Lecture 19: Oil on water; Surface waves

Immiscible Drops at an Interface

Pendant Lenses

\[ \gamma_{ab} \quad \gamma_{ac} \quad a \quad l_{ac} \]

\[ l_{bc} \]

\[ \gamma_{bc} \]

\[ R \quad c \quad b \]

\[ \beta_c < \beta_a < \beta_b \]

Stable only for drops small w.r.t. capillary length

\[ l_{ac} \sim \sqrt{\frac{\gamma_{ac}}{(\beta_a - \beta_c) \gamma}} \quad \text{and} \quad l_{bc} \sim \sqrt{\frac{\gamma_{bc}}{(\beta_b - \beta_c) \gamma}} \]

Sessile Lens: \( \beta_a < \beta_c < \beta_b \) e.g. oil on water

Oil Spill: 4 Distinct Phases
**Phase I: Inertia vs Gravity**

As previously, \( u \sim (g' h)^{1/2} \)

\[ R(t) \sim (g' V_o)^{1/4} t^{1/2} \quad \text{where} \quad j' = g' \frac{dV}{dt} \]

**Phase II: Gravity vs Viscosity**

- As previously (similar scaling, though less \( f \))

\[ R \sim \left( \frac{g' V_o^3}{V} \right)^{1/8} t^{1/8} \]

**Phase III: Line tension vs Viscosity**

- Behaviour depends on magnitude of spreading parameter \( S = \gamma_{aw} - \gamma_{ao} - \gamma_{ow} \)
For $S < 0$ : an equilibrium configuration arises; the drop assumes the form of a sussile lens

For $S > 0$ : the oil will completely cover the water, spreading to a layer of molecular thickness

Stage III A : viscous dissipation within spreading slick is dominant

\[ R(t) \]

\[ U \]

\[ S \]

\[ H \]

\[ \frac{Mv}{H} \pi R^2 \sim S \cdot 2\pi R \]

\[ UR \sim \frac{S}{H} H \sim \frac{S}{m} \frac{V}{R^2} \text{ where } H \sim \frac{V}{R^2} \]

\[ R^3 \frac{dR}{dt} \sim \frac{SV}{m} \]

\[ R \sim \left( \frac{SV}{m} \right)^{\frac{1}{2}} + \frac{1}{4} \text{ as previously} \]
Stage III B: viscous dissipation in underlying water dominant

\[ \delta \sim \sqrt{\nu t} \]

**Scaling:**

\[ \frac{\mu \sqrt{\nu}}{\delta} \cdot \pi R^2 \sim S^1 - 2 \pi R \text{ where } S \sim \sqrt{\nu t} \]

\[ R \frac{dT}{dt} \sim \left( \frac{S}{\mu} \right) \sqrt{\nu} + \frac{1}{2} \text{ where } \mu = \rho \nu \]

\[ R \sim \left( \frac{S}{\mu} \right)^{\frac{1}{2}} \nu^{\frac{1}{4}} \]

Reference: James Fay, Oil on the Sea
- Jensen, JFM (1975)
Water Waves

Assume $Re >> 1$, so that...

- motion of fluid may be described to leading order as inviscid and irrotational.
- must deduce a soln for the velocity potential $\phi$ satisfying (where $U = \nabla \phi$):

\[ \nabla^2 \phi = 0 \]

subject to kinematic + dynamic boundary conditions

B.C.s

1. $\frac{d\phi}{dz} = 0$ on $z = -h$

2. Kinematic BC: $\frac{DY}{Dt} = Uz$ on $z = \eta$
   \[ \Rightarrow \frac{dY}{dx} + \frac{dY}{dx} \frac{d\eta}{dx} = \frac{d\phi}{dz} \quad \text{on} \quad z = \eta \]

3. Dynamic B.C. (Time-dependent Bernoulli applied at free surface)

\[ \rho \frac{d\phi}{dt} + \frac{1}{2} \rho |\nabla \phi|^2 + \rho g \eta + P_\infty = f(t) \]

where $P_\infty = P_0 + \sigma \nabla \cdot n$
Here $\eta = \left( \frac{y_x, 1}{1 + y_x^2} \right)^\frac{1}{2}$ is the unit normal

$\nabla \cdot \eta = \frac{-y_{xx}}{(1 + y_x^2)^{3/2}}$ is the curvature

$P_s = P_0 - \sigma \frac{y_{xx}}{(1 + y_x^2)^{3/2}}$

Now consider small-amplitude waves and linearize the system of eqns and BCs (i.e. assume $\phi, y$ small, so neglect any terms involving $\phi^2, y^2, \phi y$ or their derivatives)

$\nabla^2 \phi = 0 \quad \text{in} \quad -h \leq z \leq 0$

B.C.s 1. $\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = -h$

2. $\frac{\partial y}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{on} \quad z = 0$

3. $\rho \frac{d \phi}{d t} + \rho g y + P_0 - \sigma y_{xx} = f(t) \quad \text{on} \quad z = 0$

Seek normal modes, solns of the form:

$\phi = \hat{\phi} e^{ik(x-ct)}$

Subbing into harmonic equation:

\[ \rho \frac{d \hat{\phi}}{d t} + \rho g \hat{\phi} + \hat{\phi}_x x_{xx} = f(t) \]

\[ \nabla^2 \hat{\phi} = 0 \quad \text{in} \quad -h \leq z \leq 0 \]

\[ \frac{\partial \hat{\phi}}{\partial z} = 0 \quad \text{on} \quad z = -h \]

\[ \frac{\partial \hat{\phi}}{\partial z} = \frac{\partial \phi}{\partial z} \quad \text{on} \quad z = 0 \]

\[ \rho \frac{d \hat{\phi}}{d t} + \rho g \hat{\phi} + \hat{\phi}_x x_{xx} = f(t) \quad \text{on} \quad z = 0 \]

\[ \text{travelling waves in } x \text{-direction with phase speed } c = \frac{\omega}{k} \text{ and wavelength } A = \frac{2\pi}{k}. \]
\[ \hat{\phi}_{zz} - k^2 \hat{\phi} = 0 \]

Solutions are \( \hat{\phi}(z) = e^{kz}, e^{-kz}, \sinh kz, \cosh kz \)

One way to satisfy B.C. 1, \( \frac{d\hat{\phi}}{dz} = 0 \) on \( z = -h \) by choosing:
\[ \hat{\phi}(z) = A \cosh k(z + h) \]
\( A \) a constant

Now, B.C. 2 \( \Rightarrow -ikc \dot{\hat{\phi}} = A k \sinh kl \)

B.C. 3 \( \Rightarrow (-ikc A \cosh kl + \rho j \dot{\hat{\psi}} + k^2 \sigma \dot{\hat{\phi}})e^{ik(x-ct)} \]
\[ = f(t) \text{ indep of } x \]

i.e. \( -ikc A \cosh kl + \rho j \dot{\hat{\psi}} + k^2 \sigma \dot{\hat{\phi}} = 0 \)

\[ \Rightarrow A = \frac{-ic \dot{\hat{\phi}}}{\sinh kl} \Rightarrow \text{sub into} \]

\[ c^2 = \left( \frac{\dot{\hat{\phi}}}{k} + \frac{\sigma k}{\rho} \right) \tanh kl \]

where \( C = \frac{\omega}{k} \) is the phase speed

Since \( C = \frac{\omega}{k} \), this yields the

Dispersion Relation:

\[ \omega^2 = (g k + \frac{\sigma k^3}{\rho}) \tanh kl \]
Note: as $h \to \infty$, $\tanh kh \to 1$ and we obtain the deep-water dispersion relation $\omega^2 = gk + \frac{\sigma k^3}{h}$

**Physical Interpretation**

- The relative importance of surface and gravity is prescribed by the Bond number
  \[ B_0 = \frac{\rho_2 g \lambda^2}{\sigma k^2} = \frac{\rho_2 \lambda^2}{4\pi^2 \sigma} = \frac{1}{\sqrt{\lambda}} \]

- For air–water, $B_0 \sim 1$ for $\lambda \sim 1.7 \text{ cm}$

- For $B_0 \gg 1$ ($\lambda \gg \lambda_c$), surface tension negligible
  $\Rightarrow$ Gravity waves

- For $B_0 \ll 1$ ($\lambda \ll \lambda_c$), gravity negligible
  $\Rightarrow$ Capillary waves

**Special Cases**

Recall: Shallow ($kh \ll 1$): $\tanh kh \approx kh - \frac{1}{3} k^3 h^3 + \ldots$
Deep ($kh \gg 1$): $\tanh kh \approx 1$

**A. Gravity Waves**: $B_0 \gg 1$, $c^2 = \frac{g}{k} \tanh kh$

- Shallow Water ($kh \ll 1$) $\Rightarrow c = \sqrt{gh}$

- All wavelengths travel at same speed
  i.e. Non-dispersive
\[ \Rightarrow \text{one can only surf in shallow water} \]

b) **Deep water** \( (kh \gg 1) \Rightarrow c = \sqrt{g/k} \)

- Long waves travel fastest
- E.g., drop large stone in a pond

\[ \boxed{\text{B. Capillary Waves} : B_0 << 1, \quad c^2 = \frac{g}{\rho} \tan k h} \]

a) **Deep water** \( kh \gg 1 \Rightarrow c = \sqrt{\frac{g}{\rho}} \)

- Short waves travel fastest
- E.g., raindrop hits a pond

b) **Shallow water** : \( kh \ll 1 \)

\[ \Rightarrow c = \sqrt{ghk^2} \]
Summary

\[ C \sim \frac{1}{k} \]

- deep gravity
- shallow gravity
- cap. waves
- const

\[ k_{\text{min}} = \left( \frac{\rho g}{\rho} \right)^{\frac{1}{2}} \]

\[ \lambda = \frac{2\pi}{k} \]

Note: 1. Four distinct scalings for \( C(H) \).

2. When \( \frac{dC}{dk} = 0 \), we have

\[ C_{\text{min}} = \left( \frac{4\rho g}{\rho} \right)^{\frac{1}{2}} \text{ for } k = \left( \frac{\rho g}{\rho} \right)^{\frac{1}{2}} \]

3. **Group Velocity**: when \( C = C(H) \), a wave is called **dispersive** since the different (Fourier) wave components (corresponding to different \( k \)) separate or disperse.

   - e.g. deep-water gravity: \( C \sim \sqrt{A} \)

   - in a dispersive system, the energy of a wave component does not propagate at the phase speed \( C = \frac{\omega}{k} \), but rather at the...
**Group Velocity**: \[ C_g = \frac{d\omega}{dk} = \frac{d}{dk} \left( CK \right) \]

E.g., **Deep Gravity Waves**: \[ C_g = \frac{d\omega}{dk} = \frac{1}{2} \sqrt{g/k} = \frac{1}{2} C \]

**Deep Capillary Waves**: \[ C_g = \frac{3}{2} C \]

4. **Flow past an obstacle**

**Note** \( C_{min} \): if \( U < C_{min} \), no steady waves generated by the obstacle.

- if \( U > C_{min} \): there are 2 \( k \)-values for which \( C = U \)

![Wave diagram](image)

i) the smaller represents a gravity wave with \[ C_g = \frac{C}{2} < C \implies \text{energy swept downstream} \]

ii) the larger \( k \) represent a capillary wave with \[ C_g = \frac{3}{2} C > C \implies \text{energy swept upstream (but quickly dissipated due to small \( k \))} \]
e.g. waves generated by a fishing line