Lecture 18. Contact line motion; spreading

Contact Line Motion: The Core/Paradox

- Contact line motion violates the NO-SLIP boundary condition in vicinity of contact line
  \( \Rightarrow \) drops can’t move!

- Air molecule adjoining boundary ahead of advancing line displaced by liquid

Resolution

Recall: continuum hypothesis is valid to ~10 molecules
  \( \Rightarrow \) expect it to fail on molecular scale, scale of contact line
A few key observations

I. Rolling Drops (E. Dussan 1977)

- Contact line advances like a tractor tread

II. Precursor Films (Hardy 1979)

- Molecular-scale films may adjoin drop, depending on surface chemistry
- Circumvents the moving-contact-line problem on the drop scale
- Shifts problem to the molecular scale

\[ \Rightarrow \text{a matter of surface chemistry rather than fluid mechanics} \]
The Moving Contact Line

Force of Traction:

\[ F(\theta_d) = \gamma_{SV} - \gamma_{SL} - \gamma \cos \theta_d = \gamma (\cos \theta_e - \cos \theta_d) \]

How does \( F \) depend on \( U \)? i.e. what is \( \theta_d(U) \)?

- Retreating contact line \((F < 0)\) can be examined with plate retraction expts., or by pushing air through a tube.
A case of advancing contact line ($E > 0$) examined by Hoffman (1975) for the case of $\Theta_e = 0$

**Observations**

\[ \Theta_e \sim V^{\frac{1}{3}} \sim C_a^{\frac{1}{2}} \]

Tanner's Law

How can we rationalize this dependence?
Flow near contact line of spreading liquid ($\theta_d > \theta_e$)

Consider $\theta_d \ll 1$, so that $\tan \theta_d = \frac{z}{x} \approx \theta_d$

$\Rightarrow z = \theta_d x$

Velocity gradient: \[
\frac{dV}{dz} \approx \frac{V}{\theta_d x}\]

Rate of viscous dissipation in corner:

\[
\hat{\Phi} = \int_{\text{corner}} m \left( \frac{dV}{dz} \right)^2 dV = m \int_0^\infty \int_0^{2\theta_d x} \frac{V^2}{\theta_d x^2} dx
\]

\[= 3\mu \int_0^\infty \frac{V^2}{\theta_d x^2} \theta_d x \, dx = \frac{3\mu V^2}{\theta_d} \int_0^\infty \frac{dx}{x}\]

*Dodgy Bit (de Gennes):*

\[
\int_0^\infty \frac{dx}{x} \sim \int \frac{\sqrt{x}}{x} = 1 \frac{\sqrt{x}}{x} \equiv \ell_D
\]

- molecular scale

where in exps, $15 \leq \ell_D \leq 20$
Energetics: \[ FV = \dot{\Phi} = \frac{3\mu kD}{\Theta d} V^2 \] 

Rate of work done by surface forces

Dissipation rate

Recall: \[ F = \frac{1}{2} (\Theta_d^2 - \Theta_e^2) \] 

In limit \( \Theta_e < \Theta_d \ll 1 \), \( \cos \Theta_e \approx 1 - \frac{\Theta_e^2}{2} \)

\[ \Rightarrow F = \frac{1}{2} (\Theta_d^2 - \Theta_e^2) \Rightarrow \text{sub into} \]

\[ \Rightarrow \frac{1}{2} (\Theta_d^2 - \Theta_e^2) V = \frac{3\mu kD}{\Theta d} V^2 \]

Contact Line Speed:

\[ V(\Theta_d) = \frac{V^*}{6kD} \Theta_d \ (\Theta_d^2 - \Theta_e^2) \]

where \( V^* \approx 30 \text{ m/s} \)

Note: 1. rationalizes Hoffman's data (obtained for \( \Theta_e = 0 \))

\[ \Rightarrow V \propto \Theta_d^3, \ \Theta d \propto V^{\frac{1}{2}} \]

2. \( V = 0 \) for \( \Theta_d = \Theta_e \) ⇒ static equilibrium

3. \( V \to 0 \) as \( \Theta_d \to 0 \): dissipation enhanced in a sharp corner
4. \( V(\theta_d) \) has a max value when

\[
\frac{\partial V}{\partial \theta_d} = \frac{V^*}{6l_d} \left( 3\theta_d^2 - \theta_e^2 \right) = 0
\]

\[\Rightarrow \theta_d = \frac{\theta_e}{\sqrt{3}}\]

\[\Rightarrow V_{\text{max}} = \frac{V^*}{6l_d} \frac{2}{3\sqrt{3}} \theta_e^3\]

\[\theta_e/\sqrt{3}\]

\[\theta_e\]

\[V\]

\[V_{\text{max}}\]

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**Example:** In water, \( V^* = 70 \text{ m/s} \), with \( \theta_e = 0.1 \text{ mld} \) and \( l_d = 20 \), we deduce

\[V_{\text{max}} = 0.2 \text{ mm/s}\]

\( \Rightarrow \) sets lower bound on extraction speed for water coating
Spreading

Recall: Gravity Currents

Stage I: $Re >> 1$

Scaling:

\[ \frac{dp}{dz} \sim \frac{\rho}{\rho} \Rightarrow \rho \sim \rho + \rho h \]

\[ \frac{dp}{dz} \sim g \Rightarrow \rho h \sim \rho g \]

Flow forced by gravity, resisted by inertia:

\[ \frac{\Delta p g h}{\rho} \sim \rho g \frac{d}{\rho} \]

\[ \Rightarrow \mathbf{v} \sim \sqrt{g' h} \text{ where } g' = g \frac{\rho}{\rho} \text{ is reduced gravity} \]

Continuity:

\[ V = \pi R^2(t) h(t) = \text{const volume} \]

\[ \Rightarrow h(t) = \frac{V}{\pi R^2(t)} \sim \frac{\sqrt{V}}{R^2(t)} \]

\[ \Rightarrow \mathbf{v} = \frac{dR}{dt} \sim \sqrt{g' V} \frac{1}{R} \]

\[ \Rightarrow R \, dR \sim \sqrt{g' V} \, dt \]

\[ \Rightarrow R(t) \sim (g' V)^{\frac{1}{4}} t^{\frac{3}{4}} \]
Note: $U \sim \sqrt{gh}$ decreases until $Re = \frac{UL}{\nu} \leq 1$, where we reach...

Stage II: $Re << 1$

- Flow forced by gravity, resisted by viscosity

\[ \frac{dp}{dz} \sim -g \Rightarrow \rho \sim \rho gh \text{ is horizontal pressure diff.} \]

\[ \frac{dp}{dz} - \nu \frac{du^2}{dz} \Rightarrow \frac{\rho gh}{\rho} \sim \nu \frac{U}{H^2} \]

\Rightarrow Eliminate $H \sim \sqrt{U/t}$ from continuity

\Rightarrow $U = \frac{dR}{dt} \sim \frac{\rho g' V^3}{\nu R^2}$

\Rightarrow $R \sim \left( \frac{\rho g' V^3}{\nu} \right)^{\frac{1}{8}} \times \frac{1}{8}$
The Spreading of Small Drops on Solids

Driven by both gravity and curvature pressures

Gravity: \( \frac{\Delta P_g}{\Delta P_c} \sim \frac{\gamma h}{R} \)

Curvature: \( \frac{\Delta P_c}{\Delta P_c} \sim \frac{\gamma h}{R^3} \)

Which dominates? \( \frac{\Delta P_g}{\Delta P_c} \sim \frac{\gamma h}{h^2} = \text{Bond} \)

\( \Rightarrow B_0 = \frac{\gamma h}{h} = \frac{\gamma h}{h^2} \sim \frac{h}{h} \Rightarrow \text{gravity becomes progressively more important} \)

Recall! drop behaviour depends on spreading parameter:

\[ S = \gamma_{sv} - \gamma_{sl} - \gamma \]

When \( S < 0 \): Partial Wetting

* spreading arises until a puddle obtains, with a flat central portion matching onto menisci that meet the boundary at \( \theta_c \)
When $S' > 0$: Complete Wetting

- Here, one expects spreading forced by tension at contact line.

\[
\frac{\mu V}{h} \cdot \pi R^2 = S' \cdot 2\pi R
\]

**Viscous resistance** \hspace{1cm} **Spreading force applied at contact line**

$$ \Rightarrow R \frac{dR}{dt} \sim \frac{S'}{\mu} h - \frac{S'}{\mu} \frac{V}{R^2} $$

$$ \Rightarrow R^3 \frac{dR}{dt} \sim \frac{SV}{\mu} $$

$$ \Rightarrow R(t) \sim \left( \frac{SV}{\mu} \right)^{\frac{1}{3}} + \frac{1}{3} $$

- But this is rarely observed.
- Instead, one sees $R \sim t^{\frac{1}{3}}$. Why?

*Hardy (1917)*: observed a precursor film placed dust/pillars ahead of spreading drop: precursor film knocks them down.
- The film is otherwise invisible, $\varepsilon \approx 20\,\text{Å}$.
- Its origins lie in the force imbalance at the contact line ($\delta > 0$).

* The stability of this precursor film results from interaction between fluid and solid (e.g. van der Waals forces).

**Physical Picture**

- Force balance at contact line changed dramatically by precursor film.

**Force at Contact Line:**

$$F = \gamma + \chi_\delta - \gamma \cos \theta_d - \theta_\delta = \delta (1 - \cos \theta_d)$$

$x \approx \theta_d^{3/2}$ for small $\theta_d$. 
Note: \( F < < S \) owing to precursor film

Now recall: \( F \bar{U} = \frac{3u \lambda d}{\Theta_1} \bar{U}^2 \)

defined speed of contact line \( \bar{U}(\Theta_1) \)

Letting \( F \rightarrow \bar{F} = \frac{\Theta_1}{2} \), we find

\[
\bar{V} = \frac{dR}{dt} = \frac{\Theta_1}{3b_0 m} \bar{F} = \frac{\bar{V}^*}{6 \lambda_0} \Theta_1^3
\]

It drop is small, it is a section of a sphere, so

\[
\bar{V} = \frac{\pi}{4} R^3 \Theta_1^3
\]

so that \( \frac{d\bar{V}}{dt} = 0 \) gives

\[
\frac{3}{R} \frac{dR}{dt} = -\frac{1}{\Theta_1} \frac{d\Theta_1}{dt}
\]

Sub in for \( \frac{dR}{dt} \) from *:

\[
\frac{1}{\Theta_1} \frac{d\Theta_1}{dt} = -\frac{\bar{V}^*}{R} \Theta_1^3
\]

Sub \( R = \Theta_1^{-\frac{1}{3}} = (\frac{\bar{V}}{\Theta_1})^{\frac{1}{3}} \) from *:

where \( \lambda = \bar{V}^{\frac{1}{3}} \)

\[
\frac{d\Theta_1}{dt} = -\frac{\bar{V}^*}{\lambda} \Theta_1^{13/3}
\]

\[
\Theta_1 = (\frac{\lambda}{\bar{V}^*})^{3/10}
\]
Thus, via \(*\), we have

\[ R \sim \mathcal{L}\left(\frac{V^* + \frac{z}{2}}{z}\right)^{\frac{1}{3}} \]

where

\[ V^* = \frac{\xi}{\mu} \]

\[ z = \frac{\sigma}{\sqrt{\beta}} \]