_2} \underset{\sim}{f} \cdot \underset{\sim}{t}\, ds$$

← integral <u>independent</u> of path (C_1, C_2 are arbitrary paths connecting P_1, P_2).

b. Since this result is a SCALAR independent of path, we may define a function $\phi(\underset{\sim}{x})$ such that

$$\phi(\underset{\sim}{x}_{P_2}) = \phi(\underset{\sim}{x}_{P_1}) + \int_{P_1}^{P_2} \overbrace{\underset{\sim}{f} \cdot d\underset{\sim}{x}}^{d\phi} \qquad d\underset{\sim}{x} = \underset{\sim}{t}\, ds$$

$$\underbrace{\phi(\underset{\sim}{x}_{P_2}) - \phi(\underset{\sim}{x}_{P_1})}_{\phi(P_2)-\phi(P_1)} = \int_{P_1}^{P_2} d\phi$$

↑ some reference value at P_1

↑ represents a path integral

↑ differential vector element along path.

provided that $\underset{\sim}{\nabla} \wedge \underset{\sim}{f} = 0$.

Since the value of the integral is independent of path (i.e., only depends on the endpoints), then a differential change in ϕ is given by

$$d\phi = \underset{\sim}{f} \cdot d\underset{\sim}{x} \quad \text{or since} \quad d\phi = \underbrace{\frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3}_{\underset{\sim}{\nabla}\phi \cdot d\underset{\sim}{x}}$$

we identify

$$\boxed{\underset{\sim}{f} = \underset{\sim}{\nabla} \phi}$$

$\phi \equiv$ potential function

(Sometimes a minus sign is inserted for convenience, i.e, people write $\underset{\sim}{f} = -\underset{\sim}{\nabla}\phi$)

c. We also know that since $\underset{\sim}{\nabla} \wedge \underset{\sim}{\nabla}\phi = 0$ for any twice continuously differentiable function ϕ, then if $\underset{\sim}{f} = \underset{\sim}{\nabla}\phi$, we have $\underset{\sim}{\nabla} \wedge \underset{\sim}{f} = 0$.

A vector function $\underset{\sim}{f}$ such that $\underset{\sim}{\nabla} \wedge \underset{\sim}{f} = 0$ is called IRROTATIONAL.

d. Application: In the field of fluid dynamics, if $\underset{\sim}{v}(\underline{x})$ = fluid velocity at $\underline{x}$, the line integral

$$\oint_C \underline{v} \cdot d\underline{x}$$

is called the CIRCULATION about C.

Crudely, it provides a measure of the net rotation or 'circular motion' that occurs around some curve C.

By Stokes theorem

$$\oint_C \underline{v} \cdot d\underline{x} = \int_S \underline{n} \cdot (\underline{\nabla} \wedge \underset{\sim}{v})\, dS = \int_S \underline{n} \cdot \underset{\sim}{\omega}\, dS$$

where $\underset{\sim}{\omega} = \underline{\nabla} \wedge \underline{v}$ ← called the VORTICITY vector

and

where S is any surface with C as a bounding edge.

Thus if the fluid is everywhere IRROTATIONAL $\underset{\sim}{\omega} = \underline{\nabla} \wedge \underline{v} = 0$, then the circulation is zero for all curves C.

Conversely, suppose it is known that the circulation is known to be zero for all curves C. Then $\int_S \underline{n} \cdot (\underline{\nabla} \wedge \underline{v})\, dS = 0$

for all S. Since this true for all possible S so that it must be true that $\underline{n} \cdot (\underline{\nabla} \wedge \underline{v}) = 0$. But since the direction of $\underline{n}$ is arbitrary given all possible surface, one has

$$\underline{\nabla} \wedge \underline{v} = 0 \quad \text{everywhere.}$$

Reading: Line & surface integrals : Hildebrand Sections 6.10 - 6.16
Greenberg Sections 9.1 - 9.6

9. Theoretical Applications of Integral Theorems

a. Derivation of the DIFFUSION EQN

Let's study a statement of <u>conservation of energy</u> for a heated material. We treat the material as <u>CONTINUOUS</u>. In other words, we are concerned with length scales much larger than atomic distances.

Let $T(\underset{\sim}{x}, t)$ = temperature at $\underset{\sim}{x}$ at time t

$\underset{\sim}{q} \equiv$ "heat flux vector" [has units: energy/(area · time)]

→ Fourier's law of heat conduction $\underset{\sim}{q} = -k \underset{\sim}{\nabla} T$ $k \equiv$ thermal conductivity ($k > 0$) (heat flows from high → low temperatures)

let $\rho \equiv$ density (mass/volume)

$C_p \equiv$ heat capacity per unit mass $\left(\frac{\text{energy} \leftarrow \text{calories}}{\text{mass} \cdot \text{degree}} \right)$

Now consider an arbitrary fixed volume in the body.

The principle of conservation of energy states that (no sources of energy in V)

$$\left\{ \begin{matrix} \text{time-rate-of-change} \\ \text{of the total energy} \\ \text{in } V \end{matrix} \right\} = \left\{ \begin{matrix} \text{rate of energy} \\ \text{input into} \\ V \text{ across } S \end{matrix} \right\} - \left\{ \begin{matrix} \text{rate of energy} \\ \text{outflow from} \\ V \text{ across } S \end{matrix} \right\}$$

or

$$\int_V \rho C_p \frac{\partial T}{\partial t} \, dV = \int_S \left(-\underline{q} \cdot \underline{n} \right) dS$$

↳ $\underline{q} \cdot \underline{n} < 0$ if $\underline{\underline{\text{input}}}$ since $\underline{n}$ is <u>outward</u> normal

$$= -\int_V \underline{\nabla} \cdot \underline{q} \, dV \quad \text{by the Divergence Theorem}$$

o.k. since $\underline{q} = -k \nabla T$

or $\underline{q} \cdot \underline{n} = -k \frac{\partial T}{\partial n}$; if $\frac{\partial T}{\partial n} > 0$ then heat $\underline{\text{input}}$

so, $-\underline{q} \cdot \underline{n}$ is heat input

or

$$\int_V \left(\rho C_p \frac{\partial T}{\partial t} + \underline{\nabla} \cdot \underline{q} \right) dV = 0$$

This must be true for any arbitrary volume element V in the body. This requires that the integrand be point-wise zero; i.e.,

$$\rho C_p \frac{\partial T}{\partial t} + \underline{\nabla} \cdot \underline{q} = 0 \quad \longrightarrow \quad \rho C_p \frac{\partial T}{\partial t} = -\underline{\nabla} \cdot \underline{q}$$

or using Fourier's law, $\underline{q} = -k \nabla T$, (and assuming k = constant, independent of temperature)

$$\rho C_p \frac{\partial T}{\partial t} = k \underline{\nabla} \cdot (\underline{\nabla} T) \quad \text{or} \quad \boxed{\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \nabla^2 T}$$

HEAT or DIFFUSION EQN

b. Derivation of Maxwell's eqn using Stokes Theorem

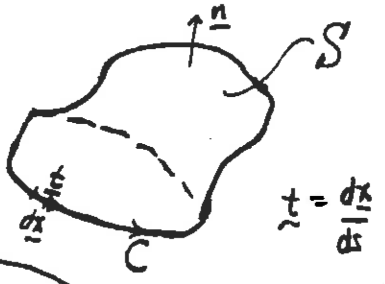
Recall from physics that Faraday (~1840) discovered that electric fields can be induced by changing magnetic fields.

It is often stated in words something like

$$\left\{\begin{array}{c}\text{time-rate-of-change}\\ \text{of the magnetic flux}\\ \text{across a surface } S\end{array}\right\} = \left\{\begin{array}{l}\text{electromotive force (emf),}\\ \text{in the circuit surrounding the}\\ \text{surface } S\text{ ; recall that the}\\ \text{emf is basically the total tangential}\\ \text{force exerted on a charge around}\\ \text{the loop or the tangential force}\\ \text{per unit charge } (= \underset{\sim}{E}\cdot\underset{\sim}{t}) \text{ integrated around}\\ \text{the circuit}\end{array}\right\}$$

$\underset{\sim}{E} =$ electric field $\underset{\sim}{E}(\underset{\sim}{x},t)$ $\quad$ $\underset{\sim}{B} =$ magnetic field $\underset{\sim}{B}(\underset{\sim}{x},t)$

or, mathematically,



$$\frac{d}{dt}\int_S \underset{\sim}{n}\cdot\underset{\sim}{B}\; dS = -\oint_C \underset{\sim}{E}\cdot d\underset{\sim}{x}$$

or by Stokes Theorem

$$= -\int_S \underset{\sim}{n}\cdot(\underset{\sim}{\nabla}\wedge\underset{\sim}{E})\,dS$$

NOTE: the negative sign is introduced to follow LENZ'S LAW

If we assume S to be an arbitrary fixed surface

then because it is fixed the order the differentiation and integration may be interchanged so that we have

$$\int_S \underset{\sim}{n}\cdot\underbrace{\left(\frac{\partial \underset{\sim}{B}}{\partial t} + \underset{\sim}{\nabla}\wedge\underset{\sim}{E}\right)}_{\text{"integrand"}} dS = 0$$

and this must be true for all surfaces S and all possible $\underset{\sim}{n}$.

Hence, one concludes $\boxed{\underset{\sim}{\nabla}\wedge\underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}}$

Also, the electric field in a medium depends on the charge distribution according to

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{E} = \rho/\varepsilon_0$$

where $\rho =$ charge density (charge/volume)

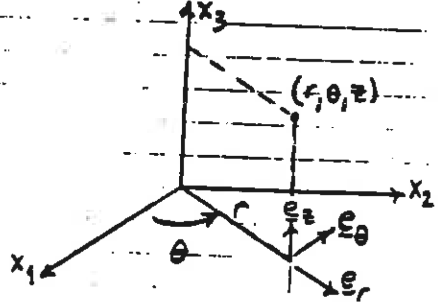
$\varepsilon_0 =$ permittivity of free space.

remark: be careful to note how r has different meanings below. But in each context, the meaning should be clear.

Reading: Greenberg, Section 8.2, 8.4, 8.5, 9.7
Hildebrand, Sections 6.17, 6.18

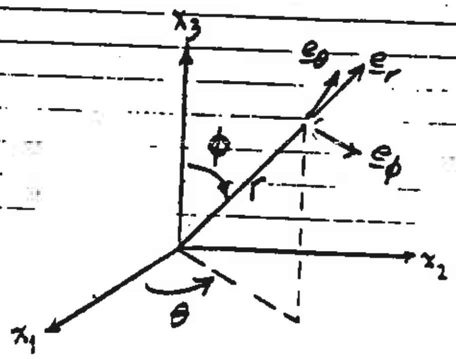
E. Vector Calculus Using Orthogonal Curvilinear Coordinates

1. So far we have only discussed the description of vectors and vector calculus relative to a cartesian coordinate system. We now wish to extend these ideas to other orthogonal coordinate systems. The two most common situations to keep in mind:



cylindrical coordinates

Note: $x_1 = r\cos\theta$
$x_2 = r\sin\theta$
$x_3 = z$



spherical coordinates

$x_1 = r\sin\phi\cos\theta$
$x_2 = r\sin\phi\sin\theta$
$x_3 = r\cos\phi$

Note: Some texts interchange role of θ, ϕ.

We see that in each of these coordinate systems the unit base vectors $(\underline{e}_r, \underline{e}_\theta, \underline{e}_z)$ or $(\underline{e}_r, \underline{e}_\theta, \underline{e}_\phi)$ vary with position in space, i.e., they are functions of spatial position and this will mean that some care will be necessary when we speak of differentiation in these coordinate systems.

2. What do we really mean by a base vector?

The unit base vector is in the direction associated with a small increment in the coordinate.

For example, consider cylindrical coordinates: $\underline{X} = r\cos\theta\,\underline{e}_1 + r\sin\theta\,\underline{e}_2 + z\,\underline{e}_3$

Then

unit length $\rightarrow$ $$\underline{e}_r = \frac{\partial\underline{X}/\partial r}{\left|\partial\underline{X}/\partial r\right|} = \frac{\cos\theta\,\underline{e}_1 + \sin\theta\,\underline{e}_2}{\sqrt{\cos^2\theta + \sin^2\theta}} \Rightarrow \underline{e}_r = \cos\theta\,\underline{e}_1 + \sin\theta\,\underline{e}_2$$

to normalize vector

Exercise: Find $\underline{e}_r, \underline{e}_\theta, \underline{e}_\phi$ in spherical coordinates

$$\underline{e}_\theta = \frac{\partial\underline{X}/\partial\theta}{\left|\partial\underline{X}/\partial\theta\right|} = \frac{-r\sin\theta\,\underline{e}_1 + r\cos\theta\,\underline{e}_2}{\sqrt{r^2(\cos^2\theta + \sin^2\theta}} \Rightarrow \underline{e}_\theta = -\sin\theta\,\underline{e}_1 + \cos\theta\,\underline{e}_2$$

Also clearly, $\underline{e}_z = \underline{e}_3$

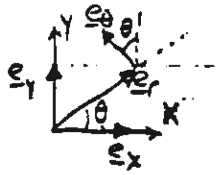
NOTICE: The base vectors are functions of position but are mutually orthogonal. Such coordinate systems are called ORTHOGONAL CURVILINEAR COORDINATES.

3. Now, it is possible to give a very careful and complete discussion of vector operations for a general orthogonal curvilinear coordinate system.

In order to at least see clearly what the idea is we first consider in detail vector operations in cylindrical coordinates.

a. $\underset{\sim}{\nabla}$ in cylindrical coordinates

1. Perhaps the first way one would think to proceed is to make a change of variables from (x,y,z) to (r,θ,z). We will do this now, but it is somewhat longwinded.



- Begin with $$\underset{\sim}{\nabla} = \underset{\sim}{e}_x \frac{\partial}{\partial x} + \underset{\sim}{e}_y \frac{\partial}{\partial y} + \underset{\sim}{e}_z \frac{\partial}{\partial z} \qquad (1)$$

From the previous page $\underset{\sim}{e}_r = \cos\theta\, \underset{\sim}{e}_x + \sin\theta\, \underset{\sim}{e}_y$; $\underset{\sim}{e}_\theta = -\sin\theta\, \underset{\sim}{e}_x + \cos\theta\, \underset{\sim}{e}_y$,

figure also can be used to verify the eqns relating base vectors

we see

$$\begin{aligned} \underset{\sim}{e}_x &= \cos\theta\, \underset{\sim}{e}_r - \sin\theta\, \underset{\sim}{e}_\theta \\ \underset{\sim}{e}_y &= \sin\theta\, \underset{\sim}{e}_r + \cos\theta\, \underset{\sim}{e}_\theta \end{aligned} \qquad (2)$$

- Then, by the chain rule $(x = r\cos\theta \quad y = r\sin\theta)$

$x(r,\theta)$
$y(r,\theta)$ $\Rightarrow$

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \end{aligned} \qquad (3)$$

where we have used $r = [x^2+y^2]^{1/2}$, $\theta = \tan^{-1}\frac{y}{x}$

to find $\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r} \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}$$

- Substituting (2) & (3) into (1) yields

$$\underset{\sim}{\nabla} = \underset{\sim}{e}_r \frac{\partial}{\partial r} + \underset{\sim}{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \underset{\sim}{e}_z \frac{\partial}{\partial z}$$

NOTE: It is VERY common for us to forget the $\frac{1}{r}$ in front of $\frac{\partial}{\partial \theta}$. Because $\underset{\sim}{\nabla}$ has units 1/length, each term must also have dimensions of 1/length.

2. Now, you might wish to bypass some algebra so an alternative is to first note that we expect

$$\underset{\sim}{\nabla} = \underset{\sim}{e}_r \, h_r(r,\theta) \frac{\partial}{\partial r} + \underset{\sim}{e}_\theta \, h_\theta(r,\theta) \frac{\partial}{\partial \theta} + \underset{\sim}{e}_z \frac{\partial}{\partial z}$$

find the functions $h_r(r,\theta)$ and $h_\theta(r,\theta)$.

Since $\underset{\sim}{e}_r, \underset{\sim}{e}_\theta, \underset{\sim}{e}_z$ are mutually orthogonal, then

$$\underset{\sim}{e}_r \cdot \underset{\sim}{\nabla} = h_r(r,\theta) \frac{\partial}{\partial r} \quad \text{and} \quad \underset{\sim}{e}_\theta \cdot \underset{\sim}{\nabla} = h_\theta(r,\theta) \frac{\partial}{\partial \theta}$$

From pg. 88, $\underset{\sim}{e}_r$ is defined as $\frac{\partial \underset{\sim}{x}}{\partial r} \Big/ \left| \frac{\partial \underset{\sim}{x}}{\partial r} \right|$

so that

$$h_r = \underset{\sim}{e}_r \cdot \underset{\sim}{\nabla} = \frac{1}{\left|\frac{\partial \underset{\sim}{x}}{\partial r}\right|} \underbrace{\frac{\partial \underset{\sim}{x}}{\partial r} \cdot \underset{\sim}{\nabla}}_{\partial/\partial r} = \frac{\partial}{\partial r} \quad \text{since} \quad \left|\frac{\partial \underset{\sim}{x}}{\partial r}\right| = 1 \quad (p. 88)$$

$$\rightsquigarrow \frac{\partial \underset{\sim}{x}}{\partial r} \cdot \underset{\sim}{\nabla} = \underset{\sim}{e}_i \frac{\partial x_i}{\partial r} \cdot \underset{\sim}{e}_j \frac{\partial}{\partial x_j} = \frac{\cancel{\partial x_i}}{\partial r} \frac{\partial}{\cancel{\partial x_i}} = \frac{\partial}{\partial r}$$

Also,

$$h_\theta = \underset{\sim}{e}_\theta \cdot \underset{\sim}{\nabla} = \frac{1}{\left|\frac{\partial \underset{\sim}{x}}{\partial \theta}\right|} \underbrace{\frac{\partial \underset{\sim}{x}}{\partial \theta} \cdot \underset{\sim}{\nabla}}_{\partial/\partial \theta} = \frac{1}{\left|\frac{\partial \underset{\sim}{x}}{\partial \theta}\right|} \frac{\partial}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial \theta}$$

from pg. 88

⇝ This was much simpler.

3. Important point: $\underset{\sim}{\nabla}$ is now known in cylindrical coords

$$\underset{\sim}{\nabla} = \underset{\sim}{e}_r \frac{\partial}{\partial r} + \underset{\sim}{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \underset{\sim}{e}_z \frac{\partial}{\partial z}$$

a. example: Suppose $\Psi = \Psi(\theta, z)$ only, independent or r.

Then

$$\underset{\sim}{\nabla} \phi = \underset{\sim}{e}_\theta \frac{1}{r} \frac{\partial \Psi}{\partial \theta} + \underset{\sim}{e}_z \frac{\partial \Psi}{\partial z}$$

and there is no r-coordinate.

4. Divergence in cylindrical coordinates

Now, lets calculate $\underline{\nabla}\cdot\underline{f}$.

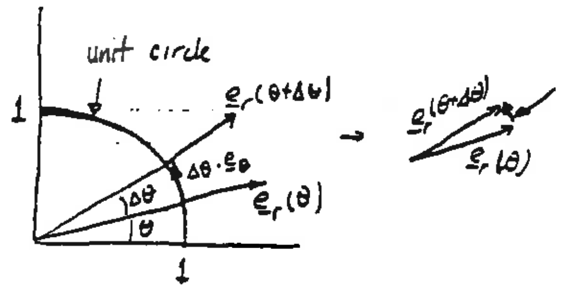
a. In cylindrical coordinates, we write

$$\underline{f}(r,\theta,z) = f_r(r,\theta,z)\,\underline{e}_r + f_\theta(r,\theta,z)\,\underline{e}_\theta + f_z(r,\theta,z)\,\underline{e}_z$$

each of the components may be a function of position

b. Formally take $\underline{\nabla}\cdot\underline{f}$ and REMEMBER that in cylindrical coordinates the unit vectors $\underline{e}_r$, $\underline{e}_\theta$ are <u>NOT</u> constant vectors but vary with θ.

NOTE: $\underline{e}_r(\theta)$; $\underline{e}_\theta(\theta)$.



$$\underline{e}_r(\theta+\Delta\theta) - \underline{e}_r(\theta) \approx \Delta\theta\ \underline{e}_\theta$$

$$\therefore \quad \boxed{\frac{\partial \underline{e}_r}{\partial\theta} = \underline{e}_\theta}$$

Similarly, geometrically show $\boxed{\frac{\partial \underline{e}_\theta}{\partial\theta} = -\underline{e}_r}$

Both these statements can be demonstrated directly from the eqns on p. 88.

c So, the calculation: proceed term-by-term - remember to take $\frac{\partial}{\partial\theta}$ of the unit vectors.

depend on θ

$$\underline{\nabla}\cdot\underline{f} = \left(\underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \underline{e}_z \frac{\partial}{\partial z}\right)\cdot\left(f_r\,\underline{e}_r + f_\theta\,\underline{e}_\theta + f_z\,\underline{e}_z\right)$$

$$= \underline{e}_r\cdot\frac{\partial}{\partial r}(f_r\underline{e}_r) + \underline{e}_r\cdot\frac{\partial}{\partial r}(f_\theta\underline{e}_\theta) + \cdots$$

↑ independent of r

$$= \frac{\partial f_r}{\partial r} + \underline{e}_\theta\cdot\frac{1}{r}\frac{\partial}{\partial\theta}\left(f_r\underline{e}_r + f_\theta\underline{e}_\theta\right) + \frac{\partial f_z}{\partial z}$$

(by inspection, these are only terms that possible survive the inner products)

$$= \frac{\partial f_r}{\partial r} + \underline{e}_\theta\cdot\underline{e}_r\frac{1}{r}\frac{\partial f_r}{\partial\theta} + \underbrace{\underline{e}_\theta\cdot\overset{\underline{e}_\theta}{\overset{\parallel}{\frac{\partial\underline{e}_r}{\partial\theta}}}}_{1}\frac{f_r}{r} + \underbrace{\underline{e}_\theta\cdot\underline{e}_\theta}_{1}\frac{1}{r}\frac{\partial f_\theta}{\partial\theta} + \underbrace{\underline{e}_\theta\cdot\overset{-\underline{e}_r}{\overset{\parallel}{\frac{\partial\underline{e}_\theta}{\partial\theta}}}}_{0}\frac{f_\theta}{r} + \frac{\partial f_z}{\partial z}$$

$$\therefore \quad \underline{\nabla}\cdot\underline{f} = \frac{\partial f_r}{\partial r} + \frac{f_r}{r} + \frac{1}{r}\frac{\partial f_r}{\partial\theta} + \frac{\partial f_z}{\partial z} \rightarrow$$

General result,

$$\boxed{\underline{\nabla}\cdot\underline{f} = \frac{1}{r}\frac{\partial}{\partial r}(rf_r) + \frac{1}{r}\frac{\partial f_\theta}{\partial\theta} + \frac{\partial f_z}{\partial z}}$$

NOTE 1
You should work through all these steps.

NOTE 2
Every calculation $\nabla^2\underline{f}$, $\underline{\nabla}\times\underline{f}$, would proceed this way

The discussion which follows on the next few pages is very general and closely follows Hildebrand. Try to understand the basic ideas involved — for example, try to think about this geometrically

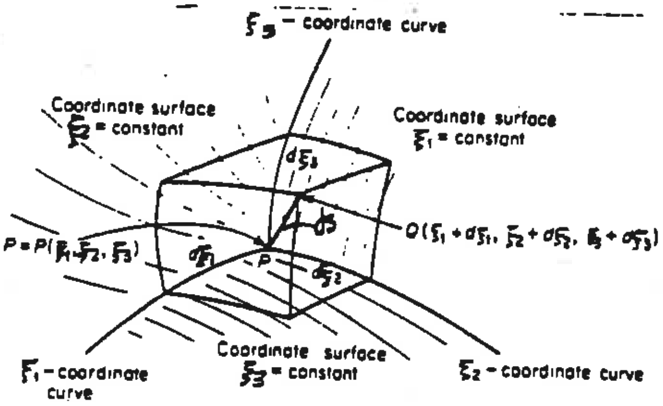
It is probably NOT wise attempting to memorize any of this.

5. Let's proceed in a more general manner for any ORTHOGONAL CURVILINEAR COORDINATE SYSTEM.

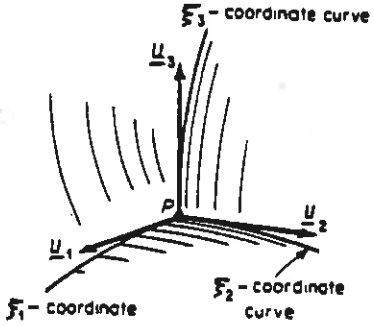
a. Let ξ_i $i = 1, 2, 3$ be a general set of orthogonal coordinates.

$x_i(\xi_j)$ or $x = (\xi_1, \xi_2, \xi_3)$

Every point in space may be specified by giving the coordinates (ξ_1, ξ_2, ξ_3). If we hold ξ_1, ξ_2 fixed and vary ξ_3 we sweep out the 'ξ_3 = curve'. (similar statements for ξ_1-curves and ξ_2-curves) We will only discuss the case where the coordinate lines (curves) are mutually orthogonal at each point in space.



Curvilinear coordinates.



(base) Unit tangent vectors.

b. Now introduce UNIT BASE VECTORS, $\underset{\sim}{u}_i$ $i = 1, 2, 3$. (right-handed)

By definition, the base vector in the i-direction indicates the direction of change in the ξ_i-coordinate, holding the other 2 coordinates fixed.

let $\underset{\sim}{x}(\xi_i)$ denote the position vector associated with the ξ_i coordinate ($i = 1, 2$ or 3)

Then, by definition,

divide by length of $\partial \underset{\sim}{x} / \partial \xi_i$ so $\underset{\sim}{u}_i$ is of unit length. →

$$\underset{\sim}{u}_i = \frac{\partial \underset{\sim}{x} / \partial \xi_i}{|\partial \underset{\sim}{x} / \partial \xi_i|} \qquad i = 1, 2 \text{ or } 3$$

NOTE: Do not sum here.

let ds = distance from pt ξ_i to $\xi_i + d\xi_i$ (or $\overline{PQ}$ in figure above)

so,

$$ds^2 = d\underset{\sim}{x} \cdot d\underset{\sim}{x} = \left(\frac{\partial \underset{\sim}{x}}{\partial \xi_j}\right) d\xi_j \cdot \frac{\partial \underset{\sim}{x}}{\partial \xi_k} d\xi_k$$

We will require that the unit base vectors be ORTHOGONAL: $\frac{\partial \underset{\sim}{x}}{\partial \xi_j} \cdot \frac{\partial \underset{\sim}{x}}{\partial \xi_k} = 0 \quad j \neq k$

$$\Rightarrow \quad ds^2 = \left|\frac{\partial \underset{\sim}{x}}{\partial \xi_1}\right|^2 d\xi_1^2 + \left|\frac{\partial \underset{\sim}{x}}{\partial \xi_2}\right|^2 d\xi_2^2 + \left|\frac{\partial \underset{\sim}{x}}{\partial \xi_3}\right|^2 d\xi_3^2$$

DEFINE: $\boxed{h_i = |\partial \underset{\sim}{x} / \partial \xi_i| \qquad i = 1, 2, 3}$ ← METRIC COEFFICIENTS or SCALE FACTORS

NOTE: $\boxed{\partial \underset{\sim}{x} / \partial \xi_i = h_i \underset{\sim}{u}_i}$ (No sum!) so h_i provide a measure of distance along ξ_i-curve

$$\therefore \boxed{ds^2 = h_1^2 d\xi_1^2 + h_2^2 d\xi_2^2 + h_3^2 d\xi_3^2 \equiv h_i^2 d\xi_i^2}$$

where now we've used summation convention; but be careful.

remark: The convention used here for defining the scale factors h_i is not universal. Some books define the scale factors as the inverse of the definition here.

c. As an example, consider cylindrical coordinates ($\xi_1 = r$, $\xi_2 = \theta$, $\xi_3 = z$)

position vector: $\underset{\sim}{x} = r\cos\theta \; \underset{\sim}{e}_1 + r\sin\theta \, \underset{\sim}{e}_2 + z \, \underset{\sim}{e}_3$

$$h_r = h_1 = \left| \frac{\partial \underset{\sim}{x}}{\partial r} \right| = \left| \cos\theta \; \underset{\sim}{e}_1 + \sin\theta \; \underset{\sim}{e}_2 \right| = 1$$

$$h_\theta = h_2 = \left| \frac{\partial \underset{\sim}{x}}{\partial \theta} \right| = \left| -r\sin\theta \; \underset{\sim}{e}_1 + r\cos\theta \; \underset{\sim}{e}_2 \right| = r$$

$$h_z = h_3 = \left| \frac{\partial \underset{\sim}{x}}{\partial z} \right| = 1$$

It follows that the square of a differential displacement, $(ds)^2$, is given by

$$(ds)^2 = dr^2 + r^2 d\theta^2 + dz^2$$

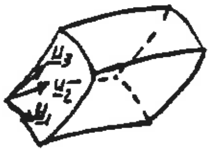
d. volume elements

A small element of volume is formed by small displacements along the ξ_1, ξ_2, and ξ_3 curves. Recall that the metric coefficients h_i provide a measure of length or distance along the ξ_i curve.

dV is a differential element of volume. It is the parallelepiped formed by the three orthogonal vector displacements

$$\frac{\partial \underset{\sim}{x}}{\partial \xi_1} d\xi_1 \, , \qquad \frac{\partial \underset{\sim}{x}}{\partial \xi_2} d\xi_2 \, , \qquad \frac{\partial \underset{\sim}{x}}{\partial \xi_3} d\xi_3$$

or $\quad h_1 \, d\xi_1 \, \underset{\sim}{u}_1 \qquad h_2 \, d\xi_2 \, \underset{\sim}{u}_2 \qquad h_3 \, d\xi_3 \, \underset{\sim}{u}_3$



so that the element of volume follows from

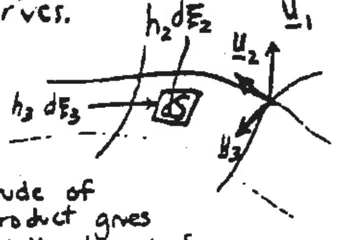
$$dV = \left(h_1 \, d\xi_1 \, \underset{\sim}{u}_1 \wedge h_2 \, d\xi_2 \, \underset{\sim}{u}_2 \right) \cdot h_3 \, d\xi_3 \, \underset{\sim}{u}_3$$

$$\boxed{dV = h_1 \, h_2 \, h_3 \; d\xi_1 \; d\xi_2 \; d\xi_3}$$

($\underset{\sim}{u}_i$ are orthogonal unit vectors)

e. surface elements (similar to above discussion)

A differential surface element is described by differential displacements along two of the coordinate curves.



For example, if we consider the surface on which $\xi_1 =$ constant ; $\underset{\sim}{u}_1 \perp S$

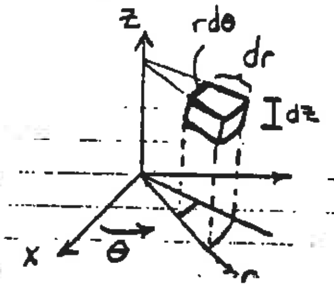
Then

$$dS = \left| h_2 \, d\xi_2 \, \underset{\sim}{u}_2 \wedge h_3 \, d\xi_3 \, \underset{\sim}{u}_3 \right| \quad \leftarrow$$

$$= h_2 h_3 \; d\xi_2 \; d\xi_3$$

magnitude of cross product gives differential element of area in a plane tangent to the surface.

Example : Again use cylindrical coordinates



$$dV = r\,dr\,d\theta\,dz$$
$$dS = r\,dr\,d\theta \quad \text{on } z = \text{constant}$$
$$dS = dr\,dz \quad \text{on } \theta = \text{constant}$$
$$dS = r\,d\theta\,dz \quad \text{on } r = \text{constant}$$

f. The representation of a vector – we can represent any vector by giving its components and unit vectors.

So,

$$\underset{\sim}{f} = f_1^{(e)}\underline{e}_1 + f_2^{(e)}\underline{e}_2 + f_3^{(e)}\underline{e}_3$$

or

$$\underset{\sim}{f} = f_1^{(u)}\underline{u}_1 + f_2^{(u)}\underline{u}_2 + f_3^{(u)}\underline{e}_3$$

where $^{(e)}$ and $^{(u)}$ remind us of which basis the components refer.

NOTE : in general, $f_i^{(u)} \neq f_i^{(e)}$

and given one of these representation the components may be calculated via

$$f_i^{(e)} = \underset{\sim}{f}\cdot\underline{e}_i \quad , \quad f_i^{(u)} = \underset{\sim}{f}\cdot\underline{u}_i$$

(we will generally drop the superscripts e and u since the meaning should be clear from the context)

It is important to keep in mind that the vector $\underset{\sim}{f}$ is the same. It is only the representations of the vector (or, equivalently, its description via components and unit vectors) which differ.

6. Gradient of a scalar in curvilinear coordinates

Let us define $\underline{\nabla}\phi$ via

$$d\phi = \underline{\nabla}\phi\cdot d\underset{\sim}{x} \longleftarrow \text{differential vector displacement}$$

It makes sense to think of $\underline{\nabla}\phi$ as

$$\underline{\nabla}\phi = \lambda_1\underline{u}_1 + \lambda_2\underline{u}_2 + \lambda_3\underline{u}_3 \Rightarrow \text{Now, what are the } \lambda\text{'s ?}$$

Well,

$$d\underset{\sim}{x} = \frac{\partial\underline{x}}{\partial\xi_1}d\xi_1 + \frac{\partial\underline{x}}{\partial\xi_2}d\xi_2 + \frac{\partial\underline{x}}{\partial\xi_3}d\xi_3 = h_1 d\xi_1\,\underline{u}_1 + h_2 d\xi_2\,\underline{u}_2 + h_3 d\xi_3\,\underline{u}_3$$

So,

$$d\phi = \underline{\nabla}\phi\cdot d\underline{x} = \lambda_1 h_1 d\xi_1 + \lambda_2 h_2 d\xi_2 + \lambda_3 h_3 d\xi_3 \overset{\boxed{\text{or}}}{=} \frac{\partial\phi}{\partial\xi_1}d\xi_1 + \frac{\partial\phi}{\partial\xi_2}d\xi_2 + \frac{\partial\phi}{\partial\xi_3}d\xi_3$$

Hence :

$$\lambda_1 = \frac{1}{h_1}\frac{\partial\phi}{\partial\xi_1} \qquad \lambda_2 = \frac{1}{h_2}\frac{\partial\phi}{\partial\xi_2} \qquad \lambda_3 = \frac{1}{h_3}\frac{\partial\phi}{\partial\xi_3}$$

Therefore, in general, the gradient may be represented as

$$\boxed{\underline{\nabla}\phi = \underline{u}_1\frac{1}{h_1}\frac{\partial\phi}{\partial\xi_1} + \underline{u}_2\frac{1}{h_2}\frac{\partial\phi}{\partial\xi_2} + \underline{u}_3\frac{1}{h_3}\frac{\partial\phi}{\partial\xi_3}} \qquad (1)$$

Example: Gradient of a scalar function in cylindrical coordinates

$$, \quad \underset{\sim}{\nabla}\phi = \underline{e}_r \frac{\partial \phi}{\partial r} + \underline{e}_\theta \frac{1}{r}\frac{\partial \phi}{\partial \theta} + \underline{e}_z \frac{\partial \phi}{\partial z}$$

It is very useful to notice that $\underline{\nabla}\phi$ represents a vector function whose components represent derivatives with respect to distance in a certain direction. Hence, in each component we must basically be dividing by a length. In the above, the r, z components clearly represent derivatives with respect to distance and in the θ-direction the distance is "$r\, d\theta$".

In other words, it is useful to think of $\underset{\sim}{\nabla}\phi$ as

$$\nabla\phi = \underset{\sim}{u}_1 \frac{\partial \phi}{\partial s_1} + \underline{u}_2 \frac{\partial \phi}{\partial s_2} + \underline{u}_3 \frac{\partial \phi}{\partial s_3}$$

where $ds_1 = h_1\, d\xi_1$ is the distance along the ξ_1 curve for $d\xi_1 > 0$ $(d\xi_2 = d\xi_3 = 0)$ and similarly for ds_2 and ds_3.

7. Divergence of a Vector function $\underset{\sim}{\nabla}\cdot \underset{\sim}{f}$ in curvilinear coordinates

a. As a preliminary, we notice the following:

Take $\phi = \xi_i$ in eqn (1) in pg $\overset{94}{\cancel{96}}$, where ξ_i is any one of the coordinates, then

$$\boxed{\underset{\sim}{\nabla}\xi_i = \frac{\underset{\sim}{u}_i}{h_i}}$$

for the i^{th} coordinate (DO NOT USE THE SUMMATION CONVENTION HERE)

Next, for this right-handed coordinate system,

$$\underset{\sim}{u}_1 = \underline{u}_2 \wedge \underline{u}_3 \implies \frac{\underline{u}_1}{h_2 h_3} = \frac{\underline{u}_2}{h_2} \wedge \frac{\underline{u}_3}{h_3} = \underset{\sim}{\nabla}\xi_2 \wedge \underset{\sim}{\nabla}\xi_3$$

IDENTITY: $\underset{\sim}{\nabla}\cdot(\underline{\nabla}\phi_1 \wedge \underline{\nabla}\phi_2) \equiv 0$ for any twice continuously differentiable functions ϕ_1, ϕ_2

Exercise: prove this identity.

So, clearly

$$\nabla\cdot(\nabla\xi_2 \wedge \underline{\nabla}\xi_3) = 0 \xrightarrow{or} \underline{\nabla}\cdot\left(\frac{\underline{u}_1}{h_2 h_3}\right) = 0$$

Likewise,

$$\underline{\nabla}\cdot\left(\frac{\underline{u}_2}{h_1 h_3}\right) = 0 \qquad \underline{\nabla}\cdot\left(\frac{\underline{u}_3}{h_1 h_2}\right) = 0.$$

b. Now consider $\underline{\nabla}\cdot\underline{f}$.

In general, $\underline{f} = f_1\,\underline{u}_1 + f_2\,\underline{u}_2 + f_3\,\underline{u}_3$
so, we can write

$$\underline{\nabla}\cdot\underline{f} = \underline{\nabla}\cdot(f_1\underline{u}_1) + \underline{\nabla}\cdot(f_2\,\underline{u}_2) + \underline{\nabla}\cdot(f_3\,\underline{u}_3)$$

$$= \underline{\nabla}\cdot\left(h_2h_3f_1\left(\frac{\underline{u}_1}{h_2h_3}\right)\right) + \underline{\nabla}\cdot\left(h_1h_3\,f_2\left(\frac{\underline{u}_2}{h_1h_3}\right)\right) + \underline{\nabla}\cdot\left(h_1h_2f_3\left(\frac{\underline{u}_3}{h_1h_2}\right)\right).$$

But, $\underline{\nabla} = \frac{\underline{u}_1}{h_1}\frac{\partial}{\partial\xi_1} + \frac{\underline{u}_2}{h_2}\frac{\partial}{\partial\xi_2} + \frac{\underline{u}_3}{h_3}\frac{\partial}{\partial\xi_3}$

so that, for example,

$$\underline{\nabla}\cdot\left[h_2h_3f_1\left(\frac{\underline{u}_1}{h_2h_3}\right)\right] = \frac{\underline{u}_1}{h_2h_3}\cdot\underline{\nabla}(h_2h_3f_1) + h_2h_3\,f_1\,\cancelto{0}{\underline{\nabla}\cdot\left(\frac{\underline{u}_1}{h_2h_3}\right)}$$

$$= \frac{1}{h_1h_2h_3}\frac{\partial}{\partial\xi_1}(h_2h_3\,f_1)$$

.... similarly for the other terms

$$\therefore \boxed{\underline{\nabla}\cdot\underline{f} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi_1}(h_2h_3f_1) + \frac{\partial}{\partial\xi_2}(h_1h_3\,f_2) + \frac{\partial}{\partial\xi_3}(h_1h_2f_3)\right]} \quad (1)$$

Example : cylindrical coordinates $h_1 = h_r = 1$ $h_2 = h_\theta = r$ $h_3 = h_z = 1$

$$\underline{f} = f_r\,\underline{e}_r + f_\theta\,\underline{e}_\theta + f_z\,\underline{e}_z \Rightarrow \underline{\nabla}\cdot\underline{f} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rf_r) + \frac{\partial f_\theta}{\partial\theta} + \frac{\partial}{\partial z}(rf_z)\right]$$

Hence,

$$\underline{\nabla}\cdot\underline{f} = \frac{1}{r}\frac{\partial}{\partial r}(rf_r) + \frac{1}{r}\frac{\partial f_\theta}{\partial\theta} + \frac{\partial f_z}{\partial z}$$

Example : spherical coordinates (r,ϕ,θ) $\underline{f} = f_r\underline{e}_r + f_\phi\underline{e}_\phi + f_\theta\,\underline{e}_\theta$

$h_1 = h_r = 1$ $h_2 = h_\phi = r$ $h_3 = h_\theta = r\sin\phi$

$$\underline{\nabla}\cdot\underline{f} = \frac{1}{r^2\sin\phi}\left[\frac{\partial}{\partial r}(r^2\sin\phi\,f_r) + \frac{\partial}{\partial\phi}(r\sin\phi\,f_\phi) + \frac{\partial}{\partial\theta}(rf_\theta)\right]$$

or

$$\underline{\nabla}\cdot\underline{f} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2f_r) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\,f_\phi) + \frac{1}{r\sin\phi}\frac{\partial f_\theta}{\partial\theta}$$

8. The Laplacian $\nabla^2\phi = \underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\phi$

Well we know how to take $\underset{\sim}{\nabla}\cdot\underset{\sim}{f}$, so let $\underset{\sim}{f} = \underset{\sim}{\nabla}\phi$

$$\Rightarrow \underset{\sim}{f} = \underset{\sim}{\nabla}\phi = \underline{u}_1 \frac{1}{h_1}\frac{\partial\phi}{\partial\xi_1} + \underset{\sim}{u}_2 \frac{1}{h_2}\frac{\partial\phi}{\partial\xi_2} + \underset{\sim}{u}_3 \frac{1}{h_3}\frac{\partial\phi}{\partial\xi_3} = f_1\underline{u}_1 + f_2\underline{u}_2 + f_3\underline{u}_3$$

Hence, substituting into eqn (1) on p. 82,

$$\boxed{\nabla^2\phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial\xi_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\phi}{\partial\xi_1}\right) + \frac{\partial}{\partial\xi_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial\phi}{\partial\xi_2}\right) + \frac{\partial}{\partial\xi_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\phi}{\partial\xi_3}\right)\right]}$$

Example : cylindrical coordinates (r,θ,z) : $h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$

$$\Rightarrow \nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

9. The curl $\underset{\sim}{\nabla}\wedge\underset{\sim}{f}$

a. Once again, we begin with a useful preliminary

We know that $\underline{\nabla}\wedge\underline{\nabla}\phi = 0$ for any scalar function ϕ so certainly

$$\underline{\nabla}\wedge\underline{\nabla}\xi_i = 0 \quad \text{for } i = 1, 2 \text{ or } 3$$

(ξ_i is one of the coordinates)

But

$$\underline{\nabla}\xi_i = \frac{\underline{u}_i}{h_i} \quad i = 1, 2 \text{ or } 3 \quad (\text{do } \underline{\text{not}} \text{ sum}) \quad \rightarrow \text{see p. 81}$$

so that

$$\underline{\nabla}\wedge(\underline{u}_i/h_i) = 0 \quad \text{for } i = 1, 2 \text{ or } 3$$

b. It is now straightforward to calculate $\underline{\nabla}\wedge\underline{f}$ as shown below :

$$\underline{f} = f_1\underline{u}_1 + f_2\underline{u}_2 + f_3\underline{u}_3$$

For example, first consider $\underline{\nabla}\wedge(f_1\underline{u}_1)$

$$\underline{\nabla}\wedge(f_1\underline{u}_1) = \underline{\nabla}\wedge\left(h_1 f_1 \frac{\underline{u}_1}{h_1}\right) = \underline{\nabla}(h_1 f_1)\wedge\frac{\underline{u}_1}{h_1} + h_1 f_1 \cancelto{0}{\underline{\nabla}\wedge(\underline{u}_1/h_1)}$$

(we have made use of the fact that $\underline{\nabla}\wedge(\phi\underline{a}) = \underline{\nabla}\phi\wedge\underline{a} + \phi\,\underline{\nabla}\wedge\underline{a}$ — you should be able to prove this)

9. the curl (continued)

Making use of the general definition of $\underline{\nabla}$, (p. 80)

$$\underline{\nabla}\wedge(f_1\,\underline{u}_1) = \underline{\nabla}(h_1 f_1)\wedge\frac{\underline{u}_1}{h_1} = \left[\frac{\underline{u}_1}{h_1}\frac{\partial}{\partial\xi_1}(h_1 f_1) + \frac{\underline{u}_2}{h_2}\frac{\partial}{\partial\xi_2}(h_1 f_1) + \frac{\underline{u}_3}{h_3}\frac{\partial}{\partial\xi_3}(h_1 f_1)\right]\wedge\frac{\underline{u}_1}{h_1}$$

$$\Rightarrow\ \underline{\nabla}\wedge(f_1\,\underline{u}_1) = \underline{u}_2\frac{1}{h_1h_3}\frac{\partial}{\partial\xi_3}(h_1f_1) - \underline{u}_3\frac{1}{h_1h_2}\frac{\partial}{\partial\xi_2}(h_1f_1)$$

since $\underline{u}_2\wedge\underline{u}_1 = -\underline{u}_3$

$\underline{u}_3\wedge\underline{u}_1 = \underline{u}_2$

The other two terms of $\underline{\nabla}\wedge\underset{\sim}{f}$ are calculated in a similar manner. The general result:

$$\underline{\nabla}\wedge\underset{\sim}{f} = \underline{u}_1\frac{1}{h_2h_3}\left[\frac{\partial}{\partial\xi_2}(h_3f_3) - \frac{\partial}{\partial\xi_3}(h_2f_2)\right] + \underline{u}_2\frac{1}{h_1h_3}\left[\frac{\partial}{\partial\xi_3}(h_1f_1) - \frac{\partial}{\partial\xi_1}(h_3f_3)\right]$$
$$+\ \underline{u}_3\frac{1}{h_1h_2}\left[\frac{\partial}{\partial\xi_1}(h_2f_2) - \frac{\partial}{\partial\xi_2}(h_1f_1)\right]$$

which can be expressed as

$$\underline{\nabla}\wedge\underset{\sim}{f} = \frac{1}{h_1h_2h_3}\det\begin{vmatrix} h_1\,\underline{u}_1 & h_2\,\underline{u}_2 & h_3\,\underline{u}_3 \\ \frac{\partial}{\partial\xi_1} & \frac{\partial}{\partial\xi_2} & \frac{\partial}{\partial\xi_3} \\ h_1f_1 & h_2f_2 & h_3f_3 \end{vmatrix}$$

→ The special case of cylindrical coordinates (r,θ,z) is straightforward.

$h_1 = h_r = 1 \qquad h_2 = h_\theta = r \qquad h_3 = h_z = 1 \qquad \underset{\sim}{f} = f_r\,\underline{e}_r + f_\theta\,\underline{e}_\theta + f_z\,\underline{e}_z$

$$\therefore\ \underline{\nabla}\wedge\underset{\sim}{f} = \underline{e}_r\left[\frac{1}{r}\frac{\partial f_z}{\partial\theta} - \frac{\partial f_\theta}{\partial z}\right] + \underline{e}_\theta\left[\frac{\partial f_r}{\partial z} - \frac{\partial f_z}{\partial r}\right] + \underline{e}_z\left[\frac{\partial f_\theta}{\partial r} - \frac{1}{r}\frac{\partial f_r}{\partial\theta}\right]$$

Similarly in spherical coordinates $(h_1 = h_r = 1,\ h_2 = h_\phi = r,\ h_3 = h_\theta = r\,s.\,\phi)$

$$\underline{\nabla}\wedge\underset{\sim}{f} = \underline{e}_r\frac{1}{r\sin\phi}\left[\frac{\partial}{\partial\phi}(\sin\phi\,f_\theta) - \frac{\partial f_\phi}{\partial\theta}\right] + \frac{\underline{e}_\phi}{r\sin\phi}\left[\frac{\partial f_r}{\partial\theta} - \sin\phi\frac{\partial}{\partial r}(r f_\theta)\right]$$
$$+\ \frac{\underline{e}_\theta}{r}\left[\frac{\partial}{\partial r}(r f_\phi) - \frac{\partial f_r}{\partial\phi}\right]$$

F. An Introduction to Tensors & Dyadics

1. Preliminary remarks

a. We now wish to generalize our ideas concerning vectors to objects called tensors. We will try both to describe some of the mathematics of tensors and show why and how they arise in physical situations

b. The idea: we have previously described scalars & vectors as

scalar - characterized by magnitude

vector - characterized by magnitude & direction

Now ⇝ 2nd order tensor - characterized by magnitude & two directions. (or a dyadic)

dyad = "two units regarded as one"

c. You have actually seen something very similar before.

For example, the vector $\underset{\sim}{a}\,(\underset{\sim}{b}\cdot\underset{\sim}{c})$ could be written

$$\underbrace{\underset{\sim}{a}\,\underset{\sim}{b}}_{\text{dyadic}}\cdot\underset{\sim}{c}$$

which has the property $\underset{\sim}{a}\,\underset{\sim}{b}\cdot\underset{\sim}{c} \equiv \underset{\sim}{a}\,(\underset{\sim}{b}\cdot\underset{\sim}{c})$

NOTE: To eliminate any ambiguity, vector operations will be defined to occur between the nearest two vectors.

This points the following important property: using index notation, the quantity $\underset{\sim}{a}\,\underset{\sim}{b}$ may be written

$$\underset{\sim}{a}\,\underset{\sim}{b} = \underbrace{a_i\, b_j}_{\text{magnitude of the ij-component}}\ \underbrace{\underset{\sim}{e}_i\,\underset{\sim}{e}_j}_{2\text{ directions}} \qquad (\text{summation convention in use}) \qquad i,j = 1,2,3$$

d. These mathematical objects that require 2 directions (or 2 indices) to be defined often correspond to PHYSICAL situations where physical properties are different in different directions.

(see page 105)

NOTE: I will almost always adopt the notation that lower case letter are vectors and upper case letter: are tensors.

2. Definition of a 2nd order dyadic

a. Define the 2nd order dyadic $\underline{\underline{T}}$ by $\underline{\underline{T}} = \underline{a}\,\underline{b}$ and assign it the following operational properties:

(i) $\underline{c} \cdot \underline{\underline{T}} = \overbrace{\underline{c} \cdot \underline{a}}\,\underline{b} = (\underline{c} \cdot \underline{a})\,\underline{b}$

Note
(vector)$\cdot$(2nd rank tensor) → a vector

(ii) $\underline{\underline{T}} \cdot \underline{c} = \underline{a}\,\overbrace{\underline{b} \cdot \underline{c}} = \underline{a}\,(\underline{b} \cdot \underline{c})$

(iii) If $\underline{\underline{T}}, \underline{\underline{S}}, \underline{\underline{R}}$ are dyadics then we also define standard linear operations:

$$\underline{c} \cdot (\underline{\underline{T}} + \underline{\underline{S}} + \underline{\underline{R}} + \cdots) = \underline{c} \cdot \underline{\underline{T}} + \underline{c} \cdot \underline{\underline{S}} + \underline{c} \cdot \underline{\underline{R}} + \cdots$$

$$(\underline{\underline{T}} + \underline{\underline{S}} + \underline{\underline{R}} + \cdots) \cdot c = \underline{\underline{T}} \cdot \underline{c} + \underline{\underline{S}} \cdot \underline{c} + \underline{\underline{R}} \cdot \underline{c} + \cdots$$

(as you would expect)

NOTE: → a vector can be thought of as a first order dyadic or tensor
and a scalar " " " " " zero " " " "

→ The inner product of dyadic and a vector produces a vector

→ ORDER IS IMPORTANT: $\underline{c} \cdot \underline{\underline{T}} \neq \underline{\underline{T}} \cdot \underline{c}$

b. Using index notation we write (vector operations occur between the nearest two vectors)

$$\underline{a} = a_i\,\underline{e}_i \qquad \underline{b} = b_j\,\underline{e}_j \qquad \Rightarrow \underline{\underline{T}} = a_i b_j\,\underline{e}_i\,\underline{e}_j$$

So

$$\underline{c} \cdot \underline{\underline{T}} = c_k\,\overbrace{\underline{e}_k \cdot a_i b_j\,\underline{e}_i}^{\delta_{ik}}\,\underline{e}_j = (c_i a_i)\,b_j\,\underline{e}_j = (\underline{c} \cdot \underline{a})\,\underline{b}$$

and

$$\underline{\underline{T}} \cdot \underline{c} = a_i b_j\,\underline{e}_i\,\underbrace{\underline{e}_j \cdot c_k\,\underline{e}_k}_{\delta_{jk}} = (a_i\,\underline{e}_i)\;b_j c_j = \underline{a}\,(\underline{b} \cdot \underline{c})$$

Alternatively, we can discuss $\underline{\underline{T}} = T_{ij}\,\underline{e}_i\,\underline{e}_j$ and speak of the 2nd order tensor $\underline{\underline{T}}$.

↑ requires 2 indices to characterize completely

NOTATION: I denote a 2nd order tensor using 2 underlines

(generally, vectors will be lower case and 2nd order tensors uppercase) a simple idea of a 'linear operator' : The work done by a force acting through a displacement $\underline{d}$ is $\underline{f}\cdot\underline{d}$ and so you could choose to think about f as a linear operator that, once fed the displacement $\underset{\sim}{d}$, yields a scalar we call work, $\underline{f}\cdot\underline{d}$.

3. 2^{nd} order tensors : An alternative way of thinking about things

a. A 2^{nd} order tensor can be thought of as a "machine" that has a vector for its input and outputs another vector :

$$\underset{\sim}{a} \longrightarrow \boxed{\text{THE TENSOR MACHINE}} \longrightarrow \underline{b}$$

which we denote

$$\underbrace{\underline{\underline{T}}\cdot\underset{\sim}{a}}_{} = \underline{b}$$

$\underline{\underline{T}}$ operating on a vector produces another vector

b. Cartesian components of a 2^{nd} rank tensor

Lets consider an arbitrary vector $\underset{\sim}{c} = c_x \underset{\sim}{e}_x + c_y \underset{\sim}{e}_y + c_z \underset{\sim}{e}_z$. Then, from a formal 'operational' viewpoint,

$$\underline{\underline{T}}\cdot\underset{\sim}{c} = c_x\, \underline{\underline{T}}\cdot\underset{\sim}{e}_x + c_y\, \underline{\underline{T}}\cdot\underset{\sim}{e}_y + c_z\, \underline{\underline{T}}\cdot\underset{\sim}{e}_z \qquad (1)$$

$\Rightarrow$ But $\underline{\underline{T}}$ operates on vectors and produces other vectors So, for example, we choose to write $\Leftarrow$

$$\underline{\underline{T}}\cdot\underset{\sim}{e}_x = T_{xx}\underset{\sim}{e}_x + T_{yx}\underset{\sim}{e}_y + T_{zx}\underset{\sim}{e}_z$$ which is a vector.

and similarly

$$\underline{\underline{T}}\cdot\underset{\sim}{e}_y = T_{xy}\underset{\sim}{e}_x + T_{yy}\underset{\sim}{e}_y + T_{zy}\underset{\sim}{e}_z$$

$$\underline{\underline{T}}\cdot\underset{\sim}{e}_z = T_{xz}\underset{\sim}{e}_x + T_{yz}\underset{\sim}{e}_y + T_{zz}\underset{\sim}{e}_z$$

Thus, since $c_x = \underset{\sim}{c}\cdot\underset{\sim}{e}_x$, $c_y = \underset{\sim}{c}\cdot\underset{\sim}{e}_y$, $c_z = \underset{\sim}{c}\cdot\underset{\sim}{e}_z$ we can write eqn (1) as

$$\begin{aligned}\underline{\underline{T}}\cdot\underset{\sim}{c} = {} & T_{xx}\underset{\sim}{e}_x(\underset{\sim}{e}_x\cdot\underset{\sim}{c}) + T_{yx}\underset{\sim}{e}_y(\underset{\sim}{e}_x\cdot\underset{\sim}{c}) + T_{zx}\underset{\sim}{e}_z(\underset{\sim}{e}_x\cdot\underset{\sim}{c}) \\ & + T_{xy}\underset{\sim}{e}_x(\underset{\sim}{e}_y\cdot\underset{\sim}{c}) + T_{yy}\underset{\sim}{e}_y(\underset{\sim}{e}_y\cdot\underset{\sim}{c}) + T_{zy}\underset{\sim}{e}_z(\underset{\sim}{e}_y\cdot\underset{\sim}{c}) \\ & + T_{xz}\underset{\sim}{e}_x(\underset{\sim}{e}_z\cdot\underset{\sim}{c}) + T_{yz}\underset{\sim}{e}_y(\underset{\sim}{e}_z\cdot\underset{\sim}{c}) + T_{zz}\underset{\sim}{e}_z(\underset{\sim}{e}_z\cdot\underset{\sim}{c})\end{aligned}$$

slight rearrangement

$$\underline{\underline{T}}\cdot\underset{\sim}{c} = \underbrace{\begin{pmatrix} T_{xx}\underset{\sim}{e}_x\underset{\sim}{e}_x + T_{xy}\underset{\sim}{e}_x\underset{\sim}{e}_y + T_{xz}\underset{\sim}{e}_x\underset{\sim}{e}_z \\ + T_{yx}\underset{\sim}{e}_y\underset{\sim}{e}_x + T_{yy}\underset{\sim}{e}_y\underset{\sim}{e}_y + T_{yz}\underset{\sim}{e}_y\underset{\sim}{e}_z \\ + T_{zx}\underset{\sim}{e}_z\underset{\sim}{e}_x + T_{zy}\underset{\sim}{e}_z\underset{\sim}{e}_y + T_{zz}\underset{\sim}{e}_z\underset{\sim}{e}_z \end{pmatrix}}_{\text{Cartesian representation of the } 2^{nd} \text{ order tensor } \underline{\underline{T}}}\cdot\underset{\sim}{c}$$

← the set of dyadics $\{\underset{\sim}{e}_x\underset{\sim}{e}_x, \underset{\sim}{e}_x\underset{\sim}{e}_y, \underset{\sim}{e}_x\underset{\sim}{e}_z, \underset{\sim}{e}_y\underset{\sim}{e}_x \cdots\}$ is a basis for the 2^{nd} order tensors.

The set of 9 quantities T_{ij} are called the Cartesian components of $\underline{\underline{T}}$.

3. 2nd order tensors (continued)

c. Some sample representations using index notation

$$\underline{c}\cdot\underline{\underline{T}} = c_i \overbrace{\underline{e}_i \cdot T_{jk}\,\underline{e}_j}^{\delta_{ij}}\,\underline{e}_k = c_i T_{ik}\,\underline{e}_k \qquad \text{(a vector)}$$

$$\underline{\underline{T}}\cdot\underline{c} = T_{ij}\,\underline{e}_i \underbrace{\underline{e}_j \cdot c_k\,\underline{e}_k}_{\delta_{jk}} = T_{ij}\,c_j\,\underline{e}_i \qquad \text{(a vector)}$$

→ RULES : Nesting convention ⇒ vector operations occur between the closest pair of unit vectors

Example :

$$\underline{\underline{T}} \wedge \underline{c} = T_{ij}\,\underline{e}_i \overbrace{\underline{e}_j \wedge c_k\,\underline{e}_k}^{} = T_{ij}\,c_k\,\underline{e}_i\,(\underline{e}_j \wedge \underline{e}_k) = T_{ij}\,c_k\,\varepsilon_{jkm}\,\underline{e}_i\,\underline{e}_m$$

→ $\boxed{\text{order of unit vectors is now important.}}$

d. Notice the similarity with matrices : A 3x3 matrix $\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}$ has 9 entries which we can think of as the components of a 2nd order tensor. Each of the components of a 2nd rank tensor, though, has two directions associated with it.

e. Notice that

$\underline{a}\cdot\underline{b}$ is a scalar

$\underline{a}\cdot\underline{\underline{T}}$, $\underline{\underline{T}}\cdot\underline{a}$ are vectors (In general, $\underline{a}\cdot\underline{\underline{T}} \neq \underline{\underline{T}}\cdot\underline{a}$)

$\underline{a}\cdot\underline{\underline{T}}\cdot\underline{b}$, $\underline{b}\cdot\underline{\underline{T}}\cdot\underline{a}$ are scalars.

Note : In the expression, $\underline{a}\cdot\underline{\underline{T}}\cdot\underline{b}$, the order in which the inner products are taken does not matter.

To see this,

$$(\underline{a}\cdot\underline{\underline{T}})\cdot\underline{b} = (a_i T_{ij}\,\overbrace{\underline{e}_j\,)\cdot b_k\,\underline{e}_k}^{\delta_{jk}} = a_i T_{ij} b_j \qquad \left(= \sum_{i=1}^{3}\sum_{j=1}^{3} a_i T_{ij} b_j\right)$$

and

$$a\cdot(\underline{\underline{T}}\cdot\underline{b}) = a_i\,\overbrace{\underline{e}_i \cdot (T_{jk} b_k\,\underline{e}_j}^{\delta_{ij}}) = a_i T_{ik} b_k$$

$$\therefore\ (\underline{a}\cdot\underline{\underline{T}})\cdot\underline{b} = \underline{a}\cdot(\underline{\underline{T}}\cdot\underline{b}) = \underline{a}\cdot\underline{\underline{T}}\cdot\underline{b}$$

However, in general, $\underline{a}\cdot\underline{\underline{T}}\cdot\underline{b} \neq \underline{b}\cdot\underline{\underline{T}}\cdot\underline{a}$

3. 2^{nd} order tensors (continued)

f. The unit tensor $\underline{\underline{I}}$ is defined as

Kronecker delta

$$\underline{\underline{I}} = \delta_{ij}\, \underline{e}_i \underline{e}_j \qquad \Rightarrow \text{analogue of the identity matrix :} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and clearly has the property

$$\underset{\sim}{a} \cdot \underline{\underline{I}} = \underset{\sim}{a}$$

$$\underline{\underline{I}} \cdot \underset{\sim}{b} = \underset{\sim}{b}$$

4. Higher order tensors - It is straightforward to construct higher order tensors $\Rightarrow$ Add indices

a. Here we will only mention <u>third</u> <u>order</u> <u>tensors</u> (or third order dyadics) e.g.,

$$\underline{\underline{\underline{S}}} = S_{ijk}\, \underline{e}_i \underline{e}_j \underline{e}_k$$

$\boxed{\text{$i,j,k = 1 \rightarrow 3$ so S_{ijk} represents 27 termo}}$

Notice, given the vector $\underset{\sim}{a} = a_l \underline{e}_l$, then

$$\underset{\sim}{a} \cdot \underline{\underline{\underline{S}}} = a_l \overbrace{\underline{e}_l \cdot S_{ijk}\, \underline{e}_i} \underline{e}_j \underline{e}_k = a_l S_{ljk}\, \underline{e}_j \underline{e}_k$$

(a 2^{nd} order tensor)

Similarly, $\underline{\underline{\underline{S}}} \cdot \underset{\sim}{a} = S_{ijk} a_k\, \underline{e}_i \underline{e}_j$ is a 2^{nd} order tensor

b. Permutation tensor $\equiv \underline{\underline{\underline{\varepsilon}}} = \varepsilon_{ijk}\, \underline{e}_i \underline{e}_j \underline{e}_k$

where ε_{ijk} is the permutation symbol introduced earlier when discussing the vector product.

Exercise : Show $\underline{\underline{I}} \wedge \underline{\underline{I}} = -\underline{\underline{\underline{\varepsilon}}}$

(be careful with order of vector operations and indices)

5. Symmetric & Anti-symmetric tensors

a. Recall the <u>transpose</u> of a matrix

If $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ then $\underset{\overset{\uparrow}{\text{transpose}}}{A^T} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{12} & A_{32} \\ A_{13} & A_{23} & A_{13} \end{bmatrix}$

Furthermore, a matrix was called SYMMETRIC if $A = A^T$ and was called ANTI-SYMMETRIC if $A = -A^T$

$$A = A^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \qquad A = -A^T = \begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix}$$

b. We now generalize these ideas to 2nd rank tensors

(i) A 2nd order tensor $\underline{\underline{T}}$ is <u>symmetric</u> if $\underline{\underline{T}} = \underline{\underline{T}}^T$ or $T_{ij} = T_{ji}$

(ii) A 2nd order tensor $\underline{\underline{T}}$ is <u>anti-symmetric</u> if $\underline{\underline{T}} = -\underline{\underline{T}}^T$ or $T_{ij} = -T_{ji}$

Some important properties (analogous to matrices shown above):

symmetric tensor ⇒ $T_{ij} = T_{ji}$ → only $\underline{6}$ independent components

anti-symmetric tensor → $T_{ij} = -T_{ji}$ → only $\underline{\underline{3}}$ independent components

↳ and we will see in the homework that an anti-symmetric tensor can be represented using a vector.

c. Every 2nd order tensor can be expressed as the sum of a symmetric and an anti-symmetric tensor.

NOTE: $\boxed{(\underline{\underline{A}}+\underline{\underline{B}})^T = \underline{\underline{A}}^T + \underline{\underline{B}}^T}$

For example,

$$\underline{\underline{T}} = \underbrace{\tfrac{1}{2}\left(\underline{\underline{T}} + \underline{\underline{T}}^T\right)}_{\text{symmetric}} + \underbrace{\tfrac{1}{2}\left(\underline{\underline{T}} - \underline{\underline{T}}^T\right)}_{\text{anti-symmetric}}$$

(this is analogous to decomposition of a function into even and odd functions, p. 56)

AN ASIDE: Tensors and the physical world

So, now you have seen that a 2^{nd} order tensor is a mathematical object that (linearly) "sends vectors into other vectors."

This idea is particularly useful when describing physical situations where the physical properties are different in different directions, and you are representing quantities characterized by a vector.

Examples:

(i) current generated due to an applied electric field $\underset{\sim}{E}$

$$\underset{\sim}{J} = \underline{\underline{\sigma}} \cdot \underline{E}$$

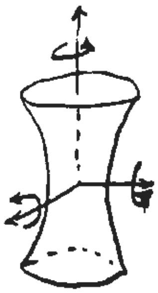
current density (current/(area·time))

electrical conductivity tensor

(ii) Angular momentum $\underset{\sim}{L}$ about a point for a body rotating with angular velocity $\underset{\sim}{\omega}$

$$\underset{\sim}{L} = \underline{\underline{I}} \cdot \underset{\sim}{\omega}$$

moment of inertia tensor



(clearly, for a given rotation speed, angular momentum varies depending on the axis of rotation)

Personally, I find (iii) & (iv) the simplest physical situation which suggest the appearance of tensorial quantities in a mathematical description of the physical world.

(iii) heat flux $\underset{\sim}{q}$ due to a thermal gradient (i.e., a temperature difference)

$$\underset{\sim}{q} = \underline{\underline{K}} \cdot \underset{\sim}{\nabla} T$$

⇐ Generalization of Fourier's law of heat conduction

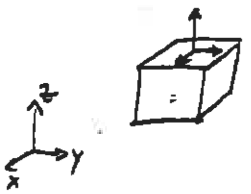
heat flux vector $\left(\frac{\text{energy}}{\text{area time}}\right)$

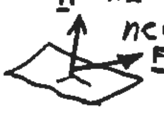
thermal conductivity tensor

→ A practical example is actually a material like wood. For a given temperature difference (or $\underset{\sim}{\nabla} T$) more energy is transferred along the grain than across the grain – the thermal conductivity of the material is different in the different directions.

(iv) STRESS TENSOR – frequently one requires information about stresses (force/area) acting on a material. To describe the state of stress at a point, e.g., $\underset{\sim}{x}$, it necessary to specify the ORIENTATION of $\underline{n}$ the surface and the vector Force → 2 directions are thus necessary.

NOTE: surface orientation is given by the unit normal vector

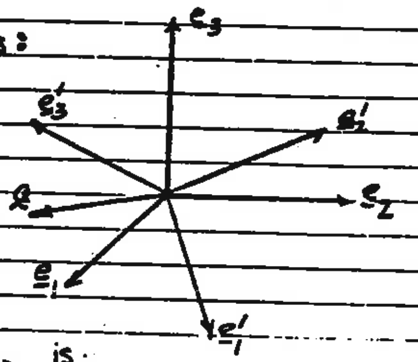




6. Transformation Rules for Tensors

→ We now examine how the components of a tensor change if we change from one set of Cartesian base vectors to another.

a. consider two sets of Cartesian base vectors:



$\underset{\sim}{e_i}$ ≡ "old" set of base vectors

e_i' ≡ "new" " " " "

o.k. = poor choice of words, Since you probably haven't seen this before we'll discuss it now

b. First, recall the transformation rule for the components of a vector.

(remember, that a vector, say $\underset{\sim}{a}$, is invariant with respect to a change of coordinate system. However, the representation of the vector in terms of its components clearly depends on the choice of the coordinate system.)

let $\underset{\sim}{a} = a_i \underset{\sim}{e_i} = a_i' \underset{\sim}{e_i'}$ (1) (the components a_i, a_i' are given by $\underset{\sim}{a} \cdot \underset{\sim}{e_i} = a_i$, $\underset{\sim}{a} \cdot \underset{\sim}{e_i'} = a_i'$)

So,

$$a_i' = (a_j \underset{\sim}{e_j}) \cdot \underset{\sim}{e_i'} = a_j \, (\underset{\sim}{e_j} \cdot \underset{\sim}{e_i'})$$

c. Definition: the DIRECTION COSINES between the axes are given by

NOTE: $l_{mn} \neq l_{nm}$

$$\underset{\text{"old set"}}{l_{mn}} = \underset{\sim}{e_m} \cdot \underset{\text{"new set"}}{\underset{\sim}{e_n'}}$$

(cosine of the angle between $\underset{\sim}{e_m}$ and $\underset{\sim}{e_n'}$)

(2) ∴ $\boxed{a_i' = a_j \, l_{ji}}$ TRANSFORMATION RULE FOR THE COMPONENTS OF A VECTOR

Similarly,

$$a_i = a_j' (\underset{\sim}{e_j'} \cdot \underset{\sim}{e_i}) = a_j' \underbrace{(\underset{\sim}{e_i} \cdot \underset{\sim}{e_j'})}_{l_{ij}}$$

So $\boxed{a_i = l_{ij} \, a_j'}$ (3) Notice how order of indices is reversed.

Notice:

Substituting (3) into (1): $(\underset{\sim}{e_i} \cdot \underset{\sim}{e_j'}) \, a_j' \, \underset{\sim}{e_i} = a_i' \, \underset{\sim}{e_i'}$

Rearranging indices allows one to conclude

$$(\underset{\sim}{e_j} \cdot \underset{\sim}{e_i'}) \underset{\sim}{e_j} = \underset{\sim}{e_i'} \rightarrow \boxed{\underset{\sim}{e_i'} = l_{ji} \, \underset{\sim}{e_j}} \quad (4)$$

Similarly,

$$\boxed{\underset{\sim}{e_i} = l_{ij} \, \underset{\sim}{e_j'}} \quad (5)$$

Exercise: Demonstrate these relations for yourself.

d. Properties of the l_{ij}

As shown on the previous page, $\underline{e}'_i = l_{ji}\underline{e}_j$, $\underline{e}_i = l_{ij}\underline{e}'_j$

So, since the base vectors in each of the coordinate systems are orthogonal,

$$\underline{e}'_n \cdot \underline{e}'_m = \delta_{nm} \rightarrow (l_{jn}\overbrace{\underline{e}_j)\cdot(l_{km}\underline{e}_k}^{\delta_{jk}}) = \delta_{nm}$$

$$\therefore \boxed{l_{kn}\, l_{km} = \delta_{nm}}$$

Similarly, $\underline{e}_n \cdot \underline{e}_m = \delta_{nm} \Rightarrow \boxed{l_{nk}\, l_{mk} = \delta_{nm}}$

Notice : one can use the above 2 eqns in useful ways. For example, beginning with eqn (2),

$$a'_i = a_j l_{ji} \rightarrow a'_i l_{ki} = a_j \underbrace{l_{ji}\, l_{ki}}_{\delta_{jk}} \quad \text{after multiplying both sides by } l_{ki}$$

$$= a_k \leftarrow \text{which is the same result as eqn (3) on pg 91}$$

#

⇒ the above illustrates the details of how the components of a vector transform when a different cartesian coordinate system is considered.

e. Now, consider the <u>transformation rule for 2nd order tensors</u>

Begin with $\underline{e}'_i = l_{ji}\,\underline{e}_j \qquad \underline{e}_i = l_{ij}\,\underline{e}'_j$ ← we know how vectors transform

The 2nd order tensor $\underline{\underline{T}}$ can then be represented as

$$\underline{\underline{T}} = T_{ij}\,\underline{e}_i\underline{e}_j = T_{ij}\,(l_{ik}\,\underline{e}'_k)(l_{jm}\,\underline{e}'_m) = \underbrace{T_{ij}l_{ik}\,l_{jm}}_{T'_{km}}\,\underline{e}'_k\underline{e}'_m = T'_{km}\underline{e}'_k\underline{e}'_m$$

$$\therefore \boxed{T'_{km} = T_{ij}\,l_{ik}\,l_{jm}}$$

Transformation rule for the components of $\underline{\underline{T}}$ in the ′ coordinate system ($\underline{e}'$) relative to the original coordinate system.

Similarly, $\underline{\underline{T}} = T'_{ij}\,\underline{e}'_i\underline{e}'_j = T'_{ij}(l_{ki}\,\underline{e}_k)(l_{mj}\,\underline{e}_m) = \underbrace{T'_{ij}\,l_{ki}\,l_{mj}}\ \underline{e}_k\underline{e}_m = T_{km}\,\underline{e}_k\underline{e}_m$

$$\therefore \boxed{T_{km} = T'_{ij}\,l_{ki}\,l_{mj}}$$ ← transformation rule from "new" to "old"

you may wish to try this as an exercise

Again, one can do the same for <u>third order</u> tensors : $\underline{\underline{\underline{S}}} = S_{ijk}\,\underline{e}_i\underline{e}_j\underline{e}_k = S_{pqr}\,\underline{e}'_p\underline{e}'_q\underline{e}'_r$ and one finds $\boxed{S'_{pqr} = S_{ijk}\,l_{ip}\,l_{jq}\,l_{kr}}$ and $\boxed{S_{ijk} = S'_{pqr}\,l_{ip}\,l_{jq}\,l_{kr}}$

7. Remark : The "definition" of 2nd order tensors

On the previous page we presented formulas which show how the components of a tensor are related in different Cartesian coordinate systems.

i.e.,

$$T'_{km} = T_{ij}\, l_{ik}\, l_{jm} \qquad T_{km} = T'_{ij}\, l_{ki}\, l_{mj}$$

where

$$\underline{\underline{T}} = T_{km}\, \underline{e}_k \underline{e}_m = T'_{km}\, \underline{e}'_k \underline{e}'_m$$

Hence, given the components of $\underline{\underline{T}}$, T_{ij}, relative to the $\underline{e}_i$-axes, the above transformation rules specify the components T'_{ij} in any other set of right-hand cartesian orthonormal base vectors, $\underline{e}'_i$.

Classical treatments of tensor often DEFINE a second order tensor as that entity whose components in any, and every, two sets of Cartesian axes transform according to the above rules. The two approaches (the first illustrated on pages 106-107 or the alternative which simply defines a tensor via the above transformation rules) are equivalent and we will call any entity whose components transform according to these rules a second order tensor.

It is nevertheless important to keep in mind that the tensor itself is defined independent of the choice of coordinate system but the representation in terms of components depends on the choice of the coordinate system

An example of this :

Suppose the "components" T_{ij} are given for all cartesian axes. Also, suppose for all vectors $\underset{\sim}{a} = a_i \underline{e}_i$ that $a_i T_{ij} \underline{e}_j$ is a vector. Then, $\underline{\underline{T}} = T_{ij}\, \underline{e}_i \underline{e}_j$ (that quantity constructed from the components T_{ij}) must be a second order tensor.

Proof : Use the above transformation which specify how the components of a second order tensor are related.

$a_i T_{ij} \underline{e}_j$ is a vector $\rightarrow$ $a'_i T'_{ij} = (a_k T_{km})\, l_{mj}$ since this is how the components of a vector transform

and since $\underset{\sim}{a}$ is a vector $\rightarrow$ $= (a'_p l_{kp})\, T_{km}\, l_{mj}$

$a_k = a'_p \delta_{kp}$

or $a'_i T'_{ij} - a'_i l_{ki} T_{km} l_{mj}$ after relabelling $p \rightarrow i$

or $a'_i \underbrace{(T'_{ij} - T_{km} l_{ki} l_{mj})}_{0} = 0$ for all vector $\underset{\sim}{a}$

$\therefore$ $T'_{ij} = T_{km}\, l_{ki}\, l_{mj}$ which is the transformation rule for the components of a second order tensor

8. Isotropic tensors

a. Definition : Any tensor which has the same components in all Cartesian axes is called an isotropic tensor.

Now, on the previous two pages, we examined how to represent the components of vectors & tensors for different cartesian coordinate axes.

b. Recall the unit tensor $\underline{\underline{I}} = \delta_{ij}\, \underline{e}_i \underline{e}_j$.

What are the components of $\underline{\underline{I}}$, δ_{ij}', relative to the $\underline{e}_i'$ axes.

Well, from pg. 107,

$$\delta_{ij}' = \delta_{mn}\, l_{mi}\, \delta_{nj}$$

$$= l_{ni}\, l_{nj} = \delta_{ij}$$

via the identities on the top of p. 92

$\therefore$ $\underline{\underline{I}}$ has the same components (δ_{ij}) in all Cartesian systems, and hence $\underline{\underline{I}}$ is called an isotropic 2nd order tensor.

⇒ It can be shown that apart from a multiplicative constant, the unit tensor $\underline{\underline{I}}$ is the only isotropic 2nd order tensor.

c. Likewise, consider the permutation tensor $\underline{\underline{\varepsilon}} = \varepsilon_{ijk}\, \underline{e}_i \underline{e}_j \underline{e}_k$

$$\underline{\underline{\varepsilon}} = \varepsilon_{ijk}\, \underline{e}_i \underline{e}_j \underline{e}_k = \varepsilon'_{ijk}\, \underline{e}_i' \underline{e}_j' \underline{e}_k'$$

in the primed coordinate system

First, a clever observation : $\varepsilon_{ijk} = \underline{e}_i \cdot (\underline{e}_j \wedge \underline{e}_k)$

From the bottom of p.107 we see that the components ε'_{pqr} and ε_{ijk} are related by

$$\varepsilon'_{pqr} = \varepsilon_{ijk}\, l_{ip}\, l_{jq}\, l_{kr} = \underline{e}_i \cdot (\underline{e}_j \wedge \underline{e}_k)\, l_{ip}\, l_{jq}\, l_{kr}$$

$$= (l_{ip}\, \underline{e}_i) \cdot [\, l_{jq}\, \underline{e}_j \wedge l_{kr}\, \underline{e}_k]$$

$$= \underline{e}_p' \cdot (\underline{e}_q' \wedge \underline{e}_r') = \varepsilon_{pqr}$$

simply because $\underline{e}_p'$, $\underline{e}_q'$, $\underline{e}_r'$ form a right-handed orthogonal coordinate system.

$\therefore$ $\underline{\underline{\varepsilon}}$ is a third order isotropic tensor.

9. Example: a change of cartesian axes $\rightarrow f_i' = f_m l_{mi}$

let $\underset{\sim}{f} = f_i \underset{=}{e}_i = f_i' \underset{=}{e}_i'$ be a vector field.

Consider the derivatives of the f_i and f_i' with respect to position, ie, consider

$$\frac{\partial f_i}{\partial x_j} \quad \text{and} \quad \frac{\partial f_i'}{\partial x_j'}$$

and we know (p. 91) that the components of the position vector satisfy

$$x_i' = x_j l_{ji} \qquad x_i = x_j' l_{ij} \rightarrow \frac{\partial x_i}{\partial x_j'} = l_{ij}$$

Clearly, via the chain rule,

$$\frac{\partial f_i'}{\partial x_j'} = \frac{\partial (f_m l_{mi})}{\partial x_k} \overset{l_{kj}}{\frac{\partial x_k}{\partial x_j'}}$$

$$= \frac{\partial f_m}{\partial x_k} l_{kj} l_{mi}$$

which is the same transformation rule found on p. 107 for the components of a 2nd order tensor.

Therefore, $\frac{\partial f_m}{\partial x_k}$ (or $\frac{\partial f_i'}{\partial x_j'}$) represents the components of a 2nd order tensor.

The notation generally used is $\underset{\sim}{\nabla} \underset{\sim}{f} = \frac{\partial f_j}{\partial x_i} \underset{=}{e}_i \underset{=}{e}_j$

9. Scalar Invariants – scalars simply have a magnitude

a. Example: consider the scalar $\underline{a} \cdot \underline{b}$

$\Rightarrow \underline{a} \cdot \underline{b} = a_i b_i = a_i' b_i'$ and this scalar quantity is an INVARIANT, in other words, it has the same value in all cartesian axes.

To demonstrate this, use the transformation rules discussed earlier.

$$a_i' b_i' = a_j l_{ji} b_k l_{ki} = a_j b_k \overbrace{l_{ji} l_{ki}}^{\delta_{jk}} = a_k b_k$$

NOTE: This does not mean $T_{11} = T_{11}'$; rather, the sums of the diagonal terms are equal $T_{11} + T_{22} + T_{33} = T_{11}' + T_{22}' + T_{33}'$

b. Example: If $\underset{=}{T} = T_{ij} \underset{=}{e}_i \underset{=}{e}_j$ is a second order tensor, then T_{ii} is an invariant.

Proof: $T_{ij}' = T_{km} l_{ki} l_{mj} \Rightarrow T_{ii}' = T_{km} \overbrace{l_{ki} l_{mi}}^{\delta_{km}} = T_{kk}$

$\therefore T_{kk} = T_{kk}' \Rightarrow T_{kk}$ is an INVARIANT.

common notation

$$\boxed{T_{kk} = \text{Trace}(\underset{\approx}{T}) \equiv \text{tr}\, \underset{=}{T}}$$

c. Invariance of the Divergence of a vector $\underline{\nabla}\cdot\underline{f}$.

Since $\underline{\nabla}\cdot\underline{f} = \frac{\partial f_i}{\partial x_i}$ and since $\frac{\partial f_i}{\partial x_j}$ are the components of a second order tensor, say $\underline{\underline{T}} = T_{ij}\,\underline{e}_i\underline{e}_j$, then it follows that because we just saw that T_{ii} is an invariant, it also must be true that $T_{ii} = \frac{\partial f_i}{\partial x_i}$ is an invariant.

Remark: SOME ADDITIONAL NOTATION

The "Double Dot product":

The following notation is sometimes used;

$$\underline{a}\,\underline{b} : \underline{c}\,\underline{d} = (\underline{a}\cdot\underline{d})\,(\underline{b}\cdot\underline{c}) \rightarrow \text{a scalar}$$

Also, for two second order tensors, $\underline{\underline{T}}$, $\underline{\underline{S}}$

$$\underline{\underline{T}} : \underline{\underline{S}} = T_{ij}\,\underline{e}_i\,\underline{e}_j : S_{kl}\,\underline{e}_k\,\underline{e}_l$$

$$= T_{ij}\,S_{kl}\,\delta_{il}\,\delta_{jk} = T_{ik}\,S_{ki}$$

note: nearest two indices are the same

So

$$\underline{\underline{I}} : \underline{\underline{T}} = T_{ii} = \underbrace{tr(\underline{\underline{T}})}$$

trace of the second order tensor; by summation convention, $tr\,\underline{\underline{T}} = T_{11} + T_{22} + T_{33}$. (See bottom of p. 110)

TOPICS 10 & 11 Are basically for your overall education and are meant to try to give you some ideas how the concepts of a tensor arise in different physical situations

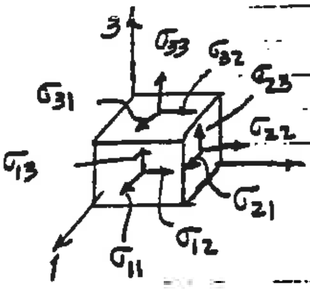
10. EXAMPLE: The Stress Tensor

← Fundamental to the description and a basic understanding of the deformation of solids and the flow of fluids

a. Why is this concept useful?
Suppose you were interested in the state of equilibrium of a material. Newton's law applied to a small piece of a material says

$$\sum \underset{\sim}{\text{Forces}} = \text{mass} \times \text{acceleration}$$

If the material is stationary and at equilibrium (acceleration = 0) then the sum of the forces on a piece of the material must balance. So, now consider the small cube of material shown below.



Of course, we will require that all the forces acting on this body must balance and it is convenient to denote the forces acting on each face. We write

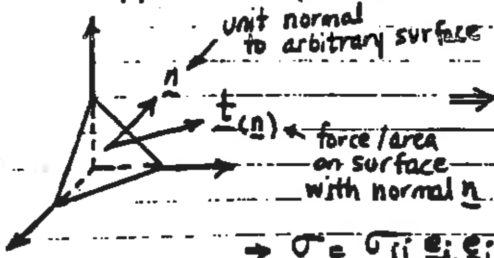
$$\underset{\sim}{t}_{(1)} = \sigma_{11}\,\underline{e}_1 + \sigma_{12}\,\underline{e}_2 + \sigma_{13}\,\underline{e}_3$$

↑ force/area on the face ⊥ $\underline{e}_1$

↑ the component of the force/area acting in the $\underline{e}_2$ direction on the face ⊥ $\underline{e}_1$.

In general, $\underset{\sim}{t}_{(i)} = \sigma_{ij}\,\underline{e}_j$ ← "stress vector" ≡ force/area on face ⊥ $\underline{e}_i$

b. Furthermore, there is a very beautiful result due to Cauchy which considers a tetrahedron-shaped material volume and applies the principle that the sum of the forces must balance.



⟹ CAUCHY'S RESULT

$$\boxed{\underset{\sim}{t}_{(\underline{n})} = \underline{n}\cdot\underline{\underline{\sigma}}}$$

for all $\underset{\sim}{n}$

← STRESS TENSOR

↑ stress vector (force/area) on surface with normal $\underline{n}$

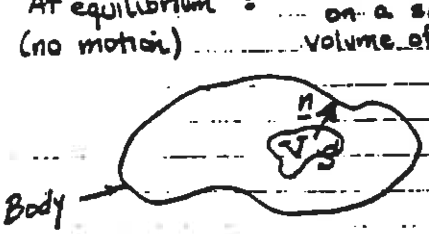
→ $\underline{\underline{\sigma}} = \sigma_{ij}\,\underline{e}_i\,\underline{e}_j$; by previous "tensor test", $\underline{\underline{\sigma}}$ must be a second order tensor.

c. Derivation of the Equilibrium Equations for a continuous material

At equilibrium (no motion): net force on a small volume of material $= \int_S \underset{\sim}{t}_{(\underline{n})}\,dS = \int_S \underline{n}\cdot\underline{\underline{\sigma}}\,dS = 0$

↑ no acceleration (condition for static equilibrium)

$$= \int_S n_i\,\sigma_{ij}\,\underline{e}_j\,dS$$



C. Equilibrium of a continuous material (continued)

We begin with

$$(1) \quad \int_S \underline{n} \cdot \underline{\underline{\sigma}} \, dS = 0 \qquad \left(= \int_S n_i \sigma_{ij} \, dS \right.$$

We now apply the Divergence Theorem, which holds equally well for tensors, i.e.,

DIVERGENCE THEOREM FOR TENSORS $\Longrightarrow$ $$\int_S \underline{n} \cdot \underline{\underline{T}} \, dS = \int_V \underline{\nabla} \cdot \underline{\underline{T}} \, dV \quad \text{or using index notation} \quad \int_S n_i T_{ijk} \, dS = \int_V \frac{\partial}{\partial x_i} T_{ijk} \, dV$$

↑ n^{th} order tenso

So, eqn (1) becomes

$$\int_V \underline{\nabla} \cdot \underline{\underline{\sigma}} \, dV = 0$$

which is true for an arbitrary volume element in a body at equilibrium

and since this must hold for all volume elements V, we conclude

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} = 0 \qquad \left(\text{or} \quad \frac{\partial}{\partial x_i} \sigma_{ij} = 0 \right)$$

If you were to write this out, $\underline{\nabla} \cdot \underline{\underline{\sigma}}$ is a vector so $\underline{\nabla} \cdot \underline{\underline{\sigma}} = 0$ represents 3 eqns, one for each component of the vector. So,

$$\frac{\partial}{\partial x_1} \sigma_{11} + \frac{\partial}{\partial x_2} \sigma_{21} + \frac{\partial}{\partial x_3} \sigma_{31} = 0$$

$$\frac{\partial}{\partial x_1} \sigma_{12} + \frac{\partial}{\partial x_2} \sigma_{22} + \frac{\partial}{\partial x_3} \sigma_{32} = 0$$

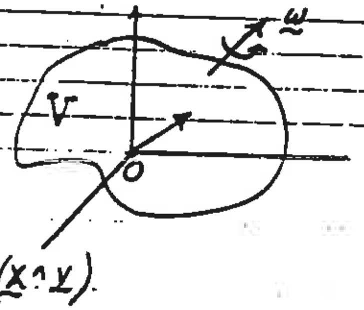
$$\frac{\partial}{\partial x_1} \sigma_{13} + \frac{\partial}{\partial x_2} \sigma_{23} + \frac{\partial}{\partial x_3} \sigma_{33} = 0.$$

Remark: In order to proceed further, you must relate the stress tensor $\underline{\underline{\sigma}}$ to the small displacements that occur in the material.

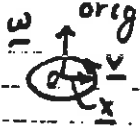
11. The Moment of Inertia Tensor

a. In the study of the mechanics of rotating rigid bodies, it is necessary to know the angular momentum and kinetic energy of the object. We will now see how a 2nd order tensor, the moment of inertia tensor, naturally arises.

b. Consider a rigid body spinning with angular velocity $\underset{\sim}{\omega}$ about a fixed point O.



Recall that for a point mass rotating at angular velocity $\underset{\sim}{\omega}$ about some origin, the actual velocity is



$$\underline{v} = \underset{\sim}{\omega} \wedge \underline{x}$$

and the angular momentum about O is $(\text{mass})\cdot(\underline{x}\wedge\underline{v})$.

So, the total angular momentum $\underset{\sim}{L}$ of the rigid body is

$$\underset{\sim}{L} = \int_V \underline{x}\wedge(\underset{\underset{\text{mass/volume}}{\uparrow}}{\rho}\,\underline{v})\,dV \overset{\underline{v}=\underline{\omega}\wedge\underline{x}}{=\!=} \int_V \rho\,\underline{x}\wedge(\underline{\omega}\wedge\underline{x})\,dV$$

where every point of the rigid body rotates with angular velocity $\underset{\sim}{\omega}$.

but...

$$\underline{x}\wedge(\underline{\omega}\wedge\underline{x}) = (\underline{x}\cdot\underline{x})\,\underset{\sim}{\omega} - \underline{x}\,(\underline{x}\cdot\underline{\omega}) = \left[(\underline{x}\cdot\underline{x})\,\underline{\underline{I}} - \underline{x}\,\underline{x}\right]\cdot\underset{\sim}{\omega}$$

Exercise: prove this identity

where $r^2 = \underline{x}\cdot\underline{x} = x_j x_j$

So,

$$\underset{\sim}{L} = \int_V \left[r^2\,\underline{\underline{I}} - \underline{x}\,\underline{x}\right]\cdot\underset{\sim}{\omega}\;dV$$

or since $\underset{\sim}{\omega} \equiv$ constant throughout V

$$\underset{\sim}{L} = \int_V \left(r^2\,\underline{\underline{I}} - \underline{x}\,\underline{x}\right)\rho\,dV\cdot\underset{\sim}{\omega} = \underset{\approx}{\mathbb{I}}\cdot\underset{\sim}{\omega}$$

where

$$\underset{\approx}{\mathbb{I}}(0) = \int_V \left(r^2\,\underline{\underline{I}} - \underline{x}\,\underline{x}\right)\rho\,dV \equiv \text{moment of inertia tensor about } O.$$

Notice: $\underline{\underline{\mathbb{I}}}$ is symmetric