Physics Today

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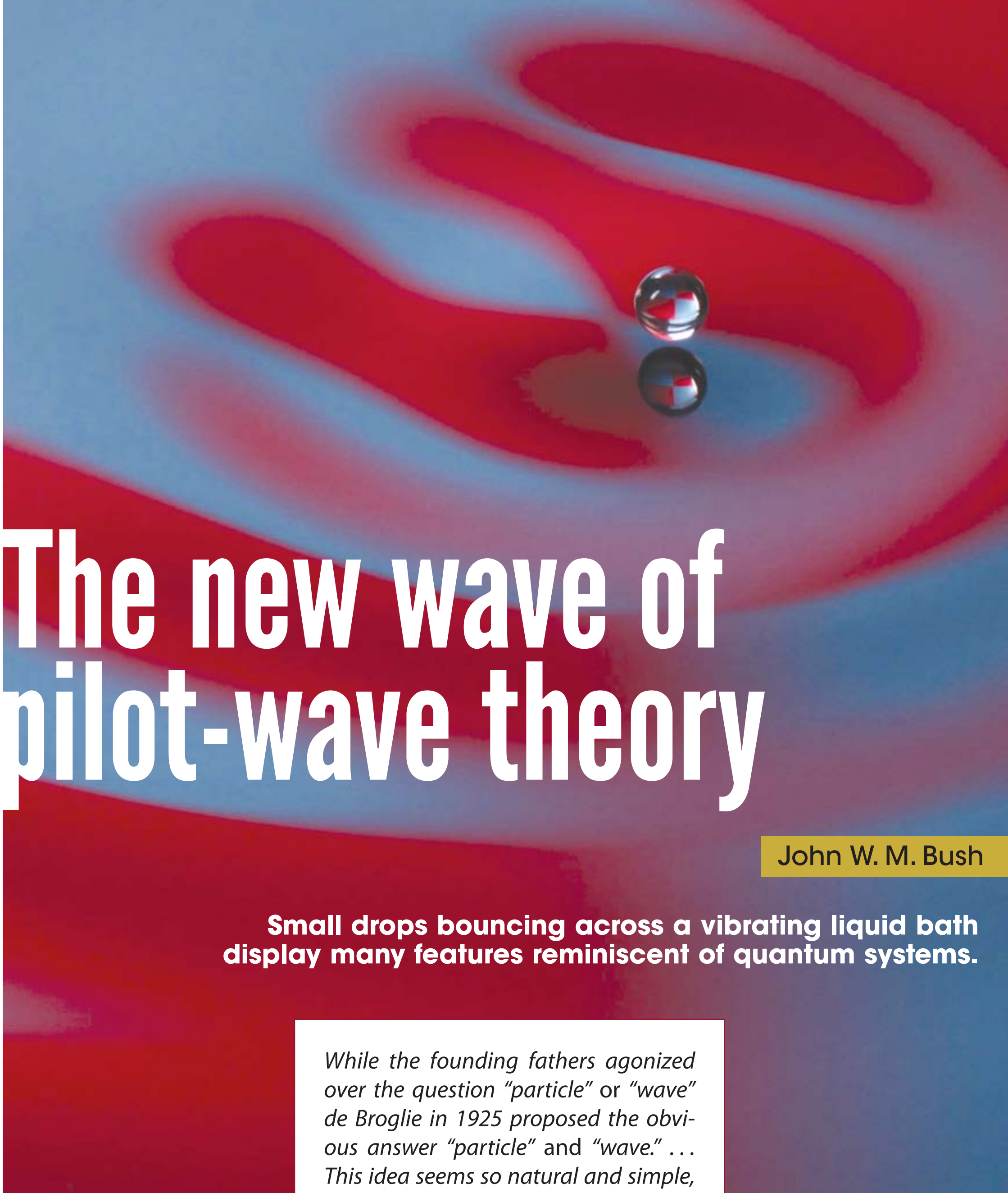
Citation: Physics Today **68**(8), 47 (2015); doi: 10.1063/PT.3.2882
View online: http://dx.doi.org/10.1063/PT.3.2882
View Table of Contents: http://scitation.aip.org/content/aip/magazine/physicstoday/68/8?ver=pdfcov
Published by the AIP Publishing



DANIEL HARRIS

The new wave of pilot-wave theory

John W. M. Bush

Small drops bouncing across a vibrating liquid bath display many features reminiscent of quantum systems.

> *While the founding fathers agonized over the question "particle" or "wave" de Broglie in 1925 proposed the obvious answer "particle" and "wave." . . . This idea seems so natural and simple, to resolve the wave–particle dilemma in such a clear and ordinary way, that it is a great mystery to me that it was so generally ignored.*
>
> —John S. Bell, *Speakable and Unspeakable in Quantum Mechanics*

If particle physics is the dazzling crown prince of science, fluid mechanics is the cantankerous queen mother: While her loyal subjects flatter her as being rich, mature, and insightful, many consider her to be démodé, uninteresting, and difficult. In her youth, she was more attractive. Her inconsistencies were taken as paradoxes that bestowed on her an air of depth and mystery. The resolution of her paradoxes left her less beguiling but more powerful, and marked her coming of age. She has since seen it all and has weighed in on topics ranging from cosmology to astronautics. Scientists are currently exploring whether she has any wisdom to offer on the controversial subject of quantum foundations.

Metaphor provides a means of using one subject to gain insight into another. Even at that imprecise level of comparison, fluid mechanics is a rich resource—for example, Isaac Newton spoke of corpuscles of light skipping through ether like stones on the surface of a pond. The mathematical description of physical systems allows for more exacting comparisons. Dynamic similarity, the cornerstone of laboratory modeling in fluid dynamics, arises between two fluid systems when a strict mathematical equivalence is achieved: The systems are governed by precisely the same equations. Thus, with meter-scale experiments, one can explore everything from astrophysical flows to the swimming of bacteria.

John W. M. Bush (http://math.mit.edu/~bush) is a professor of applied mathematics at the Massachusetts Institute of Technology in Cambridge.

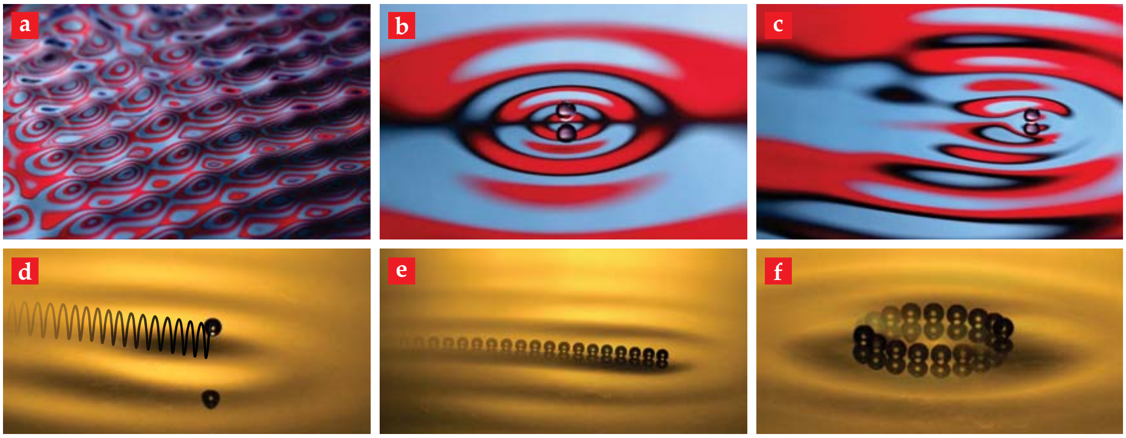


Figure 1. Riding the wave. Faraday waves **(a)** appear when a vibrating bath is driven by an acceleration γ that exceeds the Faraday threshold, γ_F. Red and blue reflected light allows for easier visualization of surface ripples. The phenomena illustrated in the following panels occur at accelerations slightly below γ_F. **(b)** A millimeter-sized droplet bounces in place on a vibrating bath. **(c)** A droplet walks along the bath surface, propelled by its own wave field. **(d)** The walking drop, whose reflection is also visible, bounces along the liquid surface, as shown by the oscillatory black line. **(e)** This strobed image of the walker illustrates the drop surfing on its own wave field. **(f)** This strobed image shows two walkers locked in orbit. (Images courtesy of Daniel Harris.)

Stronger than metaphor but weaker than mathematical equivalence is physical analogy, which may be drawn between two systems that are comparable in significant respects owing to similarities in their essential physics and underlying mathematical structure. At that level, fluid mechanics provides a framework for describing a broader class of nonfluidic systems. For example, ripple-tank experiments were used by Thomas Young to illustrate the wave nature of light.

A decade ago Yves Couder and Emmanuel Fort discovered that a millimeter-sized droplet may propel itself along the surface of a vibrating fluid bath by virtue of a resonant interaction with its own wave field; figure 1 shows several examples. This hydrodynamic system has since been shown to exhibit several features previously thought to be peculiar to the microscopic, quantum realm: single-particle diffraction, tunneling, wave-like statistics in confined geometries, quantized orbits, spin states, orbital-level splitting, and more. In this article I describe the walking-droplet system and, where possible, provide rationale for its quantum-like features. Further, I discuss the physical analogy between this hydrodynamic system and its closest relations in quantum theory, Louis de Broglie's pilot-wave theory and its modern extensions.

Louis de Broglie's pilot waves

The Copenhagen interpretation of quantum mechanics asserts that the well-established statistical description of quantum particles provided by standard quantum theory is the full story. Conversely, realist interpretations assert that a concrete dynamics underlies the statistical description; thus microscopic quantum particles follow trajectories just as do their macroscopic counterparts. The description of that hitherto unresolved dynamics would constitute a hidden-variable theory. The hydrodynamic system of Couder and Fort harkens back to a theory that predates the Copenhagen interpretation: the pilot-wave theory of de Broglie.

Given only the speed of light c and Planck's constant $\hbar$, dimensional analysis dictates that the natural frequency of a particle of mass m be proportional to the Compton frequency, $\omega_C = mc^2/\hbar$. The result is consistent with the de Broglie–Einstein relation, $mc^2 = \hbar\omega_C$, which de Broglie took as an expression of the wave nature of matter, the link between relativity and quantum mechanics.

In 1923 de Broglie proposed the first pilot-wave theory, according to which a quantum object such as an electron is a localized, vibrating particle moving in concert with a spatially extended, particle-centered pilot wave.[1] The particle vibration, or *zitterbewegung*, is characterized by an exchange, at the Compton frequency, between rest-mass energy and wave energy. De Broglie suggested that the guiding pilot wave is monochromatic, characterized by the de Broglie wavelength λ_B.

De Broglie did not detail the wave–particle interaction. He did, however, envisage the particle as a singularity following the rays of the guiding wave and moving at the wave's phase speed, $\lambda_B\omega/2\pi$, from which follows the de Broglie relation for the particle momentum, $p = 2\pi\hbar/\lambda_B$. He stressed the importance of the "harmony of phases," by which a particle's vibration stays in phase with its guiding wave; the wave and vibrating particle thus maintain a state of resonance.[1] He asserted that the pilot-wave dynamics could give rise to a statistical behavior consistent with standard quantum theory. The result was de Broglie's double-solution theory, which involved two distinct waves—the pilot wave and the statistical wave of standard quantum mechanics.

Based on his physical picture, de Broglie predicted single-particle diffraction and interference, the experimental confirmation of which led to his being awarded the Nobel Prize in Physics in 1929. As an electron passes through a narrow gap, he imagined, its guiding wave is diffracted, which leads to a divergence of particle paths. The pilot-wave picture provided de Broglie with a framework for rationalizing other quantum phenomena, including the uncertainty relations and the Bohr–Sommerfeld rule for quantized orbits. It garnered notable supporters, including Albert Einstein, who likewise sought to reconcile quantum mechanics and relativity through consideration of the wave nature of matter and declared that de Broglie had "lifted a corner of the great veil." Nevertheless, the Copenhagen interpretation gained ascendancy, its position bolstered by the proofs of John von Neumann and others that erroneously suggested the impossibility of hidden-variable theories and so discouraged their development. (See, for example, reference 2 and the article by Reinhold Bertlmann, PHYSICS TODAY, July 2015, page 40.)

David Bohm's variant

One year after the 1925 formulation of the Schrödinger equation, Erwin Madelung demonstrated that a particular transformation of the wavefunction provides a means of recasting the equation into hydrodynamic form. The corresponding system is a shallow, inviscid fluid layer evolving under the action of surface tension, if one associates the probability density with the fluid depth and the quantum velocity of probability (proportional to the gradient of the wavefunction phase) with the depth-averaged fluid velocity. Planck's constant $\hbar$ then plays a role analogous to surface tension in shallow-water hydrodynamics.

In 1952 David Bohm demonstrated that if one interprets the quantum velocity of probability as a particle velocity, one obtains statistical predictions consistent with those of standard quantum mechanics.[3] Bohmian mechanics has just a single wave: The guiding wave and the statistical wave are one and the same. It is thus significantly less rich dynamically than de Broglie's double-solution theory. Bohm's formulation played an important historical role in providing a counterexample to the impossibility proofs that held sway at the time. It also reminds us that any hidden-variable theory consistent with the statistical predictions of quantum mechanics has the feature of quantum nonlocality.

Jean-Pierre Vigier and others extended the original Bohmian mechanics by including in the dynamics an additional stochastic element that arises through the particle's interaction with a subquantum realm.[4] According to these theories, the quantum velocity of probability represents a mean velocity about which the real particles jostle, just as, because of their Brownian motion, gas molecules jostle about streamlines in a gas flow. In his later years, de Broglie also sought the origins of his pilot wave in a stochastic subquantum realm.[1]

Vacuum-based pilot-wave theories

Nowadays de Broglie's realm is called the quantum vacuum; it is a turbulent sea, roiling with waves associated with a panoply of force-mediating fields such as the photon and Higgs fields. Insofar as they interact with quantum particles, all such fields are candidates for de Broglie's pilot wave. Stochastic electrodynamics posits electromagnetic energy in the quantum vacuum at 0 K, in the form of a stochastic, fluctuating "zero-point field" (ZPF), whose spectrum has an energy $U(\omega) = \hbar\omega/2$ per normal mode. The ZPF gives an alternative rationale for numerous quantum mechanical phenomena, including the Casimir effect, van der Waals forces, and the blackbody radiation spectrum.[5]

According to Luis de la Peña and colleagues, the ZPF might also excite a quantum particle's *zitterbewegung*.[6] As the particle translates, the vibration interacts selectively with waves in the vacuum field, a resonant interaction that amplifies an electromagnetic pilot wave. In this picture, the de Broglie pilot wave consists of a carrier wave with the Compton wavelength $\lambda_C = 2\pi\hbar/mc$, modulated over the speed-dependent de Broglie wavelength. The coupling between the particle and the electromagnetic field results in nonlocal dynamics. For multiparticle systems exhibiting quantum nonlocality, entanglement would be an expression of wave-induced correlations.[6] The concept of a particle as a ZPF-driven oscillating charge with a resonance at ω_C was further explored by Bernard Haisch and colleagues, who suggested that ZPF–particle interactions might offer insight into the origins of inertial mass; in that way, they sought to link the ZPF, the quantum wave nature of matter, and relativistic mechanics.[7]

Bouncing and walking on a liquid

Surface tension σ specifies the energy per area of a liquid–gas surface. Owing to surface tension, nonplanar distortions of an initially flat liquid surface are energetically costly, as are aspherical distortions of small drops. According to dimensional analysis, an inviscid drop with mass m and surface tension σ will have a natural frequency proportional to $\sqrt{\sigma/m}$. The drop is an oscillator: When perturbed, it will oscillate at its natural frequency, exchanging surface energy and kinetic energy.

When a fluid bath is vibrated vertically, its surface becomes unstable to a field of so-called Faraday waves, such as shown in figure 1a, when the driving acceleration γ exceeds the Faraday threshold γ_F. As the threshold is crossed, the first waves to appear are subharmonic (having a frequency equal to half the driving frequency) with a wavelength λ_F determined by the dispersion relation for water waves. The phenomena of interest arise when a millimeter-sized drop is placed on a bath of silicone oil vibrating below the Faraday threshold, so the bath's surface would remain flat if not for the drop. The drop may avoid coalescing with the bath owing to the persistence of a thin air layer between drop and bath during impact (see figure 1b). As the bouncing drop strikes the bath, the dominant force resisting its intrusion is that generated by the surface tension: The bath surface behaves roughly like a linear spring that reverses its direction.

When the bouncing frequency becomes commensurate with the bath's most unstable Faraday mode (the first to appear as γ is progressively increased), the drop generates at each impact a radially expanding wavefront behind which is triggered a decaying field of Faraday waves. Couder and Fort

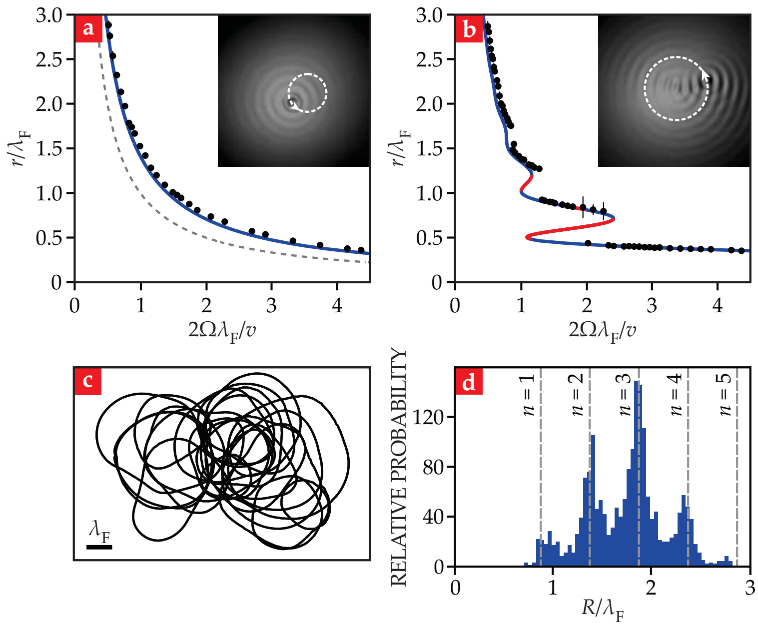


Figure 2. A walker in a rotating frame. The orbital radius *r* of a walker (normalized by the Faraday wavelength λ_F) as a function of bath rotation rate Ω and walking speed *v* demonstrates qualitatively different behavior at **(a)** low memory and **(b)** intermediate memory. Data are filled circles. Theoretical curves are solid lines; blue segments correspond to stable regions, red to unstable regions. The data and theoretical curve in panel a are offset from the dashed curve, which represents $r = v/2\Omega$. The displacement is due to an increased effective mass for the drop,[18] induced by its pilot wave. **(c)** At high memory the walker trajectory is chaotic, but most of the trajectory arcs have a radius of curvature *R* roughly of the form $R = (n + 1)\,\lambda_F/2$, resulting in the multimodal statistics shown in **(d)**. (For additional detail see refs. 11 and 12.)

discovered that the bouncing state can then destabilize into a dynamic walking state; figures 1c–1e illustrate the walking drop. A critical feature of the walker dynamics is the "path memory" that arises due to the longevity of the pilot wave.[8] At impact, the walker receives a lateral propulsive force proportional to the local slope of the interface. For low memory—that is, when γ is well below γ_F—the waves are quickly damped. As γ approaches γ_F (high memory), the waves are relatively persistent, and the force acting on the walker depends on the drop's distant past and its environment, both of which are encoded in the pilot wave. The walker's quantum-like features emerge at high memory, when this dynamical nonlocality is most pronounced.

The bouncing and walking behavior of the droplets has been well characterized both experimentally and theoretically.[9] Periodic or chaotic bouncing states can arise, depending on the vibrational frequency and amplitude, drop size, and fluid properties. A drop will bounce most readily when the driving frequency is commensurate with its natural frequency $\sqrt{\sigma/m}$, and it will walk most readily when resonance is achieved between the bouncing drop and its pilot wave. Theorists have developed increasingly sophisticated models to describe both the bouncing dynamics and the wave generation. The most recent models have predictive power: Given the control parameters, they specify if a droplet will coalesce, bounce, or walk. When the droplet walks, the theory determines both its gait and its speed. When a walker is in a resonant state, for which the drop and wave are synchronized, the so-called stroboscopic approximation describes its horizontal motion in terms of a droplet surfing on its pilot wave and provides rationale for the stability of various dynamical states, including the rectilinear-walking and circular-orbiting states shown in figures 1d–1f.

Orbital dynamics

Fort and colleagues demonstrated in 2010 that when a vibrating bath is caused to rotate at a constant angular speed Ω, a walker with speed *v* travels in a circular orbit with radius $r \approx v/2\Omega$, provided the memory is sufficiently low.[10] As the memory increases, however, the orbiting walker begins to interact with its own wake. As figure 2 shows, the result is a kind of orbital quantization. Guided by the identical forms of the Coriolis force acting on a mass in a rotating frame and the Lorentz force acting on a charge in a uniform magnetic field, Fort and company drew the physical analogy between the angular quantization arising in their system and that of the Landau levels that arise when a charged particle orbits in a magnetic field. The Faraday wavelength λ_F in the fluid system plays the role of the de Broglie wavelength λ_B for the orbiting charged particle.

A few years later, my colleagues and I examined the high-memory regime of the rotating system both experimentally[11] and theoretically.[12] At high memory, virtually all orbital states become unstable, typically via a period-doubling transition to chaos. Figure 2c shows a typical chaotic trajectory, which tends to move along arcs whose radii of curvature correspond to those of the unstable orbits. Thus the trajectory's radius of curvature is characterized by multimodal statistics, with peaks arising at integer multiples of $\lambda_F/2$, as shown in figure 2d. One can thus rationalize the quantum-like statistics in terms of chaotic pilot-wave dynamics: In its chaotic state, the walker switches between accessible unstable orbital states.

Antonin Eddi and coworkers studied walker pairs locked in circular orbits by their wave field; figure 1f shows an example. When the bath was rotated, they found that the orbital radius increased or decreased according to the relative sense of the orbital motion of the walkers and the bath; the phenomenon is reminiscent of Zeeman splitting.[13] The stroboscopic model indicates the possibility of hydrodynamic spin states and level splitting for single walkers.[12] As memory increases, even in the absence of rotation, orbital solutions arise in which the wave force balances the radial inertial force. The stability of such hydrodynamic spin states is a subject of current interest.

By encapsulating a magnetic suspension in a droplet and applying a spatially varying magnetic field, Couder's group investigated a walker in a central force field—that of a two-dimensional simple harmonic oscillator.[14] Their study revealed a variety of complex orbital forms that, as shown in figure 3, are quantized in both mean radius and angular momentum. As for the case of the rotating bath, in the chaotic regime prevalent at high memory, the walker drifts between accessible unstable orbital states and manifests multimodal statistics.

Interactions with boundaries

A couple of years ago, Couder's group and mine joined forces to examine a walker in a circular corral.[15] We found that at low memory, the walker orbits along the bounding walls. As memory increases, the walker executes progressively more complex trajectories that include wobbling circular orbits, drifting elliptical orbits, and epicycles. At the highest memory we examined, the trajectories become irregular and chaotic, as shown in figure 4. Despite the complexity of the trajectories, a coherent statistical behavior emerges. Specifically, the histogram (figure 4b) of the walker's position is roughly prescribed by the amplitude of the Faraday mode that arises when the bath is driven just above the Faraday threshold. If one views the concentric circular peaks evident in the histogram as orbital states of the system between which the walker switches when it is in a chaotic state, the multimodal statistics may again be rationalized in terms of chaotic pilot-wave dynamics.[16] That the walker system is closer to de Broglie's mechanics than to Bohm's is particularly evident in the corral, where the distinction between the complex pilot wave that guides the walker and the axially symmetric statistical wave is clear.

Couder's group examined the dynamics of a walker impinging on a region with a submerged barrier and found that the walker occasionally demonstrates a kind of tunneling (see figure 5a). As the simulation in figure 5b reveals, the incident pilot-wave field is partially reflected at the boundary but also has an evanescent tail. The reflected wave typically causes the approaching walker to be reflected; however, the droplet occasionally crosses the barrier, with a probability that decreases exponentially with barrier width. This statistical behavior, reminiscent of quantum tunneling, is presumably rooted in the chaotic pilot-wave dynamics but has yet to be rationalized theoretically. Characterizing the interaction between walkers and boundaries is a central goal of ongoing theoretical work.

Couder and Fort demonstrated that when a walker passes through a slit—more precisely, over

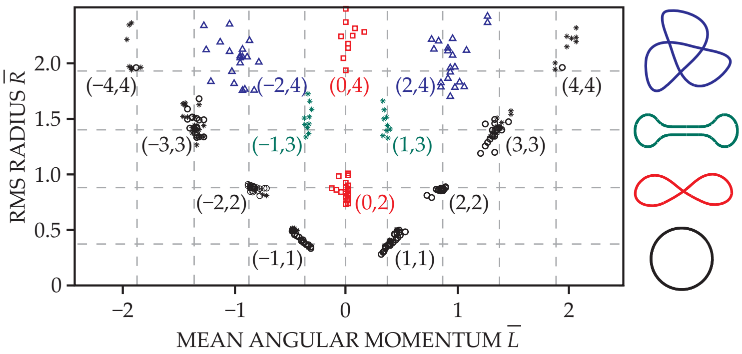


Figure 3. A walker in a harmonic potential. The observed orbits of a walker subject to a two-dimensional harmonic-oscillator force assume many shapes, including circles (black), lemniscates (red), dumbbells (green), and trefoils (blue). A double quantization emerges when the orbits of a walker of mass m and speed v are classified by $\overline{R}$, the root-mean-square radius (normalized by the Faraday wavelength λ_F) and $\overline{L}$, the mean angular momentum (normalized by $mv\lambda_F$). (Adapted from S. Perrard et al., *Nat. Commun.* **5**, 3219, 2014.)

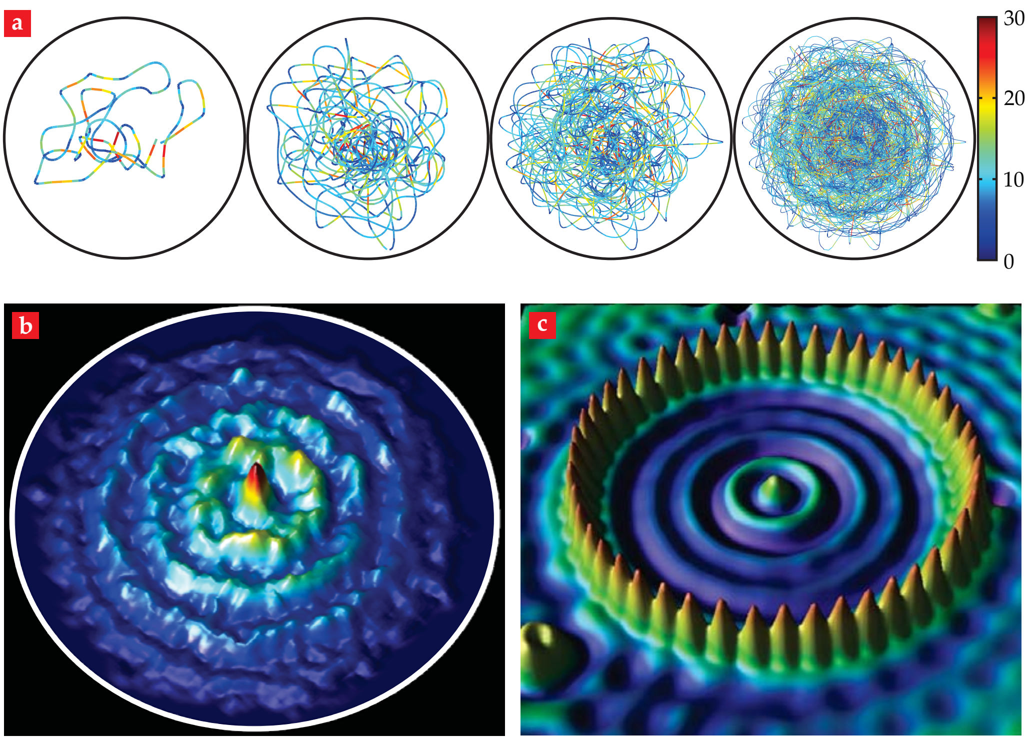


Figure 4. A walker in a circular corral. The high-memory trajectories of increasing length depicted in **(a)** are color coded according to droplet speed in millimeters per second, as indicated on the sidebar. The corral has a radius of 14.3 mm. **(b)** The histogram of the walker's position corresponds roughly to the amplitude of the corral's most unstable Faraday mode. (Panels a and b adapted from ref. 15.) The emerging statistics of the confined walker are reminiscent of **(c)** the statistics of electrons in a quantum corral. (Panel c courtesy of Donald Eigler.)

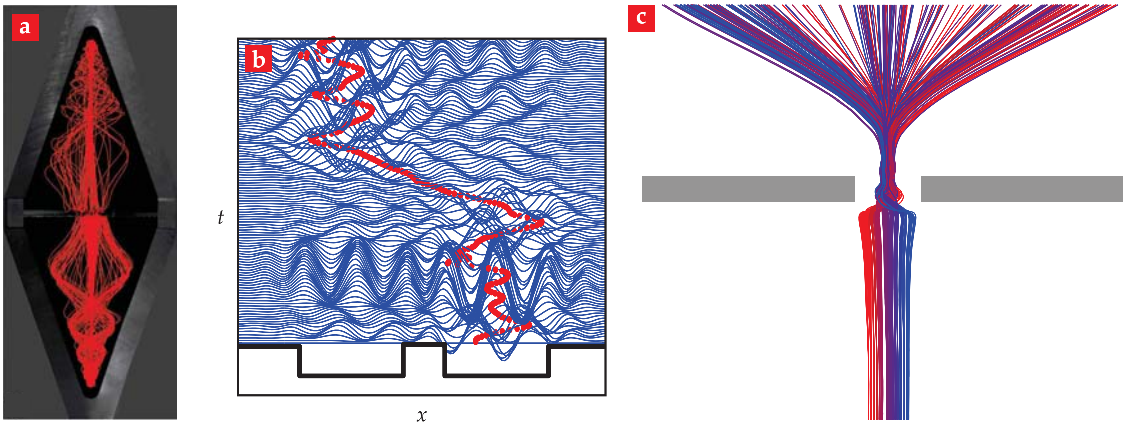


Figure 5. A walker encountering obstacles. When a walker in a confined geometry encounters a region with a submerged barrier, it can occasionally "tunnel" across the barrier. **(a)** The experimental trajectories shown here represent 110 trials in which a walker encountered a submerged barrier; it crossed the obstacle 14 times. The rhombus-shaped frame is 45 mm across at its waist. (Adapted from A. Eddi et al., *Phys. Rev. Lett.* **102**, 240401, 2009; courtesy of Yves Couder.) **(b)** In this simulation, the walker encountering the barrier is confined to a line. Space and time coordinates *x* and *t* trace the evolution of both the walker (red) and the pilot wave (blue). Note the tunneling event midway through the simulation. (Courtesy of André Nachbin and Paul Milewski.) **(c)** Observed walker trajectories are splayed by the spatial confinement associated with a single 14.7-mm-wide slit. (Courtesy of Daniel Harris and Giuseppe Pucci.)

a gap in a submerged barrier—its rectilinear motion is disturbed, as illustrated in figure 5c, owing to the disruption of its pilot-wave field.[17] The behavior is analogous to that envisioned by de Broglie in his prediction of electron diffraction and in his rationalization of the position–momentum uncertainty relation. In the double-slit geometry, the pilot wave passes through both slits even though the droplet passes through only one. The walker thus interacts with both slits because of the spatial delocalization of its pilot wave.

Bridging the micro–macro chasm

The walker represents an example of an oscillating particle moving in resonance with its own wave field. The droplet moves in a state of energetic equilibrium with the vibrating bath, navigating a wave field sculpted by its motion. The walker system continues to extend the range of classical systems to include features previously thought to be exclusive to the quantum realm. What might one infer if unaware that it is a driven, dissipative pilot-wave system?

One would be puzzled by the prevalence of quantization and multimodal statistics. Inferring a consistent trajectory equation would be possible only in certain limits. Doing so in the limit of weak walker acceleration would suggest that the droplet's effective mass depends on its speed.[18] Multiple-particle interactions would be characterized by inexplicable scattering events and bound states, and baffling correlations. If one could detect a walker only by interacting with the fluid bath, the measurement process would become intrusive. If a detector confined the walker spatially, one would infer a position–momentum uncertainty relation. If detection required collisions with other droplets, disruption of the pilot wave would destroy any coherent statistical behavior that might otherwise arise.

The walker dynamics are markedly different from those described by Bohmian mechanics but similar to those imagined in de Broglie's double-solution theory. An oscillating particle is piloted by its self-generated wave field, and its wave energy is related to its frequency by $\hbar$ in de Broglie's mechanics and by surface tension σ for the walker. In both, the pilot wave and statistical wave have the same wavelength but different geometric form. In the walker system, the wavelength is fixed by the bath's driving frequency, whereas in its quantum predecessor, it is speed dependent, prescribed by the de Broglie relation, $p = 2\pi\hbar/\lambda_B$. One common criticism of de Broglie's pilot-wave theory—that it is too complicated to ever work—would seem to be put to rest by its hydrodynamic variant. The walker system demonstrates that when an oscillating particle moves in resonance with a monochromatic pilot wave, its nonlocal dynamics may give rise to quantization and multimodal statistics.

De Broglie did not specify the physical origins of the pilot wave, but modern extensions of his mechanics have sought an electromagnetic pilot wave originating in the quantum vacuum. A marked difference between the walker system and the vacuum-based models is that in the latter the length scales of the pilot wave and statistical wave are different. Nevertheless, the physical analogy between the two systems is intriguing: The vibrating bath plays the role of the zero-point field in driving the system; the drop's bouncing, that of the particle's *zitterbewegung* in triggering the pilot wave.[9] Like the electromagnetic pilot-wave system envisaged by de la Peña and company,[6] the walker system is a driven, dissipative system in which an oscillator is excited by background vibration and moves in a field structured by its motion, giving rise to a pilot-wave dynamics that is nonlocal in both space and time.

The resolution of the fluid mechanical paradoxes invariably arose through the elucidation of unimagined dynamics at an unanticipated scale. The hydrodynamic pilot-wave system, when considered in light of vacuum-based pilot-wave theories, would seem to suggest the possibility of an unresolved dynamics on the Compton scale, that a successful nonlocal hidden-variable theory might be based on the physical picture of particles interacting with the vacuum and propagating in an equilibrium state of energy exchange within it. Since the Compton frequency—which is approximately 10^{21} Hz for an electron, for example—sets the time and length scales for particle-pair production from the vacuum, the resolution of such Compton-scale dynamics poses serious experimental challenges.

One is invited to believe that the macroscopic and microscopic worlds are separated by a philosophical chasm so deep that there is no hope of ever crossing it. In the depths of the abyss, profound and obscure, lurk the impossibility proofs. To those who cannot see beyond strict mathematical equivalence, one must concede that bouncing droplets are not quantum particles. To those with the relatively modest goal of establishing philosophical equivalence through physical analogy, the walking-droplet system has extended a speculative bridge from the macroscopic toward the microscopic and linked with a structure emerging from the other side: the modern extensions of de Broglie's pilot-wave theory. Rickety though this bridge may be, under construction, it offers striking new views to workers from both sides.

I gratefully acknowledge the financial support of NSF and thank Mason Biamonte, Yves Couder, Emmanuel Fort, Daniel Harris, Anand Oza, Giuseppe Pucci, and Rodolfo Ruben Rosales for valuable input.

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